Turkish J. Ineq., 10 (1) (2026), Pages 168 –179 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

A FRACTIONAL INTEGRAL APPROACH TO INEQUALITIES FOR EXPONENTIALLY CONVEX FUNCTIONS VIA RAINA OPERATORS

JAVANSHIR ZEYNALOV[1], ABDULLA HASANOV[1], AYISHA AHMADOVA[1], GOZAL JAFARLI[1], AND AHMET OCAK AKDEMİR[2]

Abstract. This study derives new integral inequalities for exponentially convex functions using Raina fractional integral operators. First, some fundamental properties under the concept of exponentially convexity are examined, and suitable integral structures are constructed for this class of functions. Then, new inequalities are derived using the generalized framework provided by Raina fractional integral operators. The results obtained encompass classical integral inequalities and many known results in the literature as special cases under appropriate parameter choices. In this respect, the study offers new contributions to both the fractional analysis and convexity theory literature and demonstrates the effectiveness of Raina operators on integral inequalities.

1. Introduction

The theory of integral inequalities, a significant area of research within mathematical analysis, plays a fundamental role in approximate calculations, numerical analysis, optimization, and many fields of applied science. In particular, Hermite-Hadamard, Jensen, Holder, and Minkowski type inequalities are closely related to convexity theory and offer powerful tools for controlling the average behavior of functions. These inequalities are widely used not only in theoretical analysis but also in differential equations, probability theory, economics, and engineering applications.

The concept of convexity is central to the theory of inequalities, and results obtained from classical convex functions have been extended over time to different generalizations. In this context, various types of convexity, such as $s-$convexity, $m-$convexity, $(\alpha, m)-$convexity, and exponentially convexity, have gained significant importance in the literature. Exponentially convex functions, in particular, emerge as a natural generalization of logarithmic and exponential structures, offering more flexible modeling possibilities in both

Key words and phrases. Convex function, Raina integral operator, Mittag-Leffler function.

2010 *Mathematics Subject Classification.* Primary: 26D15. Secondary: 26A51.

Received: 10/05/2026 *Accepted*: 13/07/2026.

Cite this article as: J. Zeynalov, A. Hasanov, A. Ahmadova, G. Jafarli, A.O. Akdemir, A Fractional Integral Approach to Inequalities for Exponentially Convex Functions via Raina Operators, Turkish Journal of Inequalities, 10(1) (2026), 168-179.

theoretical and applied problems. Therefore, the study of integral inequalities under different classes of convexity has become a research topic of intense interest in recent years ([2, 6, 8, 10, 13, 14, 16–18, 21–23]).

On the other hand, fractional analysis is a branch of analysis that is based on the generalization of classical derivative and integral concepts and stands out as a powerful tool, especially in modeling systems exhibiting non-local behavior and memory effects. Fractional integral operators, unlike classical integral operators, allow for the construction of more realistic models by taking into account past-dependent effects and spatial interactions. In addition to classical fractional operators such as Riemann-Liouville and Caputo, new fractional integral operators containing more general kernel functions have been defined in recent years, and numerous new inequalities have been obtained using these operators.

Within this general framework, Raina fractional integral operators emerge as an important structure bridging the gap between special function theory and fractional analysis. These operators offer a broader class encompassing classical fractional operators through generalized hypergeometric functions and Mittag-Leffler type kernels. In this respect, Raina operators are a highly effective tool for generalizing integral inequalities and obtaining new upper bounds.

In recent years, numerous studies have focused on investigating Hermite-Hadamard type inequalities under different convexity classes using fractional integral operators. These studies have contributed both to the extension of classical results and to the derivation of sharper and more general inequalities. However, studies that consider exponentially convex functions together with Raina fractional integral operators are still limited, indicating a significant research gap in the literature. Now, we are in a position to give some important concepts that will be beneficial for our main findings.

Definition 1.1. (Exponentially Convex Function): (See [3]) $f:\ K\subseteq\mathbb{R}\to\mathbb{R}$ be a function and said to be exponentially convex function if satisfies the following condition

$$f((1-\iota_1)\eta+\iota_1\theta)\le(1-\iota_1)\frac{f(\eta)}{e^{\nu\eta}}+\iota_1\frac{f(\theta)}{e^{\nu\theta}}$$

for all η,$\theta\in I$,$\iota_1\in$ [0,1] and $\nu\in\mathbb{R}$.

Definition 1.2. ($(m_1,m_2)-$**Convex Function):** (See [11]) For $m_1,m_2\in(0,1]$, a function $g:[0,v]\to\mathbb{R}$ be a $(m_1,m_2)-$convex function if

$$g(m_1wx+m_2(1-w)y)\le m_1wg(x)+m_2(1-w)g(y)$$

for all $x,y\in[0,v]$ and $w\in[0,1]$.

Definition 1.3. (Exponentially $(m_1,m_2)-$**Convex Function):** (See [4]) A mapping g :$I=[0,v]\to\mathbb{R},v>0$ is called an exponentially $(m_1,m_2)-$convex function, if

$$g(m_1wa_1+m_2(1-w)a_2)\le m_1w\frac{f(a_1)}{e^{\lambda a_1}}+m_2(1-w)\frac{f(a_2)}{e^{\lambda a_2}}$$

for all $m_1a_1,m_2a_2\in I$,$w\in[0,1]$ and $\lambda\in\mathbb{R},m_1,m_2\in(0,1]$.

Definition 1.4. (Mittag-Leffler Function): (See [7]) The function

$$E_\alpha(\iota) = \sum_{k=0}^{\infty} \frac{\iota^\kappa}{\Gamma(a\kappa+1)} \qquad (Re(\alpha) > 0, \iota \in \mathbb{C})$$

is called as Mittag-Leffler function.

The Maclaurin series expansion of the exponentially function is given by

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

By generalizing this series, for $\Re(\alpha) > 0$, one obtains the function

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}.$$

This function can be regarded as a natural generalization of the exponentially function and plays a fundamental role in fractional calculus. In particular, when $\alpha = 1$, the classical exponentially function is recovered, i.e.,

$$E_1(x) = e^x.$$

Therefore, the function $E_\alpha(x)$ is known in the literature as the one-parameter Mittag–Leffler function and is widely used in fractional differential equations and integral operators.

Definition 1.5. (Raina Fractional Integral Operator)
Let $\rho, \lambda > 0$ and let $\sigma = \{\sigma(k)\}_{k=0}^{\infty}$ be a bounded sequence of positive real numbers. Then, the function

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} x^k, \quad (|x| < \infty),$$

is called the generalized function associated with the Raina operator.

Let $\phi(\tau)$ be an integrable function. The left-sided generalized fractional integral operator is defined by

$$(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\phi)(x) = \int_a^x (x-\tau)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}(\omega(x-\tau)^\rho)\phi(\tau)\,d\tau, \quad (x > a > 0),$$

and the right-sided generalized fractional integral operator is defined by

$$(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\phi)(x) = \int_x^b (\tau-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}(\omega(\tau-x)^\rho)\phi(\tau)\,d\tau, \quad (0 < x < b).$$

These operators were introduced by Raina in [19].

For more information about fractional calculus see the papers [1, 9, 12, 19, 20, 24].

The main objective of this study is to obtain new integral inequalities for exponentially convex functions using Raina fractional integral operators. In this context, the properties of suitable integral structures under exponentially convexity are first examined, and then new Hermite-Hadamard type inequalities are derived using this structure. It is shown that the obtained results can be reduced to classical inequalities under appropriate parameter choices, thus revealing the generalizing nature of the results. Furthermore, the contribution of this study is highlighted by comparing the obtained inequalities with similar results in the literature.

This study contributes to obtaining new and more general inequalities through Raina fractional integral operators by combining fractional analysis with convexity theory. In this respect, it is believed that the results offer an important framework that can be used in both theoretical mathematics and applied fields.

2. Main results

Lemma 2.1. (See [25]) *Let $\xi_1 < \xi_2$ and let $\upsilon : [\xi_1, \xi_2] \to \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) such that $\upsilon' \in L[\xi_1, \xi_2]$. Then the following identity holds:*

$$\begin{aligned}
&\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{1}{4\kappa_1(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}(\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1})} \\
&\quad\times\left[(\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1) + (\mathcal{J}^{\sigma_1,\kappa_1,g}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)\right] \\
&= \frac{\xi_2-\xi_1}{4(\psi(\xi_2)-\psi(\xi_1))^{\frac{\lambda_1}{\kappa_1}}\mathcal{F}^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}(\omega_1(\psi(\xi_2)-\psi(\xi_1))^{\rho_1})} \\
&\quad\times\int_0^1 \Sigma^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1,g}(\zeta)\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)\,d\zeta.
\end{aligned}$$

Theorem 2.1. *Let $\lambda_1, \rho_1 > 0$, $\omega_1 \in \mathbb{R}$ and let $\sigma_1 = \{\sigma_1(k)\}_{k=0}^{\infty}$ be a sequence of non-negative real numbers. Let $\xi_1 < \xi_2$ and let $\upsilon : [\xi_1, \xi_2] \to \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) such that $\upsilon' \in L[\xi_1, \xi_2]$. If $|\upsilon'|$ is $(m_1, m_2)-$exponentially convex on $[\xi_1, \xi_2]$, $m_1, m_2 \in (0, 1)$, where $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\right| \\
&\leq \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right. \\
&\qquad\times\left(\frac{\zeta\left|\upsilon'(\frac{\xi_1}{m_1})\right|}{e^{\omega_1\frac{\xi_1}{m_1}}} + \frac{(1-\zeta)\left|\upsilon'(\frac{\xi_2}{m_2})\right|}{e^{\omega_1\frac{\xi_2}{m_2}}}\right)d\zeta \\
&\quad+\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}] \\
&\qquad\left.\times\left(\frac{\zeta\left|\upsilon'(\frac{\xi_1}{m_1})\right|}{e^{\omega_1\frac{\xi_1}{m_1}}} + \frac{(1-\zeta)\left|\upsilon'(\frac{\xi_2}{m_2})\right|}{e^{\omega_1\frac{\xi_2}{m_2}}}\right)d\zeta\right].
\end{aligned}$$

Proof. By Lemma 2.1 and the properties of the absolute value, we have

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\right| \\
&\le \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\Bigg[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}] \\
&\qquad\times\left|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)\right|d\zeta \\
&\quad+\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}] \\
&\qquad\times\left|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)\right|d\zeta\Bigg].
\end{aligned}$$

Since $|\upsilon'|$ is $(m_1,m_2)-$exponentially convex on $[\xi_1,\xi_2]$, we get

$$\left|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)\right|\le\frac{\zeta\left|\upsilon'(\frac{\xi_1}{m_1})\right|}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)\left|\upsilon'(\frac{\xi_2}{m_2})\right|}{e^{\omega_1\frac{\xi_2}{m_2}}}.$$

Substituting this estimate into the previous inequality yields

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\right| \\
&\le \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\Bigg[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}] \\
&\qquad\times\left(\frac{\zeta\left|\upsilon'(\frac{\xi_1}{m_1})\right|}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)\left|\upsilon'(\frac{\xi_2}{m_2})\right|}{e^{\omega_1\frac{\xi_2}{m_2}}}\right)d\zeta \\
&\quad+\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}] \\
&\qquad\times\left(\frac{\zeta\left|\upsilon'(\frac{\xi_1}{m_1})\right|}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)\left|\upsilon'(\frac{\xi_2}{m_2})\right|}{e^{\omega_1\frac{\xi_2}{m_2}}}\right)d\zeta\Bigg].
\end{aligned}$$

This is the desired result. Hence, the proof is completed. □

Theorem 2.2. *Let $\lambda_1,\rho_1>0$, $\omega_1\in\mathbb{R}$ and let $\sigma_1=\{\sigma_1(k)\}_{k=0}^{\infty}$ be a sequence of non-negative real numbers. Let $\xi_1<\xi_2$ and let $\upsilon:[\xi_1,\xi_2]\to\mathbb{R}$ be a differentiable function on (ξ_1,ξ_2) such that $\upsilon'\in L[\xi_1,\xi_2]$. If $|\upsilon'|^q$ is $(m_1,m_2)-$exponentially convex on $[\xi_1,\xi_2]$, $m_1,m_2\in(0,1)$,* *where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{\upsilon(\xi_1) + \upsilon(\xi_2)}{2} - \frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2) + (\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]} \right| \\
&\le \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]} \left[\left(\int_0^1 \left| (1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}] \right|^p d\zeta \right)^{\frac{1}{p}} \right. \\
&\qquad \times \left(\int_0^1 \left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1\frac{\xi_1}{m_1}}} + \frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1\frac{\xi_2}{m_2}}} \right] d\zeta \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left| \zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}] \right|^p d\zeta \right)^{\frac{1}{p}} \\
&\qquad \left. \times \left(\int_0^1 \left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1\frac{\xi_1}{m_1}}} + \frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1\frac{\xi_2}{m_2}}} \right] d\zeta \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Proof. By Lemma 2.1 and the properties of the absolute value, we have

$$\begin{aligned}
&\left| \frac{\upsilon(\xi_1) + \upsilon(\xi_2)}{2} - \frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2) + (\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]} \right| \\
&\le \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]} \left[\int_0^1 (1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}] \right. \\
&\qquad \times \left| \upsilon'(\zeta\xi_1 + (1-\zeta)\xi_2) \right|^q d\zeta \\
&\quad + \int_0^1 \zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}] \\
&\qquad \left. \times \left| \upsilon'(\zeta\xi_1 + (1-\zeta)\xi_2) \right|^q d\zeta \right].
\end{aligned}$$

Applying Hölder's inequality to each integral gives

$$\begin{aligned}
&\left| \frac{\upsilon(\xi_1) + \upsilon(\xi_2)}{2} - \frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2) + (\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]} \right| \\
&\le \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]} \left[\left(\int_0^1 \left| (1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}] \right|^p d\zeta \right)^{\frac{1}{p}} \right. \\
&\qquad \times \left(\int_0^1 \left| \upsilon'(\zeta\xi_1 + (1-\zeta)\xi_2) \right|^q d\zeta \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left| \zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}] \right|^p d\zeta \right)^{\frac{1}{p}} \\
&\qquad \left. \times \left(\int_0^1 \left| \upsilon'(\zeta\xi_1 + (1-\zeta)\xi_2) \right|^q d\zeta \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Since $|\upsilon'^q|$ is $(m_1, m_2)-$exponentially convex on $[\xi_1, \xi_2]$, we obtain

$$\left|\upsilon'(\zeta\xi_1 + (1-\zeta)\xi_2)\right|^q \leq \frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1 \frac{\xi_1}{m_1}}} + \frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1 \frac{\xi_2}{m_2}}}.$$

Therefore,

$$\left(\int_0^1 \left|\upsilon'(\zeta\xi_1 + (1-\zeta)\xi_2)\right|^q d\zeta\right)^{\frac{1}{q}} \leq \left(\int_0^1 \left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1 \frac{\xi_1}{m_1}}} + \frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1 \frac{\xi_2}{m_2}}}\right] d\zeta\right)^{\frac{1}{q}}.$$

Substituting this estimate into the previous inequality, we get

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2) + (\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\right| \\
&\leq \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\left[\left(\int_0^1 \left|(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}\right]\right|^p d\zeta\right)^{\frac{1}{p}}\right. \\
&\qquad \times \left(\int_0^1 \left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1 \frac{\xi_1}{m_1}}} + \frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1 \frac{\xi_2}{m_2}}}\right] d\zeta\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left|\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}\right]\right|^p d\zeta\right)^{\frac{1}{p}} \\
&\qquad \left.\times \left(\int_0^1 \left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1 \frac{\xi_1}{m_1}}} + \frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1 \frac{\xi_2}{m_2}}}\right] d\zeta\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This is the desired result. Hence, the proof is completed. □

Theorem 2.3. *Let $\lambda_1, \rho_1 > 0$, $\omega_1 \in \mathbb{R}$ and let $\sigma_1 = \{\sigma_1(k)\}_{k=0}^{\infty}$ be a sequence of non-negative real numbers. Let $\xi_1 < \xi_2$ and let $\upsilon : [\xi_1, \xi_2] \to \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) such that $\upsilon' \in L[\xi_1, \xi_2]$. If $|\upsilon'|^q$ is $(m_1, m_2)-$exponentially convex on $[\xi_1, \xi_2]$, $m_1, m_2 \in (0,1)$, where $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2} - \frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2) + (\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\right| \\
&\leq \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\left[\left(\int_0^1 (1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}\right] d\zeta\right)^{1-\frac{1}{q}}\right. \\
&\quad \times \left(\int_0^1 (1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}\right]\left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1 \frac{\xi_1}{m_1}}} + \frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1 \frac{\xi_2}{m_2}}}\right] d\zeta\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}\right] d\zeta\right)^{1-\frac{1}{q}} \\
&\quad \left.\times \left(\int_0^1 \zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}\right]\left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1 \frac{\xi_1}{m_1}}} + \frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1 \frac{\xi_2}{m_2}}}\right] d\zeta\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. By Lemma 2.1 and the properties of the absolute value, we have

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\right|\\
&\leq \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}]}\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\right.\\
&\qquad\qquad\times|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q\,d\zeta\\
&\quad+\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]\\
&\qquad\qquad\left.\times|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q\,d\zeta\right].
\end{aligned}$$

Using the power-mean inequality, we get

$$\begin{aligned}
&\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\,|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|\,d\zeta\\
&\leq\left(\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]d\zeta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}]\,|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q\,d\zeta\right)^{\frac{1}{q}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]\,|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|\,d\zeta\\
&\leq\left(\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]d\zeta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}]\,|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q\,d\zeta\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|\upsilon'^q|$ is $(m_1,m_2)-$exponentially convex on $[\xi_1,\xi_2]$, we have

$$|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q\leq\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1\frac{\xi_2}{m_2}}}.$$

Substituting this estimate into the preceding inequalities yields

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\right| \\
&\leq \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\left[\left(\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}\right]d\zeta\right)^{1-\frac{1}{q}}\right. \\
&\quad\times\left(\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}\right]\left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1\frac{\xi_2}{m_2}}}\right]d\zeta\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}\right]d\zeta\right)^{1-\frac{1}{q}} \\
&\quad\left.\times\left(\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}\right]\left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1\frac{\xi_2}{m_2}}}\right]d\zeta\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This is the desired inequality. Hence, the proof is completed. □

Theorem 2.4. *Let $\lambda_1,\rho_1>0$, $\omega_1\in\mathbb{R}$ and let $\sigma_1=\{\sigma_1(k)\}_{k=0}^{\infty}$ be a sequence of non-negative real numbers. Let $\xi_1<\xi_2$ and let $\upsilon:[\xi_1,\xi_2]\to\mathbb{R}$ be a differentiable function on (ξ_1,ξ_2) such that $\upsilon'\in L[\xi_1,\xi_2]$. If $|\upsilon'|^q$ is $(m_1,m_2)-$exponentially convex on $[\xi_1,\xi_2]$, $m_1,m_2\in(0,1)$, where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\right| \\
&\leq \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\left[\frac{1}{p}\int_0^1\left|(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}\right]\right|^p d\zeta\right. \\
&\quad+\frac{1}{q}\int_0^1\left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1\frac{\xi_2}{m_2}}}\right]d\zeta \\
&\quad+\frac{1}{p}\int_0^1\left|\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}\right]\right|^p d\zeta \\
&\quad\left.+\frac{1}{q}\int_0^1\left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1\frac{\xi_2}{m_2}}}\right]d\zeta\right].
\end{aligned}$$

Proof. By Lemma 2.1 and the properties of the absolute value, we have

$$\begin{aligned}
&\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\right| \\
&\leq \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\left[\int_0^1(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}\right]\left|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)\right|^q d\zeta\right. \\
&\quad\left.+\int_0^1\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}\right]\left|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)\right|^q d\zeta\right].
\end{aligned}$$

Applying Young's inequality

$$ab\leq\frac{a^p}{p}+\frac{b^q}{q},\qquad \frac{1}{p}+\frac{1}{q}=1,$$

to both integrals, we obtain

$$\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\right|$$
$$\leq \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\left[\frac{1}{p}\int_0^1\left|(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}\right]\right|^p d\zeta\right.$$
$$+\frac{1}{q}\int_0^1|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta$$
$$+\frac{1}{p}\int_0^1\left|\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}\right]\right|^p d\zeta$$
$$\left.+\frac{1}{q}\int_0^1|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q d\zeta\right].$$

Since $|\upsilon'^q|$ is (m_1,m_2) −exponentially convex on $[\xi_1,\xi_2]$, we have

$$|\upsilon'(\zeta\xi_1+(1-\zeta)\xi_2)|^q\leq\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1\frac{\xi_2}{m_2}}}.$$

Therefore,

$$\left|\frac{\upsilon(\xi_1)+\upsilon(\xi_2)}{2}-\frac{(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_1+;\omega_1}\upsilon)(\xi_2)+(\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,\xi_2-;\omega_1}\upsilon)(\xi_1)}{2(\xi_2-\xi_1)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\right|$$
$$\leq \frac{\xi_2-\xi_1}{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\right]}\left[\frac{1}{p}\int_0^1\left|(1-\zeta)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}(1-\zeta)^{\rho_1}\right]\right|^p d\zeta\right.$$
$$+\frac{1}{q}\int_0^1\left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1\frac{\xi_2}{m_2}}}\right]d\zeta$$
$$+\frac{1}{p}\int_0^1\left|\zeta^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(\xi_2-\xi_1)^{\rho_1}\zeta^{\rho_1}\right]\right|^p d\zeta$$
$$\left.+\frac{1}{q}\int_0^1\left[\frac{\zeta|\upsilon'(\frac{\xi_1}{m_1})|^q}{e^{\omega_1\frac{\xi_1}{m_1}}}+\frac{(1-\zeta)|\upsilon'(\frac{\xi_2}{m_2})|^q}{e^{\omega_1\frac{\xi_2}{m_2}}}\right]d\zeta\right].$$

This is the desired inequality. Hence, the proof is completed. □

Conclusion

In this study, new integral inequalities for exponentially convex functions are obtained using Raina fractional integral operators. First, the fundamental properties of functions under the concept of exponentially convexity are examined, and integral structures suitable for this class of functions are defined. Then, new Hermite-Hadamard type inequalities are derived using the generalized framework provided by Raina fractional integral operators. These inequalities can be reduced to classical results under appropriate parameter choices and encompass many known inequalities in the literature as special cases.

The results clearly demonstrate that fractional integral operators, especially Raina-type operators, are a highly effective tool in generalizing integral inequalities. Thanks to the flexible parameter structure of these operators, it is possible to obtain results valid for a wider class of functions by going beyond the classical convexity assumptions. This provides both a more comprehensive theoretical analysis opportunity and increases the modeling capacity for different application areas.

In terms of future research, it is possible to extend the approach discussed in this study under different types of convexity (e.g., h-convexity, Godunova-Levin type convexity, and logarithmic convexity). Furthermore, the investigation of variable-order versions of Raina fractional integral operators can contribute to obtaining more general and realistic models. In addition, the application of the obtained results in fields such as numerical analysis, optimization, and differential equations is an important research direction.

In conclusion, this study contributes to the development of new integral inequalities through Raina fractional integral operators by combining fractional analysis with convexity theory, and provides a solid foundation for further research in this field.

References

[1] A.O. Akdemir, S. Aslan, M.A. Dokuyucu, E. Celik, *Exponentially convex functions on the coordinates and novel estimations via Riemann–Liouville fractional operator*, Journal of Function Spaces, **2023**(1) (2023), 1–9.

[2] M. Alomari, M. Darus, *The Hadamard's inequality for s-convex function of two variables on the co-ordinates*, International Journal of Mathematical Analysis, **2**(13) (2008), 629–638.

[3] M.U. Awan, M.A. Noor, K.I. Noor, *On exponential convexity and related inequalities,* Applied Mathematics and Information Sciences, **12**(2) (2018), 405–409.

[4] S.I. Butt, B. Bayraktar, J. Nasir, *Novel Ostrowski type inequalities via exponentially* $(m_1, m_2)-$ *convex functions and their applications,* Annals of the University of Craiova-Mathematics and Computer Science Series, **51**(2) (2024), 488–504.

[5] S.S. Dragomir, S. Fitzpatrick, *The Hadamard inequalities for s-convex functions in the second sense*, Demonstratio Mathematica, **32**(4) (1999), 687–696.

[6] S.S. Dragomir, *On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese Journal of Mathematics, **5**(4) (2001), 775–788.

[7] R. Gorenflo, J. Loutchko, Y. Luchko, *Mittag-Leffler Functions, Related Topics and Applications*, Springer, 2020.

[8] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, 1952.

[9] J.E. Hernandez, M. Vivas-Cortez, *Hermite-Hadamard inequalities type for Raina fractional integral operator using η-convex functions,* Revista de Matematica Teoria y Aplicaciones, **26**(1) (2019), 1–20.

[10] H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Mathematicae, **48**(1) (1994), 100–111.

[11] H. Kadakal, $(\alpha, m_1, m_2)-$*convexity and some inequalities of Hermite-Hadamard type*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **68**(2) (2019), 2128–2142.

[12] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[13] D.S. Mitrinovic, P.M. Vasic, *Analytic Inequalities*, Springer, 1970.

[14] D.S. Mitrinovic, J.E. Pecaric, A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[15] C. Niculescu, L.-E. Persson, *Convex Functions and Their Applications*, Springer, 2006.

[16] M.E. Ozdemir, H. Kavurmaci, A.O. Akdemir, M. Avci, *Inequalities for convex and s-convex functions on* $\Delta = [a, b] \times [c, d]$, Journal of Inequalities and Application, **2012** (2012), 1–19.

[17] M.E. Ozdemir, M.A. Latif, A.O. Akdemir, *On some Hadamard-type inequalities for product of two s-convex functions on the co-ordinates,* Journal of Inequalities and Applications, **2012** (2012), 1–13.

[18] J.E. Pecaric, Y.L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, 1992.

[19] R.K. Raina, *Certain operators involving the generalized hypergeometric function*, Indian Journal of Pure and Applied Mathematics, **18**(8) (1987), 713–720.

[20] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, 1993.

[21] M.Z. Sarikaya, E. Set, M.E. Ozdemir, S.S. Dragomir, *New some Hadamard's type inequalities for co-ordinated convex functions*, arXiv preprint arXiv:1005.0700, 2010.

[22] M.Z. Sarikaya, *On the Hermite–Hadamard-type inequalities for co-ordinated convex function via fractional integrals*, Integral Transforms and Special Functions, **25**(2) (2014), 134–147.

[23] E. Set, M. Sardari, M.E. Ozdemir, J. Rooin, *On generalizations of the Hadamard inequality for* (α, m)*-convex functions*, Kyungpook Mathematical Journal, **52**(3) (2012), 307–317.

[24] M. Tariq, S.K. Sahoo, S.K. Ntouyas, *Some refinements of Hermite–Hadamard type integral inequalities involving refined convex function of the Raina type*, Axioms, **12**(2) (2023), 124.

[25] T. Tunc, H. Budak, F. Usta, M.Z. Sarikaya, *On new generalized fractional integral operators and related fractional inequalities,* Konuralp Journal of Mathematics, **8**(2) (2020), 268–278.

[26] S. Varosanec, *On h-convexity*, Journal of Mathematical Analysis and Applications, **326**(1) (2007), 303–311.

[1]Department of Mathematics,
Nakhcivan State University,
Nakhcivan-Azerbaijan
Email address: cavansirzeynalov@ndu.edu.az
Email address: abdullahesenov@ndu.edu.az
Email address: ayishaahmadova@ndu.edu.az
Email address: gozelceferli@ndu.edu.az

[2]Department of Mathematics,
Ağri Ibrahim Cecen University,
Ağri, Turkey
Email address: aocakakdemir@gmail.com