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HERMITE–HADAMARD TYPE INEQUALITIES FOR $*-$CONVEX AND $*-$P CLASS FUNCTIONS VIA NON-NEWTONIAN CALCULUS

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Abstract. In this paper, we establish Hermite–Hadamard type inequalities for $*-$convex functions as well as for functions belonging to the $*-$P class within the framework of Non-Newtonian calculus. By employing a key identity involving $*-$differentiable functions, we derive several new inequalities and obtain a variety of corollaries corresponding to classical special cases such as log-convex, GA-convex, GG-convex, AH-concave, HA-convex, and HH-concave functions. The results not only generalize existing inequalities but also provide a unified approach that extends several known results in the literature.

1. INTRODUCTION

Convexity plays a fundamental role in mathematical analysis and optimization theory. It provides a powerful framework for studying the behavior of functions, particularly in establishing inequalities and estimating integral bounds. A function is said to be convex if its graph lies below the line segment connecting any two points on the graph.

Mathematically, a function $f : [a, b] \to \mathbb{R}$ is convex if the following inequality holds:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Convex functions satisfy several important integral inequalities. One of the most classical results in this context is the Hermite–Hadamard inequality, which provides bounds for the integral average of a convex function.

If $f : [a, b] \to \mathbb{R}$ is convex, then the following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

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A comprehensive account of the early developments on Hermite–Hadamard type inequalities can be found in [8]. The classical result has since been extended to broader convexity classes such as s-convex [20], m-convex [21], and (α, m)-convex functions [22], together with identity-based estimates for differentiable mappings [19]. More recent studies employ various fractional integral operators – conformable [23], variable-order [28], Atangana–Baleanu [7], k-Atangana–Baleanu [18], and Katugampola [29] – while Simpson [6], Hermite–Hadamard–Jensen–Mercer [25], and Steffensen [24] type results have been obtained for several classes of convex functions.

In the following, we present the definition of functions belonging to the class $P(I)$.

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to belong to the class $P(I)$ (or briefly, $f \in P(I)$) if f is nonnegative and satisfies

$$f(tx + (1-t)y) \leq f(x) + f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0, 1]$.

Convexity can also be extended in a multiplicative sense, leading to the concept of log-convexity, which plays an important role in various inequalities.

Definition 1.2. [8] A function $f : I \to [0, \infty)$ is said to be log-convex (or AG-convex) if $\ln f$ is convex. Equivalently, for all $x, y \in I$ and $t \in [0, 1]$, the following inequality holds:

$$f\left(tx + (1-t)y\right) \leq [f(x)]^t \, [f(y)]^{1-t}.$$

Log-convex functions satisfy a refined version of the Hermite–Hadamard inequality expressed in logarithmic form [8].

Theorem 1.1. *Let $f : I \to [0, \infty)$ be a log-convex function on $[a, b]$. Then the following inequality holds:*

$$\ln\left[f\left(\frac{a+b}{2}\right)\right] \leq \frac{1}{b-a}\int_a^b \ln f(x)\,dx \leq \frac{\ln f(a) + \ln f(b)}{2}.$$

It is useful to recall the notion of a mean function, which plays a central role in generalized convexity concepts. [2]

Definition 1.3. A mapping $M : (0, \infty) \times (0, \infty) \to (0, \infty)$ is called a mean function if it satisfies the following properties:

(i) $M(x, y) = M(y, x)$,
(ii) $M(x, x) = x$,
(iii) $x < M(x, y) < y$ whenever $x < y$,
(iv) $M(ax, ay) = aM(x, y)$ for all $a > 0$.

Some classical examples of mean functions are listed below:

- Arithmetic mean: $A(x, y) = \dfrac{x+y}{2}$,
- Geometric mean: $G(x, y) = \sqrt{xy}$,

- Harmonic mean: $H(x,y) = \dfrac{1}{A\left(\frac{1}{x}, \frac{1}{y}\right)}$.

Using mean functions, one can define a generalized notion of convexity as follows [2].

Definition 1.4. Let $f : I \to (0,\infty)$ be a continuous function, where $I \subset (0,\infty)$. Let M and N be two mean functions. Then f is said to be MN-convex (respectively, MN-concave) if

$$f\left(M(x,y)\right) \leq (\geq)\, N\left(f(x), f(y)\right)$$

for all $x, y \in I$.

For some publications on convexity with respect to means, one may refer to [1,9–12].

In [19] Kavurmaci et al. gave the following inequality for convex functions:

Theorem 1.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ \leq\ & \frac{(x-a)^2}{b-a}\left[\frac{|f'(x)|+2|f'(a)|}{6}\right] + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|+2|f'(b)|}{6}\right] \end{aligned}$$

for each $x \in [a,b]$.

2. Preliminaries on Non-Newtonian Calculus and Non-Newtonian Convexity

Non-Newtonian calculus, introduced by Grossman and Katz [16], generalizes classical analysis by replacing standard arithmetic operations with alternative ones generated by a pair of bijective functions α and β, where α acts on arguments and β acts on values. The resulting $*$-calculus encompasses classical calculus as the special case $\alpha = \beta = I$, and yields geometric calculus when $\beta(x) = e^x$. This flexible structure makes it particularly well-suited for modeling phenomena of a multiplicative or exponential nature that resist classical treatment. Extensive background on non-Newtonian and multiplicative calculus is available in [3–5, 13–17, 26, 27].

Given a generator α defined on a subset $A \subseteq \mathbb{R}$, we construct the α-arithmetic over the realm A by defining the following operations:

$$\begin{array}{lll} \alpha\text{-addition} & y \dot{+} z & = \alpha\left(\alpha^{-1}(y) + \alpha^{-1}(z)\right) \\ \alpha\text{-subtraction} & y \dot{-} z & = \alpha\left(\alpha^{-1}(y) - \alpha^{-1}(z)\right) \\ \alpha\text{-multiplication} & y \dot{\times} z & = \alpha\left(\alpha^{-1}(y) \times \alpha^{-1}(z)\right) \\ \alpha\text{-division} & y \,\dot{/}\, z & = \alpha\left(\frac{\alpha^{-1}(y)}{\alpha^{-1}(z)}\right), \quad z \neq 0 \\ \alpha\text{-order} & y \,\dot{<}\, z & \Longleftrightarrow \alpha^{-1}(y) < \alpha^{-1}(z) \end{array} \tag{2.1}$$

We say that α generates α-arithmetic. As special cases, the identity function I recovers classical arithmetic, while $\alpha(x) = e^x$ yields geometric arithmetic. Given two generators α and β, the pair $* = (\alpha\text{-arithmetic}, \beta\text{-arithmetic})$ defines the $*$-calculus, where α-arithmetic governs function arguments and β-arithmetic governs function values. For a real number a, we use the shorthand notations $\dot{a} = \alpha(a)$, $\ddot{a} = \beta(a)$, and $\bar{a} = \alpha^{-1}(a)$.

Definition 2.1 ($*$-Limit)**.** Let f be a function and $a \in A$. A value $b \in B$ is called the $*$-limit of f at a if, for every sequence $\{a_n\}$ of arguments of f distinct from a that α-converges to a, the sequence $\{f(a_n)\}$ β-converges to b. In this case we write

$$*\text{-}\lim_{x\to a} f(x) = b. \tag{2.2}$$

Moreover, if $\bar{f}(x) = \beta^{-1}\{f(\alpha(x))\}$, then

$$*\text{-}\lim_{x\to a} f(x) = \beta\left\{\lim_{x\to \bar{a}} \bar{f}(x)\right\}. \tag{2.3}$$

Definition 2.2 ($*$-Continuity)**.** A function f is $*$-continuous at a point $a \in A$ if and only if a is in the domain of f and

$$*\text{-}\lim_{x\to a} f(x) = f(a). \tag{2.4}$$

Definition 2.3 ($*$-Derivative)**.** Let $\iota(x) = \beta\{\alpha^{-1}(x)\}$ denote the isomorphism from α-arithmetic to β-arithmetic. The $*$-derivative of f at a point a is defined as

$$f^*(a) = *\text{-}\lim_{x\to a}\left\{\frac{f(x) \ddot{-} f(a)}{\iota(x) \ddot{-} \iota(a)} \ddot{:}\right\}, \tag{2.5}$$

provided the limit exists. When both f^* and the classical derivative f' exist, they are related by

$$f^*(x) = \beta\left\{f'\Big(\alpha^{-1}(x)\Big)\right\}. \tag{2.6}$$

Definition 2.4 (Transformation Operator T [14])**.** Let $\bar{f} : \mathbb{R} \to \mathbb{R}$ be a classically defined function. Its $*$-calculus counterpart $f : \mathbb{R}_\alpha \to \mathbb{R}_\beta$ is obtained via the operator T defined by

$$f(x) = (T\bar{f})(x) = \beta\Big(\bar{f}(\alpha^{-1}(x))\Big). \tag{2.7}$$

The inverse operator T^{-1} recovers $\bar{f}$ from f via

$$\bar{f}(x) = (T^{-1}f)(x) = \beta^{-1}(f(\alpha(x))). \tag{2.8}$$

We call f the $*$-calculus counterpart of $\bar{f}$.

Definition 2.5 ($*$-Composition [14])**.** For functions $f, g : \mathbb{R}_\alpha \to \mathbb{R}_\beta$, the $*$-composition $f\tilde{\circ}g$ is defined by

$$(f\tilde{\circ}g)(x) = f\Big(\alpha\Big(\beta^{-1}(g(x))\Big)\Big) = f\langle g(x)\rangle, \tag{2.9}$$

where $\langle\cdot\rangle$ denotes the composition delimiter. If $\bar{f} \circ \bar{g}$ exists in classical calculus, then

$$f\tilde{\circ}g = T[\bar{f} \circ \bar{g}]. \tag{2.10}$$

Definition 2.6 ($*$-Integral)**.** The $*$-integral of a $*$-continuous function f on $\dot{[}r, s\dot{]}$ is defined via the $*$-average $\overset{*}{M}{}_r^s f$ of f on $\dot{[}r, s\dot{]}$ as

$$*\int_r^s f = [\iota(x) \ddot{-} \iota(a)] \ddot{\times} \overset{*}{M}{}_r^s f. \tag{2.11}$$

When f is $*$-continuous on $\left[\dot{r}, \dot{s}\right]$ and $\bar{f}$ is continuous on $[\bar{r}, \bar{s}]$, the $*$-integral relates to the classical integral by

$$*\int_{r}^{s} f(x)\, d_{\alpha}x = \beta\left\{\int_{\bar{r}}^{\bar{s}} \bar{f}(x)\, dx\right\}, \tag{2.12}$$

where $\bar{f}(x) = \beta^{-1}\{f(\alpha(x))\}$.

Theorem 2.1 (Second Fundamental Theorem of $*$-Calculus [16]). *Let h be a $*$-differentiable function whose $*$-derivative h^{*} is $*$-continuous on $[r, s]$. Then*

$$*\int_{r}^{s} h^{*}(x)\, d_{\alpha}x = h(s) \ddot{-} h(r). \tag{2.13}$$

Theorem 2.2 ($*$-Chain Rule [14]). *Let $I, J \subseteq \mathbb{R}_{\alpha}$ be open intervals, $f : I \to \mathbb{R}_{\beta}$, $g : J \to \mathbb{R}_{\beta}$, with $T^{-1}\{f(I)\} \subseteq J$. If f is $*$-differentiable at $x_0 \in I$ and g is $*$-differentiable at $T^{-1}\{f(x_0)\}$, then $g\tilde{\circ}f$ is $*$-differentiable at x_0 with*

$$(g\tilde{\circ}f)^{*}(x_0) = (g^{*}\tilde{\circ}f)(x_0) \ddot{\times} f^{*}(x_0). \tag{2.14}$$

Equivalently, if $(\bar{g} \circ \bar{f})'(x)$ exists, then

$$(g\tilde{\circ}f)^{*}(x_0) = T(\bar{g} \circ \bar{f})'(x_0). \tag{2.15}$$

Theorem 2.3 ($*$-Substitution Rule [14]). *Let $\varphi : [a, b] \subseteq \mathbb{R}_{\alpha} \to I \subseteq \mathbb{R}_{\beta}$ be $*$-differentiable with a $*$-continuous derivative, and let $f : I \to \mathbb{R}_{\beta}$ be $*$-continuous. Then*

$$*\int_{a}^{b} (f\tilde{\circ}\varphi)(x) \ddot{\times} \varphi^{*}(x)\, d_{\alpha}x = *\int_{\iota^{-1}(\varphi(a))}^{\iota^{-1}(\varphi(b))} f(u)\, d_{\alpha}u. \tag{2.16}$$

Theorem 2.4 ($*$-Integration by Parts [14]). *Let $f, g : [a, b] \subseteq \mathbb{R}_{\alpha} \to I \subseteq \mathbb{R}_{\beta}$ be $*$-differentiable with $*$-continuous derivatives. Then*

$$*\int_{a}^{b} f(x) \ddot{\times} g^{*}(x)\, d_{\alpha}x = (f \ddot{\times} g)(b) \ddot{-} (f \ddot{\times} g)(a) \ddot{-} *\int_{a}^{b} f^{*}(x) \ddot{\times} g(x)\, d_{\alpha}x. \tag{2.17}$$

Definition 2.7. [17, 26] Let I_{α} be an interval in $\mathbb{R}_{\alpha}$. Then, $f : I_{\alpha} \to \mathbb{R}_{\beta}$ is said to be $*$-convex on $\left[\dot{a}, \dot{b}\right]$ if

$$f\left(\dot{t}\dot{\times}\dot{a}\dot{+}\left(\dot{1}\dot{-}\dot{t}\right)\dot{\times}\dot{b}\right) \ddot{\leq} \ddot{t}\ddot{\times}f\left(\dot{a}\right)\ddot{+}\left(\ddot{1}\ddot{-}\ddot{t}\right)\ddot{\times}f\left(\dot{b}\right) \tag{2.18}$$

holds, where $t \in [0, 1]$ and $a, b \in \mathbb{R}$ with $a < b$. If the above inequality is reversed, then f is said to be $*$-concave. (2.18) can be stated in classical arithmetic as

$$\beta^{-1}\left\{f\left(\alpha\left(ta + (1 - t)\, b\right)\right)\right\} \leq t\beta^{-1}\left\{f\left(\alpha(a)\right)\right\} + (1 - t)\,\beta^{-1}\left\{f\left(\alpha(b)\right)\right\}. \tag{2.19}$$

From the definition, one can easily see that if $f : [a, b] \subseteq I_{\alpha} \to \mathbb{R}$ and $\bar{f} : [\bar{a}, \bar{b}] \subseteq I \to \mathbb{R}$ are functions such that

$$\bar{f}(x) = \beta^{-1}\left\{f\left(\alpha(x)\right)\right\},$$

then f is $*$-convex on $[a, b]$ if and only if $\bar{f}$ is classically convex on $[\bar{a}, \bar{b}]$.

Theorem 2.5. [14] *Let $f : [a, b] \subseteq I_{\alpha} \to \mathbb{R}$ be a $*$-convex function on $[a, b]$. Then the following hold true:*

(a) *If $\alpha(x) = x$ and $\beta(x) = e^{x}$, then f is log-convex or AG-convex on $[a, b]$.*

(b) *If* $\alpha(x) = e^x$ *and* $\beta(x) = x$*, then* f *is GA-convex on* $[a, b]$*.*

(c) *If* $\alpha(x) = \beta(x) = e^x$*, then* f *is GG-convex on* $[a, b]$*.*

(d) *If* $\alpha(x) = x$ *and* $\beta(x) = \dfrac{1}{x}$ *and* $f([a, b]) \subseteq \mathbb{R}_+$*, then* f *is AH-concave on* $[a, b]$*.*

(e) *If* $\alpha(x) = \dfrac{1}{x}$ *and* $\beta(x) = x$*, then* f *is HA-convex or harmonically convex on* $[a, b]$*.*

(f) *If* $\alpha(x) = \beta(x) = \dfrac{1}{x}$ *and* $f([a, b]) \subseteq \mathbb{R}_+$*, then* f *is HH-concave on* $[a, b]$*.*

Definition 2.8. Let I_α be an interval in $\mathbb{R}_\alpha$. Then, $f : I_\alpha \to \mathbb{R}_\beta$ is said to be $*$-p class on $\left[\dot{a}, \dot{b}\right]$ if

$$f\left(\dot{t}\dot{\times}\dot{a}\dot{+}\left(\dot{1}\dot{-}\dot{t}\right)\dot{\times}\dot{b}\right)\ddot{\leq} f\left(\dot{a}\right)\ddot{+}f\left(\dot{b}\right) \tag{2.20}$$

holds, where $t \in [0, 1]$ and $a, b \in \mathbb{R}$ with $a < b$. If the above inequality is reversed, then f is said to be $*$-p class concave. (2.20) can be stated in classical arithmetic as

$$\beta^{-1}\left\{f\left(\alpha\left(ta + (1 - t)\, b\right)\right)\right\} \leq \beta^{-1}\left\{f\left(\alpha(a)\right)\right\} + \beta^{-1}\left\{f\left(\alpha(b)\right)\right\}. \tag{2.21}$$

From the definition, one can easily see that if $f : [a, b] \subseteq I_\alpha \to \mathbb{R}$ and $\bar{f} : [\bar{a}, \bar{b}] \subseteq I \to \mathbb{R}$ are functions such that

$$\bar{f}(x) = \beta^{-1}\left\{f\left(\alpha(x)\right)\right\},$$

then f is $*$-p class on $[a, b]$ if and only if $\bar{f}$ is classically p class on $[\bar{a}, \bar{b}]$.

3. MAIN RESULTS

In this section, we establish two Hermite–Hadamard type inequalities for $*$-differentiable functions, one for $*$-convex and one for $*$-P functions. Since $*$-convexity is a general notion of convexity that unifies many classical convexity concepts through suitable choices of the generators α and β, the resulting inequalities simultaneously contain, as special cases, corresponding results for several classes of functions such as log-convex, GA-, GG-, AH-, HA-, and HH-convex functions.

The following lemma provides the fundamental identity upon which our main theorems are based.

Lemma 3.1. *Let* $f : I \subseteq \mathbb{R}_\alpha \to \mathbb{R}_\beta$ *be a* $*$*-differentiable mapping on* I° *and* $\dot{a}, \dot{b} \in I^\circ$ *with* $a \leq b$*. Then*

$$\begin{aligned}
&\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)\ddot{\times}f\left(\dot{b}\right)\ddot{+}\left(\ddot{x}\ddot{-}\ddot{a}\right)\ddot{\times}f\left(\dot{a}\right)}{\left(\ddot{b}\ddot{-}\ddot{a}\right)} : \ddot{-}\frac{\ddot{1}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)} : \ddot{\times} * \int_{\dot{a}}^{\dot{b}} f\left(x\right) d_\alpha x \\
= \; & \frac{\left(\ddot{x}\ddot{-}\ddot{a}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)} : \ddot{\times} * \int_{\dot{0}}^{\dot{1}} \left(\ddot{\bar{t}}\ddot{-}\ddot{1}\right)\ddot{\times}f^*\left(t\dot{\times}\dot{x} + \left(\dot{1}\dot{-}t\right)\dot{\times}\dot{a}\right) d_\alpha t \\
+ \; & \frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)} : \ddot{\times} * \int_{\dot{0}}^{\dot{1}} \left(\ddot{1}\ddot{-}\ddot{\bar{t}}\right)\ddot{\times}f^*\left(t\dot{\times}\dot{x} + \left(\dot{1}\dot{-}t\right)\dot{\times}\dot{b}\right) d_\alpha t.
\end{aligned}$$

Proof. We begin by computing the sum $I_1 \ddot{+} I_2$, where

$$
\begin{aligned}
& * \int_{\dot{0}}^{\dot{1}} \left(\ddot{t} \ddot{-} \ddot{1}\right) \ddot{\times} f^* \left(t \dot{\times} \dot{x} + \left(\dot{1} \dot{-} t\right) \dot{\times} \dot{a}\right) d_\alpha t \\
& \ddot{+} * \int_{\dot{0}}^{\dot{1}} \left(\ddot{1} \ddot{-} \ddot{t}\right) \ddot{\times} f^* \left(t \dot{\times} \dot{x} + \left(\dot{1} \dot{-} t\right) \dot{\times} \dot{b}\right) d_\alpha t \\
= \quad & I_1 \ddot{+} I_2.
\end{aligned}
$$

Applying $*$-composition (2.9), we have

$$
f^* \left(t \dot{\times} \dot{x} + \left(\dot{1} \dot{-} t\right) \dot{\times} \dot{a}\right) = f^* \tilde{\circ} \beta \left(\bar{t} x + \left(1 - \bar{t}\right) a\right), \tag{3.1}
$$

and by the $*$-chain rule (2.14), we obtain

$$
f^* \left\langle \beta \left(\bar{t} x + \left(1 - \bar{t}\right) a\right) \right\rangle = \frac{\left[f \tilde{\circ} \beta \left(\bar{t} x + \left(1 - \bar{t}\right) a\right)\right]^*}{\left(\beta \left(\bar{t} x + \left(1 - \bar{t}\right) a\right)\right)^*} = \frac{\left[f \tilde{\circ} \beta \left(\bar{t} x + \left(1 - \bar{t}\right) a\right)\right]^*}{\beta \left\{x - a\right\}} : \tag{3.2}
$$

Hence I_1 takes the form

$$
I_1 = \beta \left\{\frac{1}{x - a}\right\} \ddot{\times} * \int_{\dot{0}}^{\dot{1}} \left(\ddot{t} \ddot{-} \ddot{1}\right) \ddot{\times} \left[f \tilde{\circ} \beta \left(\bar{t} x + \left(1 - \bar{t}\right) a\right)\right]^* d_\alpha t.
$$

An application of $*$-integration by parts (Theorem 2.4) gives

$$
I_1 = \beta \left\{\frac{1}{x - a}\right\} \ddot{\times} \left(f\left(\dot{a}\right) \ddot{-} \int_{\dot{0}}^{\dot{1}} f \tilde{\circ} \beta \left(\bar{t} x + \left(1 - \bar{t}\right) a\right) d_\alpha t \right).
$$

Using the $*$-substitution rule (Theorem 2.3), we have

$$
I_1 = \beta \left\{\frac{1}{x - a}\right\} \ddot{\times} f\left(\dot{a}\right) \ddot{-} \beta^{\ddot{2}} \left\{\frac{1}{x - a}\right\} \ddot{\times} \int_{\dot{a}}^{\dot{x}} f\left(u\right) d_\alpha u.
$$

Proceeding analogously for I_2, we obtain

$$
I_2 = \beta \left\{\frac{1}{b - x}\right\} \ddot{\times} f\left(\dot{b}\right) \ddot{-} \beta^{\ddot{2}} \left\{\frac{1}{b - x}\right\} \ddot{\times} \int_{\dot{x}}^{\dot{b}} f\left(u\right) d_\alpha u.
$$

Multiplying I_1 and I_2 by their corresponding coefficients and adding, we get the desired result. □

Theorem 3.1. *Let $f: I \subset \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be a ∗−differentiable mapping on I° and $\dot{a},\dot{b} \in I^\circ$ with $\dot{a}\dot{\leq}\dot{b}$. If $|f^*|_\beta$ is ∗−convex on $\left[\dot{a},\dot{b}\right]$, then the following inequality holds:*

$$\left|\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)\ddot{\times}f\left(\dot{b}\right)\ddot{+}\left(\ddot{x}\ddot{-}\ddot{a}\right)\ddot{\times}f\left(\dot{a}\right)}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{-}\frac{\ddot{1}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}*\int_{\dot{a}}^{\dot{b}}f\left(x\right)d_\alpha x\right|_\beta$$
$$\ddot{\leq}\frac{\left(\ddot{x}\ddot{-}\ddot{a}\right)^{\ddot{2}}}{\ddot{6}\ddot{\times}\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}\left(\left|f^*\left(\dot{x}\right)\right|_\beta\ddot{+}\ddot{2}\ddot{\times}\left|f^*\left(\dot{a}\right)\right|_\beta\right)$$
$$\ddot{+}\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\ddot{6}\ddot{\times}\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}\left(\left|f^*\left(\dot{x}\right)\right|_\beta\ddot{+}\ddot{2}\ddot{\times}\left|f^*\left(\dot{b}\right)\right|_\beta\right).$$

where $\dot{x} \in [\dot{a},\dot{b}]$.

Moreover, if the function $\bar{f}(x) = \beta^{-1}\{f(\alpha(x))\}$ exists and is integrable on $[a,b]$ with ∗-convexity of $|f^*|_\beta$ on $[\dot{a},\dot{b}]$, then the following inequality holds:

$$\left|\frac{(b-x)\bar{f}(b)+(x-a)\bar{f}(a)}{b-a}-\frac{1}{b-a}\int_a^b\bar{f}(x)dx\right| \tag{3.3}$$
$$\leq\frac{(x-a)^2}{6(b-a)}\left[\left|\bar{f}'(x)\right|+2\left|\bar{f}'(a)\right|\right]+\frac{(b-x)^2}{6(b-a)}\left[\left|\bar{f}'(x)\right|+2\left|\bar{f}'(b)\right|\right]$$

where $x \in [a,b]$.

Proof. Applying Lemma 3.1 and taking the β-modulus, we have

$$I=\left|\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)\ddot{\times}f\left(\dot{b}\right)\ddot{+}\left(\ddot{x}\ddot{-}\ddot{a}\right)\ddot{\times}f\left(\dot{a}\right)}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{-}\frac{\ddot{1}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}*\int_{\dot{a}}^{\dot{b}}f\left(x\right)d_\alpha x\right|_\beta$$
$$\ddot{\leq}\left|\frac{\left(\ddot{x}\ddot{-}\ddot{a}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}*\int_{\dot{0}}^{\dot{1}}\left(\ddot{\ddot{t}}\ddot{-}\ddot{1}\right)\ddot{\times}f^*\left(t\dot{\times}\dot{x}+\left(\dot{1}\dot{-}t\right)\dot{\times}\dot{a}\right)d_\alpha t\right|_\beta$$
$$\ddot{+}\left|\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}*\int_{\dot{0}}^{\dot{1}}\left(\ddot{1}\ddot{-}\ddot{\ddot{t}}\right)\ddot{\times}f^*\left(t\dot{\times}\dot{x}+\left(\dot{1}\dot{-}t\right)\dot{\times}\dot{b}\right)d_\alpha t\right|_\beta$$
$$\ddot{\leq}\frac{\left(\ddot{x}\ddot{-}\ddot{a}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}*\int_{\dot{0}}^{\dot{1}}\left(\ddot{1}\ddot{-}\ddot{\ddot{t}}\right)\ddot{\times}\left|f^*\left(t\dot{\times}\dot{x}+\left(\dot{1}\dot{-}t\right)\dot{\times}\dot{a}\right)\right|_\beta d_\alpha t$$
$$\ddot{+}\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}*\int_{\dot{0}}^{\dot{1}}\left(\ddot{1}\ddot{-}\ddot{\ddot{t}}\right)\ddot{\times}\left|f^*\left(t\dot{\times}\dot{x}+\left(\dot{1}\dot{-}t\right)\dot{\times}\dot{b}\right)\right|_\beta d_\alpha t.$$

Since $|f^*|_\beta$ is $*$-convex on $[\dot{a},\dot{b}]$, we have

$$
\begin{aligned}
I \ddot{\leq}\, & \frac{(\ddot{x}\ddot{-}\ddot{a})^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)} \ddot{:}\, \ddot{\times} \left(*\int_{\dot{0}}^{\dot{1}} \left(\ddot{1}\ddot{-}\ddot{t}\right) \ddot{\times} \ddot{t}\, d_\alpha t \right) \ddot{\times} \left|f^*(\dot{x})\right|_\beta \\
\ddot{+}\, & \frac{(\ddot{x}\ddot{-}\ddot{a})^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)} \ddot{:}\, \ddot{\times} \left(*\int_{\dot{0}}^{\dot{1}} \left(\ddot{1}\ddot{-}\ddot{t}\right)^{\ddot{2}} d_\alpha t \right) \ddot{\times} \left|f^*(\dot{a})\right|_\beta \\
\ddot{+}\, & \frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)} \ddot{:}\, \ddot{\times} \left(*\int_{\dot{0}}^{\dot{1}} \left(\ddot{1}\ddot{-}\ddot{t}\right) \ddot{\times} \ddot{t}\, d_\alpha t \right) \ddot{\times} \left|f^*(\dot{x})\right|_\beta \\
\ddot{+}\, & \frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)} \ddot{:}\, \ddot{\times} \left(*\int_{\dot{0}}^{\dot{1}} \left(\ddot{1}\ddot{-}\ddot{t}\right)^{\ddot{2}} d_\alpha t \right) \ddot{\times} \left|f^*\left(\dot{b}\right)\right|_\beta .
\end{aligned}
$$

Evaluating the $*$-integrals and combining like terms, we obtain

$$
\begin{aligned}
I \ddot{\leq}\, & \frac{(\ddot{x}\ddot{-}\ddot{a})^{\ddot{2}}}{\ddot{6}\ddot{\times}\left(\ddot{b}\ddot{-}\ddot{a}\right)} \ddot{:}\, \ddot{\times} \left(\left|f^*(\dot{x})\right|_\beta \ddot{+} \ddot{2} \ddot{\times} \left|f^*(\dot{a})\right|_\beta \right) \\
\ddot{+}\, & \frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\ddot{6}\ddot{\times}\left(\ddot{b}\ddot{-}\ddot{a}\right)} \ddot{:}\, \ddot{\times} \left(\left|f^*(\dot{x})\right|_\beta \ddot{+} \ddot{2} \ddot{\times} \left|f^*\left(\dot{b}\right)\right|_\beta \right),
\end{aligned}
$$

which completes the proof. □

Remark 3.1. If we set $\alpha = \beta = I$ in (3.3)then $|f|$ becomes convex on $[a,b]$ and we obtain the inequality in Theorem 1.2.

Corollary 3.1. *If we set $\alpha = I$, $\beta = \exp$ in (3.3) then f becomes log-convex, equivalently AG-convex on $[a,b]$ and we have*

$$
\begin{aligned}
& \left| \frac{(b-x)\ln f(b) + (x-a)\ln f(a)}{b-a} - \frac{1}{b-a}\int_a^b \ln f(x)\, dx \right| \\
\leq\ & \frac{(x-a)^2}{6(b-a)} \left[\left|\frac{f'(x)}{f(x)}\right| + 2\left|\frac{f'(a)}{f(a)}\right| \right] + \frac{(b-x)^2}{6(b-a)} \left[\left|\frac{f'(x)}{f(x)}\right| + 2\left|\frac{f'(b)}{f(b)}\right| \right]
\end{aligned}
$$

where $x \in [a,b]$.

Corollary 3.2. *If we take $\alpha = \exp$ and $\beta = I$ in (3.3) and apply the change of variables*

$$
x \mapsto \ln x, \qquad a \mapsto \ln a, \qquad b \mapsto \ln b,
$$

then $|f|$ becomes GA-convex on $[a,b]$, and we obtain the following inequality:

$$\begin{aligned}&\left|\frac{\ln(b/x)\,f(b)+\ln(x/a)\,f(a)}{\ln(b/a)}-\frac{1}{\ln(b/a)}\int_a^b\frac{f(t)}{t}\,dt\right|\\ \leq\;&\frac{\ln^2(x/a)}{6\ln(b/a)}\left[x|f'(x)|+2a|f'(a)|\right]\\ &+\frac{\ln^2(b/x)}{6\ln(b/a)}\left[x|f'(x)|+2b|f'(b)|\right]\end{aligned}$$

where $x\in[a,b]$.

Corollary 3.3. *If we take $\alpha=\beta=\exp$ in (3.3) and apply the change of variables*

$$x\mapsto\ln x,\qquad a\mapsto\ln a,\qquad b\mapsto\ln b,$$

then $|f|$ becomes GG-convex on $[a,b]$, and we obtain the following inequality:

$$\begin{aligned}&\left|\frac{\ln(b/x)\ln f(b)+\ln(x/a)\ln f(a)}{\ln(b/a)}-\frac{1}{\ln(b/a)}\int_a^b\frac{\ln f(u)}{u}\,du\right|\\ \leq\;&\frac{\ln^2(x/a)}{6\ln(b/a)}\left[\left|\frac{xf'(x)}{f(x)}\right|+2\left|\frac{af'(a)}{f(a)}\right|\right]\\ &+\frac{\ln^2(b/x)}{6\ln(b/a)}\left[\left|\frac{xf'(x)}{f(x)}\right|+2\left|\frac{bf'(b)}{f(b)}\right|\right]\end{aligned}$$

Corollary 3.4. *If we set $\alpha=I,\ \beta(x)=\frac{1}{x}$ in (3.3) then $|f|$ becomes AH-concave on $[a,b]$ and we have*

$$\begin{aligned}&\left|\frac{b-x}{(b-a)f(b)}+\frac{x-a}{(b-a)f(a)}-\frac{1}{b-a}\int_a^b\frac{dx}{f(x)}\right|\\ \leq\;&\frac{(x-a)^2}{6(b-a)}\left[\frac{|f'(x)|}{[f(x)]^2}+\frac{2|f'(a)|}{[f(a)]^2}\right]\\ &+\frac{(b-x)^2}{6(b-a)}\left[\frac{|f'(x)|}{[f(x)]^2}+\frac{2|f'(b)|}{[f(b)]^2}\right]\end{aligned}$$

where $x\in[a,b]$.

Corollary 3.5. *If we take $\alpha=\frac{1}{x}$ and $\beta=x$ in (3.3) and apply the change of variables*

$$x\mapsto\frac{1}{x},\qquad a\mapsto\frac{1}{a},\qquad b\mapsto\frac{1}{b},$$

then $|f|$ becomes HA-convex on $[a,b]$, and we obtain the following inequality:

$$\left|\frac{\frac{(b-x)f(b)}{bx}+\frac{(x-a)f(a)}{ax}}{\frac{b-a}{ab}}-\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt\right|$$
$$\leq \frac{(x-a)^2b}{6(b-a)\,ax^2}\left[x^2|f'(x)|+2a^2|f'(a)|\right]$$
$$+\frac{(b-x)^2a}{6(b-a)\,bx^2}\left[x^2|f'(x)|+2b^2|f'(b)|\right]$$

where $x\in[a,b]$.

Corollary 3.6. *If we take $\alpha=\beta=\frac{1}{x}$ in (3.3) and apply the change of variables*

$$x\mapsto\frac{1}{x},\qquad a\mapsto\frac{1}{a},\qquad b\mapsto\frac{1}{b},$$

then $|f|$ becomes HH-concave on $[a,b]$, and we obtain the following inequality:

$$\left|\frac{a(b-x)}{x(b-a)f(b)}+\frac{b(x-a)}{x(b-a)f(a)}-\frac{ab}{b-a}\int_a^b\frac{dt}{t^2f(t)}\right|$$
$$\leq\frac{(x-a)^2b}{6(b-a)ax^2}\left[\frac{x^2|f'(x)|}{[f(x)]^2}+\frac{2a^2|f'(a)|}{[f(a)]^2}\right]$$
$$+\frac{(b-x)^2a}{6(b-a)bx^2}\left[\frac{x^2|f'(x)|}{[f(x)]^2}+\frac{2b^2|f'(b)|}{[f(b)]^2}\right]$$

where $x\in[a,b]$.

Theorem 3.2. *Let $f:I\subset\mathbb{R}_\alpha\to\mathbb{R}_\beta$ be a $*-$differentiable mapping on I° and $\dot{a},\dot{b}\in I^\circ$ with $\dot{a}\dot{\leq}\dot{b}$. If $|f^*|_\beta$ is $*$-p class on $\left[\dot{a},\dot{b}\right]$, then the following inequality holds:*

$$\left|\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)\ddot{\times}f\left(\dot{b}\right)\ddot{+}\left(\ddot{x}\ddot{-}\ddot{a}\right)\ddot{\times}f\left(\dot{a}\right)}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{-}\frac{\ddot{1}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}*\int_{\dot{a}}^{\dot{b}}f\left(x\right)d_\alpha x\right|_\beta$$
$$\ddot{\leq}\frac{\left(\ddot{x}\ddot{-}\ddot{a}\right)^{\ddot{2}}}{\ddot{2}\ddot{\times}\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}\left(\left|f^*\left(\dot{x}\right)\right|_\beta\ddot{+}\left|f^*\left(\dot{a}\right)\right|_\beta\right)$$
$$\ddot{+}\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\ddot{2}\ddot{\times}\left(\ddot{b}\ddot{-}\ddot{a}\right)}\ddot{:}\ddot{\times}\left(\left|f^*\left(\dot{x}\right)\right|_\beta\ddot{+}\left|f^*\left(\dot{b}\right)\right|_\beta\right).$$

where $\dot{x}\in[\dot{a},\dot{b}]$.

Moreover, if the function $\bar{f}(x) = \beta^{-1}\left\{f\left(\alpha(x)\right)\right\}$ exists and is integrable on $[a,b]$ with $*$-P class property of $|f^*|_\beta$ on $[\dot{a},\dot{b}]$, then the following inequality holds:

$$\left|\frac{(b-x)\,\bar{f}\,(b)+(x-a)\,\bar{f}\,(a)}{b-a}-\frac{1}{b-a}\int\limits_a^b \bar{f}(x)dx\right| \tag{3.4}$$
$$\leq \frac{(x-a)^2}{2\,(b-a)}\left[\left|\bar{f}'(x)\right|+\left|\bar{f}'(a)\right|\right]+\frac{(b-x)^2}{2\,(b-a)}\left[\left|\bar{f}'(x)\right|+\left|\bar{f}'(b)\right|\right]$$

where $x\in[a,b]$.

Proof. Applying Lemma 3.1 and taking the β-modulus, we have

$$I=\left|\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)\ddot{\times}f\left(\dot{b}\right)\dot{+}\left(\ddot{x}\ddot{-}\ddot{a}\right)\ddot{\times}f\left(\dot{a}\right)}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}:\ddot{-}\frac{\ddot{1}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}:\ddot{\times}*\int\limits_{\dot{a}}^{\dot{b}}f\left(x\right)d_\alpha x\right|_\beta$$
$$\ddot{\leq}\left|\frac{\left(\ddot{x}\ddot{-}\ddot{a}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}:\ddot{\times}*\int\limits_{\dot{0}}^{\dot{1}}\left(\ddot{t}\ddot{-}\ddot{1}\right)\ddot{\times}f^*\left(t\dot{\times}\dot{x}+\left(\dot{1}\dot{-}t\right)\dot{\times}\dot{a}\right)d_\alpha t\right|_\beta$$
$$\ddot{+}\left|\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}:\ddot{\times}*\int\limits_{\dot{0}}^{\dot{1}}\left(\ddot{1}\ddot{-}\ddot{t}\right)\ddot{\times}f^*\left(t\dot{\times}\dot{x}+\left(\dot{1}\dot{-}t\right)\dot{\times}\dot{b}\right)d_\alpha t\right|_\beta$$
$$\ddot{\leq}\frac{\left(\ddot{x}\ddot{-}\ddot{a}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}:\ddot{\times}*\int\limits_{\dot{0}}^{\dot{1}}\left(\ddot{1}\ddot{-}\ddot{t}\right)\ddot{\times}\left|f^*\left(t\dot{\times}\dot{x}+\left(\dot{1}\dot{-}t\right)\dot{\times}\dot{a}\right)\right|_\beta d_\alpha t$$
$$\ddot{+}\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}:\ddot{\times}*\int\limits_{\dot{0}}^{\dot{1}}\left(\ddot{1}\ddot{-}\ddot{t}\right)\ddot{\times}\left|f^*\left(t\dot{\times}\dot{x}+\left(\dot{1}\dot{-}t\right)\dot{\times}\dot{b}\right)\right|_\beta d_\alpha t.$$

Since $|f^*|_\beta$ is $*$-P class on $[\dot{a},\dot{b}]$, we have

$$I\ddot{\leq}\frac{\left(\ddot{x}\ddot{-}\ddot{a}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}:\ddot{\times}\left(*\int\limits_{\dot{0}}^{\dot{1}}\left(\ddot{1}\ddot{-}\ddot{t}\right)\,d_\alpha t\right)\ddot{\times}\left|f^*\left(\dot{x}\right)\right|_\beta$$
$$\ddot{+}\frac{\left(\ddot{x}\ddot{-}\ddot{a}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}:\ddot{\times}\left(*\int\limits_{\dot{0}}^{\dot{1}}\left(\ddot{1}\ddot{-}\ddot{t}\right)\,d_\alpha t\right)\ddot{\times}\left|f^*\left(\dot{a}\right)\right|_\beta$$
$$\ddot{+}\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}:\ddot{\times}\left(*\int\limits_{\dot{0}}^{\dot{1}}\left(\ddot{1}\ddot{-}\ddot{t}\right)\,d_\alpha t\right)\ddot{\times}\left|f^*\left(\dot{x}\right)\right|_\beta$$
$$\ddot{+}\frac{\left(\ddot{b}\ddot{-}\ddot{x}\right)^{\ddot{2}}}{\left(\ddot{b}\ddot{-}\ddot{a}\right)}:\ddot{\times}\left(*\int\limits_{\dot{0}}^{\dot{1}}\left(\ddot{1}\ddot{-}\ddot{t}\right)\,d_\alpha t\right)\ddot{\times}\left|f^*\left(\dot{b}\right)\right|_\beta.$$

Evaluating the $*$-integrals and combining like terms, we obtain

$$
\begin{aligned}
I \ddot{\leq} & \frac{\left(\ddot{x} \ddot{-} \ddot{a}\right)^{\ddot{2}}}{\ddot{2} \ddot{\times} \left(\ddot{b} \ddot{-} \ddot{a}\right)} : \ddot{\times} \left(\left|f^{*}\left(\dot{x}\right)\right|_{\beta} \ddot{+} \left|f^{*}\left(\dot{a}\right)\right|_{\beta} \right) \\
& \ddot{+} \frac{\left(\ddot{b} \ddot{-} \ddot{x}\right)^{\ddot{2}}}{\ddot{2} \ddot{\times} \left(\ddot{b} \ddot{-} \ddot{a}\right)} : \ddot{\times} \left(\left|f^{*}\left(\dot{x}\right)\right|_{\beta} \ddot{+} \left|f^{*}\left(\dot{b}\right)\right|_{\beta} \right),
\end{aligned}
$$

which completes the proof. $\square$

Corollary 3.7. *If we set $\alpha = \beta = I$ in (3.4), then $|f|$ becomes P class on $[a, b]$ and we have*

$$
\begin{aligned}
& \left| \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \\
\leq\ & \frac{(x-a)^2}{2(b-a)} \left[|f'(x)| + |f'(a)| \right] \\
& + \frac{(b-x)^2}{2(b-a)} \left[|f'(x)| + |f'(b)| \right]
\end{aligned}
$$

where $x \in [a, b]$.

Corollary 3.8. *If we set $\alpha = I$, $\beta = \exp$ in (3.4) and $|f|$ is $*$-P class on $[a, b]$, then we have*

$$
\begin{aligned}
& \left| \frac{(b-x) \ln f(b) + (x-a) \ln f(a)}{b-a} - \frac{1}{b-a} \int_a^b \ln f(x)\, dx \right| \\
\leq\ & \frac{(x-a)^2}{2(b-a)} \left[\left| \frac{f'(x)}{f(x)} \right| + \left| \frac{f'(a)}{f(a)} \right| \right] \\
& + \frac{(b-x)^2}{2(b-a)} \left[\left| \frac{f'(x)}{f(x)} \right| + \left| \frac{f'(b)}{f(b)} \right| \right]
\end{aligned}
$$

where $x \in [a, b]$.

Corollary 3.9. *If we take $\alpha = \exp$ and $\beta = I$ in (3.4) and apply the change of variables*

$$
x \mapsto \ln x, \qquad a \mapsto \ln a, \qquad b \mapsto \ln b,
$$

and $|f|$ is $*$-p class on $[a, b]$, then we have

$$
\begin{aligned}
& \left| \frac{\ln(b/x)\, f(b) + \ln(x/a)\, f(a)}{\ln(b/a)} - \frac{1}{\ln(b/a)} \int_a^b \frac{f(t)}{t}\, dt \right| \\
\leq\ & \frac{\ln^2(x/a)}{2 \ln(b/a)} \left[x|f'(x)| + a|f'(a)| \right] \\
& + \frac{\ln^2(b/x)}{2 \ln(b/a)} \left[x|f'(x)| + b|f'(b)| \right]
\end{aligned}
$$

where $x \in [a, b]$.

Corollary 3.10. *If we take* $\alpha = \beta = \exp$ *in* (3.4) *and apply the change of variables*

$$x \mapsto \ln x, \qquad a \mapsto \ln a, \qquad b \mapsto \ln b,$$

and $|f|$ *is* $*$-P *class on* $[a, b]$, *then we have*

$$\begin{aligned} &\left| \frac{\ln(b/x)\ln f(b) + \ln(x/a)\ln f(a)}{\ln(b/a)} - \frac{1}{\ln(b/a)} \int_a^b \frac{\ln f(u)}{u}\, du \right| \\ \leq\ & \frac{\ln^2(x/a)}{2\ln(b/a)} \left[\left| \frac{xf'(x)}{f(x)} \right| + \left| \frac{af'(a)}{f(a)} \right| \right] \\ &+ \frac{\ln^2(b/x)}{2\ln(b/a)} \left[\left| \frac{xf'(x)}{f(x)} \right| + \left| \frac{bf'(b)}{f(b)} \right| \right] \end{aligned}$$

where $x \in [a, b]$.

Corollary 3.11. *If we set* $\alpha = I,\ \beta(x) = \frac{1}{x}$ *in* (3.4) *and* $|f|$ *is* $*$-P *class on* $[a, b]$, *then we have*

$$\begin{aligned} &\left| \frac{\dfrac{b-x}{f(b)} + \dfrac{x-a}{f(a)}}{b-a} - \frac{1}{b-a} \int_a^b \frac{1}{f(x)}\, dx \right| \\ \leq\ & \frac{(x-a)^2}{2(b-a)} \left[\left| \frac{f'(x)}{[f(x)]^2} \right| + \left| \frac{f'(a)}{[f(a)]^2} \right| \right] \\ &+ \frac{(b-x)^2}{2(b-a)} \left[\left| \frac{f'(x)}{[f(x)]^2} \right| + \left| \frac{f'(b)}{[f(b)]^2} \right| \right] \end{aligned}$$

where $x \in [a, b]$.

Corollary 3.12. *If we take* $\alpha(x) = \frac{1}{x}$ *and* $\beta = I$ *in* (3.4) *and apply the change of variables*

$$x \mapsto \frac{1}{x}, \qquad a \mapsto \frac{1}{a}, \qquad b \mapsto \frac{1}{b},$$

and $|f|$ *is* $*$-P *class on* $[a, b]$, *then we have*

$$\begin{aligned} &\left| \frac{\dfrac{(b-x)f(b)}{bx} + \dfrac{(x-a)f(a)}{ax}}{\dfrac{b-a}{ab}} - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2}\, dt \right| \\ \leq\ & \frac{b(x-a)^2}{2ax^2(b-a)} \left[|f'(x)| + |f'(a)| \right] \\ &+ \frac{a(b-x)^2}{2bx^2(b-a)} \left[|f'(x)| + |f'(b)| \right] \end{aligned}$$

where $x \in [a, b]$.

Corollary 3.13. *If we take* $\alpha(x) = \frac{1}{x}$ *and* $\beta(x) = \frac{1}{x}$ *in* (3.4) *and apply the change of variables*

$$x \mapsto \frac{1}{x}, \qquad a \mapsto \frac{1}{a}, \qquad b \mapsto \frac{1}{b},$$

and $|f|$ *is* $*$*-P class on* $[a, b]$*, then we have*

$$\begin{aligned} &\left| \frac{a(b-x)}{x(b-a)f(b)} + \frac{b(x-a)}{x(b-a)f(a)} - \frac{ab}{b-a}\int_a^b \frac{1}{t^2 f(t)}\,dt \right| \\ \leq\ & \frac{b(x-a)^2}{2ax^2(b-a)}\left[\frac{|f'(x)|}{[f(x)]^2} + \frac{|f'(a)|}{[f(a)]^2}\right] \\ &+ \frac{a(b-x)^2}{2bx^2(b-a)}\left[\frac{|f'(x)|}{[f(x)]^2} + \frac{|f'(b)|}{[f(b)]^2}\right] \end{aligned}$$

where $x \in [a, b]$.

4. Conclusion

Within the framework of non-Newtonian calculus, we obtained Hermite–Hadamard type inequalities for $*$-convex and $*$-P functions by means of a new identity for $*$-differentiable mappings. A notable feature of the approach is that, through appropriate choices of the generators α and β, the established inequalities recover their counterparts for several well-known classes of functions, which reflects the unifying character of the $*$-calculus setting. The identity and techniques presented here can be developed further to obtain Ostrowski-, Simpson-, and Steffensen-type results, as well as their fractional and co-ordinated analogues, within the non-Newtonian framework.

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References

[1] A.O. Akdemir, M.E. Özdemir, F. Sevinç, *Some inequalities for GG-convex functions*, Turkish J. Inequal., **2**(2) (2018), 78–86.
[2] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, *Generalized convexity and inequalities,* J. Math. Anal. Appl., **335** (2007), 1294–1308.
[3] A.E. Bashirov, E. Kurpinar, A. Ozyapici, *Multiplicative calculus and its applications*, J. Math. Anal. Appl., **337**(1) (2008), 36–48.
[4] F. Başar, B. Hazarika, *Non-Newtonian Sequence Spaces with Applications*, 1st ed., Chapman and Hall/CRC, New York, 2025.
[5] K. Boruah, B. Hazarika, *G-calculus*, TWMS J. Appl. Eng. Math., **8** (2018), 94–105.
[6] S.I. Butt, H. Inam, M.A. Dokuyucu, *New fractal Simpson estimates for twice local differentiable generalized convex mappings,* Appl. Comput. Math., **23**(4) (2024), 474–503.
[7] B. Çelik, A.O. Akdemir, E. Set, S. Aslan, *Ostrowski-Mercer type integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators,* TWMS J. Pure Appl. Math., **15**(2) (2024), 269–285.

[8] S.S. Dragomir, C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000, 357 p.

[9] S.S. Dragomir, *Inequalities of Hermite-Hadamard type for HA-convex functions,* Moroccan J. Pure Appl. Anal., **3**(1) (2017), 83–101.

[10] S.S. Dragomir, *Inequalities of Hermite-Hadamard type for GA-convex functions*, RGMIA Res. Rep. Coll., **18** (2015), Preprint.

[11] S.S. Dragomir, *Inequalities of Hermite-Hadamard type for HH-convex functions*, Acta Comment. Univ. Tartu. Math., **22**(2) (2018), 179–190.

[12] S.S. Dragomir, *Inequalities of Hermite-Hadamard type for GG-convex functions*, An. Univ. Vest Timiş., Ser. Mat.-Inform., **57**(2) (2019), 34–52.

[13] A. Ekinci, S. Baş, T. Korpinar, Z. Korpinar, *New geometric magnetic energy according to geometric Frenet formulas*, Opt. Quantum Electron., **56**(1) (2024), Article 81.

[14] A. Ekinci, A.O. Akdemir, M. Gürbüz, E. Set, *Hermite-Hadamard type inequalities in the framework of non-Newtonian convex functions*, Appl. Comput. Math., **24**(4) (2025), 717–732.

[15] D. Filip, C. Piatecki, *An overview on the non-Newtonian calculus and its potential applications to economics*, HAL archives-ouvertes.fr, hal-00945788, 2014.

[16] M. Grossman, R. Katz, *Non-Newtonian Calculus*, Lee Press, 1972.

[17] U. Kadak, Y. Gürefe, *A generalization on weighted means and convex functions with respect to the non-Newtonian calculus*, Int. J. Anal., **2016** (2016), Article ID 4216873, 1–9.

[18] E. Karaduman, A. Bıdıl, *Integral inequalities involving k-Atangana-Baleanu fractional integral operators,* Turkish J. Sci., **10**(3) (2025), 143–158.

[19] H. Kavurmacı, M. Avcı, M.E. Özdemir, *New inequalities of Hermite-Hadamard type for convex functions with applications,* J. Inequal. Appl., **2011** (2011), Article 86.

[20] U.S. Kirmaci, M.K. Bakula, M.E. Özdemir, J. Pečarić, *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comput., **193**(1) (2007), 26–35.

[21] M.E. Özdemir, M. Avcı, E. Set, *On some inequalities of Hermite-Hadamard type via m-convexity*, Appl. Math. Lett., **23**(9) (2010), 1065–1070.

[22] M.E. Özdemir, A.O. Akdemir, A. Ekinci, *New Hadamard-type inequalities for functions whose derivatives are (α, m)-convex functions*, Tbilisi Math. J., **7**(2) (2014), 61–72.

[23] E. Set, A. Gözpınar, A. Ekinci, *Hermite-Hadamard type inequalities via conformable fractional integrals,* Acta Math. Univ. Comenianae, **86**(2) (2017), 309–320.

[24] S. Şahin, A. Ekinci, *Steffensen-type integral inequalities for functions with s-convex absolute n-th derivatives*, Turkish J. Sci., **10**(2) (2025), 59–72.

[25] M.Ç. Tatar, Ç. Yıldız, *Some new estimates for Hermite-Hadamard-Jensen-Mercer type inequalities*, Turkish J. Sci., **9**(3) (2024), 225–240.

[26] E. Ünlüyol, Y. Erdaş, S. Salaş, *Hermite-Hadamard-Fejér inequality and some new inequalities via ∗-calculus*, Sigma J. Eng. Nat. Sci., **10**(3) (2019), 287–299.

[27] E. Ünlüyol, S. Salaş, İ. Işcan, *A new view of some operators and their properties in terms of the non-Newtonian calculus*, Topol. Algebra Appl., **5**(1) (2017), 49–54.

[28] A. Yalçın, E. Gül, A.O. Akdemir, *Hermite-Hadamard type inequalities for co-ordinated convex functions with variable-order fractional integrals*, Appl. Comput. Math., **24**(2) (2025), 326–343.

[29] J. Zeynalov, Y. Erdaş, G. Zeynelzede, Z. Nağiyeva, A.O. Akdemir, *Some new estimations for different kinds of convex functions via Katugampola fractional operator*, Turkish J. Inequal., **9**(2) (2025), 79–89.

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