Turkish J. Ineq., 10 (1) (2026), Pages 142 –151 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

A NEW ORDER RELATION USING ℓ-DIFFERENCE FOR SET-VALUED OPTIMIZATION

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ABSTRACT. In this work, a new pre-order relation is defined using ℓ-difference. After establishing the basic properties of this order relation, several relations are obtained between the new order relation and well known order relations in the literature. In addition, it is demonstrated that this ordering relation is a generalization of the ordering relation used in vector spaces. Solutions of set-valued optimizations using the set approach are examined according to the newly defined ordering relation.

1. INTRODUCTION

The aim of optimizations is to find the best among the options. Such problems are common for researchers in many fields such as mathematics, statistics, engineering, economics, computer science and business. Optimization problems are divided into subclasses according to the structure of the problem. If we characterize our options one-to-one with real numbers, the encountered problem can be mathematically characterized with the help of a real-valued function. Such problems are called scalar or real-valued optimizations. When the objective function and constraints are convex, the optimization is called a convex optimization problem. If vector-valued functions are used to mathematically characterize the encountered problem, the optimizations used to find the best points of this function are called vector-valued optimizations. If each option contains multiple and independent data, set-valued optimization problems (shortly, SVOP) are used to characterize such problems. For example, set-valued optimization can be used to determine the best team among the teams in a league.

Set-valued optimizations have recently attracted much attention. One of the reasons for this is their wide application areas. Another reason is that set-valued transformations are an extension of vector and scalar-valued functions, the results obtained for set-valued

Key words and phrases. Optimization, Set-valued map, Set-valued optimization, Order relation
2010 *Mathematics Subject Classification.* Primary: 49K05. Secondary: 49K35.
Received: 27/04/2026 *Accepted*: 01/06/2026.
Cite this article as: H. Büyükçavuşoğlu Erçolak, E. Karaman, M.K. Yıldız, T. Yalçın Uzun, A new order relation using ℓ-difference for set-valued optimization, Turkish Journal of Inequalities, 10(1) (2026), 142-151.

optimizations can be applied to vector and scalar optimizations. Two types of approaches are commonly used to obtain the solution of set-valued optimizations. These are the set approach and the vector approach. When application areas are considered, the set approach yields more meaningful results than the vector approach. Therefore, finding the solutions of set-valued optimizations according to the set approach has attracted much attention.

When a set-valued optimization is considered to the set approach, an order relation is required over sets. This method relies on comparing all elements in the constraint set of the set-valued transformation according to this ordering relation. First, Kuroiwa discovered that the set approach depends on a set relation based on the ordering cone [17,18]. Kuroiwa gave the solutions of the SVOP with respect to the set approach. Subsequently, some researchers obtained new order relations for set-valued optimization [7, 8, 13]. Set-valued optimizations will be considered with respect to the set approach in the remainder of this study. It is very difficult to manually find the solution set of set-valued optimizations. In order to solve or obtain the optimality conditions for SVOP, some methods are used such as vectorization [6, 11], scalarization [1, 3, 4, 13], directional derivative [5, 12, 20], subdifferential [10, 22], embedding space [9, 16], and so on. The most known scalarization functions are Gerstewitz and the oriented function of Hiriart-Urruty and their generalizations [1,3,4,11,15]. These methods and more can be found in the references in [2, 14].

In this study, we research and find a new order relation for set-valued optimizations depend on the set approach. This will contribute to researchers working on set-valued optimizations and in areas, that need solutions to this problem. To achieve this, the less well-known ℓ-difference on sets was used.

The next of this work is organized as follows: Some basic notations and definitions are recalled in the second part. In the last part of work, a new order relation namely, p relation is defined, and its properties are discovered.

2. Preliminaries

Some notations and definitions used in the work are presented in this section.

Let $P, R \subseteq \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ be given. $\lambda P := \{\lambda a : a \in P\}$. It is denoted that the algebraic sum of P and R by

$$P + R := \{\bigcup_{b \in R} (P + b)\} = \{a + b : a \in P \text{ and } b \in R\}.$$

Definition 2.1. Let $P, R \subseteq \mathbb{R}^n$. The set defined as

$$P -_m R := \{x \in \mathbb{R}^n : x + R \subseteq P\}$$

is called the Minkowski (or Pontryagin) difference of sets P and R, that is

$$P -_m R := \bigcap_{b \in R} (P - b).$$

If $\alpha x \in K$ for all $x \in K$, $\alpha \geq 0$, then K is called a cone. The cone K is called a convex pointed cone if $K + K = K$ (convexity) and $K \cap (-K) = 0_{\mathbb{R}^n}$ (pointness). The cone K with nonempty interior will be assume a closed, convex pointed cone in the rest of this work. $\mathcal{P}(\mathbb{R}^n)$ denotes the family of nonempty subsets of $\mathbb{R}^n$. Similarly, $\mathcal{C}(\mathbb{R}^n)$ will be the family of nonempty compact and convex subsets of $\mathbb{R}^n$ in the next. The topological interior and convex hull of any set $P \in \mathcal{P}(\mathbb{R}^n)$ are indicated by $int(P)$ and $convP$, respectively. Let $P \in \mathcal{P}(\mathbb{R}^n)$ be given, if there exists positive t such that $P \subseteq \mathbf{B}(0_{\mathbb{R}^n}, t) + K$, then P is called K-bounded where $\mathbf{B}(x,t)$ is open neighborhood at x with radius t, and $\bar{\mathbf{B}}(x,t)$ is closed neighborhood at x with radius t. It is obvious that if a set is bounded then it is also K-bounded and $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) : x_i \geq 0, i = 1, 2, \ldots, n\}$.

Let $P, R, S \in \mathcal{P}(\mathbb{R}^n)$ such that R be bounded and S be closed and convex sets. It is defined as cancellation law on sets [19, 21]

$$P + R \subseteq S + R \text{ implies } P \subseteq S.$$

In the same way

$$P + R = S + R \text{ implies } P = S.$$

Definition 2.2. [21] Let $P, R \in \mathcal{C}(\mathbb{R}^n)$ be given. Then, P is called a summand of R if there exists a $S \in \mathcal{C}(\mathbb{R}^n)$ such that $P + S = R$.

We have the following properties for $P, R, S \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda, \gamma \geq 0$

(i) $P \subseteq R \Rightarrow P -_m S \subseteq R -_m S$
(ii) $P \subseteq R \Rightarrow S -_m R \subseteq S -_m P$
(iii) $(P \cup R) + S = (P + S) \cup (R + S)$
(iv) $(P \cap R) + S \subseteq (P + S) \cap (R + S)$
(v) $\lambda P + \lambda R = \lambda(P + R)$
(vi) $(\lambda + \gamma)P \subseteq \lambda P + \gamma P$
(vii) $(\lambda + \gamma)P = \lambda P + \gamma P$ (P is convex set)
(viii) $P \subseteq (P + R) -_m R$
(ix) $(P -_m R) + R \subseteq P$
(x) $(P -_m R) + S \subseteq (P + S) -_m R$
(xi) $(P -_m R) -_m S = P -_m (R + S)$
(xii) $P -_m R \subseteq S \Leftrightarrow P \subseteq S -_m R$
(xiii) $P \subseteq (P + R) -_m R$
(xiv) $P -_m (P -_m R) = R$ if and only if R is an intersection of translates of P
(xv) $P + R \subseteq S \Leftrightarrow P \subseteq S -_m R$
(xvi) $(P + K) -_m R = P -_m R$.

If $P, R \in \mathcal{C}(\mathbb{R}^n)$, then

(i) $(P + R) -_m R = P$
(ii) $cl(P + R) \subseteq cl(S + R) \Rightarrow P \subseteq S$
(iii) $(P \cap R) + S = (P + S) \cap (R + S)$ (if $P \cup R$ is convex)
(iv) $(P \cup R) + (P \cap R) = P + R$ (if $P \cup R$ is convex)
(v) $(P -_m R) + R = P$ (if R is summand of P)

Definition 2.3. [8, 20] Let $P, R \in \mathcal{P}(\mathbb{R}^n)$. The set defined as $P -_\ell R := \{x \in \mathbb{R}^n : x + R \subseteq P + K\}$ or

$$P -_\ell R := \{x \in \mathbb{R}^n : x + R \subseteq P + K\} = (P + K) -_m R$$

is called the ℓ-difference of sets P and R.

Order relation which is commonly used in set-valued optimizations is lower set order defined as:

$$P \preceq^{l} R :\Leftrightarrow R \subseteq P + K.$$

Definition 2.4. [8] Let $P, R \in \mathcal{P}(\mathbb{R}^n)$ be given.

(i) ℓ_1 relation is defined as:

$$P \preceq^{\ell_1} R :\Leftrightarrow (R -_l P) \cap K \neq \emptyset.$$

(ii) The strictly ℓ_1 order relation is defined as:

$$P \prec^{\ell_1} R :\Leftrightarrow (R -_l P) \cap intK \neq \emptyset.$$

Definition 2.5. [13] Let $P, R \in \mathcal{P}(\mathbb{R}^n)$. m_1 relation is defined as:

$$P \preceq^{m_1} R :\Leftrightarrow (R -_m P) \cap K \neq \emptyset.$$

Proposition 2.1. [8] *The order relation $\preceq^{\ell_1}$ has the next properties;*

(i) $\preceq^{\ell_1}$ *is compatible with the addition,*
(ii) $\preceq^{\ell_1}$ *is compatible with the positive scalar multiplication,*
(iii) $\preceq^{\ell_1}$ *is reflexive,*
(iv) $\preceq^{\ell_1}$ *is transitive,*
(v) $\preceq^{\ell_1}$ *isn't antisymmetric.*

The order relation $\preceq^{\ell_1}$, $\preceq^{m_1}$ and $\preceq^{l}$ are pre-order relations on $\mathcal{P}(\mathbb{R}^n)$[4, 8, 13].

3. A new Order Relation for Sets Compare

In this section, a pre-order relation is defined using ℓ-difference and some of its properties are examined.

Proposition 3.1. *Let $P, R \in \mathcal{P}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$ be given. Then, $-_l$ have the following algebraic properties:*

(i) $x + (P -_l R) = (x + P) -_l R$,
(ii) $(P + K) -_l R = P -_l R$,
(iii) $(P -_l R) + K = P -_l R$,
(iv) $P -_l 0_{\mathbb{R}^n} = P + K$,
(v) $R -_l P = R -_l (P + K)$.

Proof. (i) Let $y \in x + (P -_l R)$. Then, $y - x \in P -_l R$. So, $y - x + R \subseteq P + K$. Then, $y + R \subseteq x + P + K$ implies that $y \in (x + P) -_l R$. For the other direction, let $y \in (x + P) -_l R$. That means $y + R \subseteq x + P + K$. From here, $y - x + R \subseteq P + K$. Then, $y - x \in P -_l R$ implies that $y \in x + P -_l R$. Therefore, $x + P -_l R = (x + P) -_l R$.

(ii) Let $y \in (P + K) -_l R$. That means $y + R \subseteq P + K + K = P + K$. From here $y + R \subseteq P + K$. That is $y \in P -_l R$. For the other direction, let $x \in P -_l R$. Then, $x + R \subseteq P + K$ and, $x + R \subseteq P + K + K$. That is $x \in (P + K) -_l R$.

(iii)

$$\begin{aligned}(P -_l R) + K &= \{x + k : x + R \subseteq P + K, k \in K\}\\ &= \{u : u - k + R \subseteq P + K, k \in K\}\\ &= \{u : u + R \subseteq P + K + k = P + K\}\\ &= P -_l R.\end{aligned}$$

(iv)

$$\begin{aligned}P -_l 0_{\mathbb{R}^n} &= \{x : x + 0_{\mathbb{R}^n} \subseteq P + K\}\\ &= \{x : x \in P + K\}\\ &= P + K.\end{aligned}$$

(v)

$$\begin{aligned}R -_l P &= \{x : x + P \subseteq R + K\}\\ &= \{x : x + P + K \subseteq R + K + K = R + K\}\\ &= \{x : x + P + K \subseteq R + K\}\\ &= R -_l (P + K).\end{aligned}$$

□

Definition 3.1. Let $P, R \in \mathcal{P}(\mathbb{R}^n)$ be given. Then,

(i) p order relation is defined as:

$$P \preceq^p R :\Leftrightarrow R -_l P \subseteq K,$$

(ii) The strictly p order relation is defined as:

$$P \prec^p R :\Leftrightarrow R -_l P \subseteq int(K).$$

These relations are different from ℓ_1 order relations. One advantage of the p-relation is that we can order elements that cannot be ordered with the ℓ_1 relation. The p -relation is stronger than ℓ_1 order relation because ℓ_1 relation can contain any element outside the cone. Although the p-order relation is more restrictive and mainly suitable for compact convex sets, it provides a robust and global dominance structure, which is desirable in set optimization.

Equaivalence relation for the p-order relation can be defined as:

$$P \sim^p R \Leftrightarrow P \preceq^p R \text{ and } R \preceq^p P.$$

We can derive the equivalence classes of P with respect to the p-order relation denoted by $[P]^p$.

Proposition 3.2. *Let $P, R, S \in \mathcal{C}(\mathbb{R}^n)$ be any sets. The relation $\preceq^p$ on $\mathcal{C}(\mathbb{R}^n)$ is*

(i) *compatible with the addition, that is $P \preceq^p R$ implies $P + S \preceq^p R + S$, also the other direction is satisfied, that it $P + S \preceq^p R + S$ implies $P \preceq^p R$,*

(ii) *compatible with positive scalar multiplication, that is $P \preceq^p R$ implies $\lambda P \preceq^p \lambda R$ for all scalars $\lambda \geq 0$.*

Proof. (i) Let $P, R, S \in \mathcal{C}(\mathbb{R}^n)$ and $P \preceq^p R$ be given. Since $P \preceq^p R$, we have $R -_l P \subseteq K$. That is, for each $x \in R -_l P$ such that $x \in K$. For all $y \in (R + S) -_l (P + S)$, $y + P + S \subseteq R + S + K$. Then $y + P \subseteq R + K$ is obtained from cancellation law on sets. That is $y \in R -_l P$. Since $R -_l P \subseteq K$, $y \in K$. Therefore, $(R + S) -_l (P + S) \subseteq K$.

Thus, we obtain $P + S \preceq^p R + S$. This indicates that $\preceq^p$ is compatible with the addition. The other direction can be obtained using cancellation law.

(ii) Let $P \preceq^p R$. Since $P \preceq^p R$, we have $R -_l P \subseteq K$. This implies that, for all $x \in R -_l P$ such that $x \in K$. For all $\lambda \geq 0$ and $x \in \lambda R -_l \lambda P$, then $x + \lambda P \subseteq \lambda R + K = \lambda R + \lambda K$. Hence, for all $\lambda \geq 0$ we can find $\bar{x}\mathbb{R}^n$ such that $x = \lambda\bar{x}$ then $\lambda\bar{x}\lambda P \subseteq \lambda R + \lambda K$. Since $\lambda \geq 0$, then $\bar{x} + P \subseteq R + K$. Then, we obtain $\bar{x} \in R -_l P$ such that $\bar{x} \in K$. Since $x = \lambda\bar{x}$, then $x \in K$.Therefore, we obtain $\lambda R -_l \lambda P \subseteq K$ and $\lambda P \preceq^p \lambda R$. Hence, $\preceq^p$ is compatible with the positive scalar multiplication.

□

Proposition 3.3. *The relation $\preceq^p$ on $\mathcal{C}(\mathbb{R}^n)$ has the next properties;*

(i) *$\preceq^p$ is reflexive,*

(ii) *$\preceq^p$ is transitive,*

(iii) *$\prec^p$ is transitive.*

Proof. (i) Let $P \in \mathcal{C}(\mathbb{R}^n)$. We show that $P \preceq^p P$. Let's choose an $x \in \mathbb{R}^n$ such that $x + P \subseteq P + K$, then $x \in P -_l P$. Because of cancelation law, $x + P \subseteq P + K$ implies that $x \in K$. Then, $x \in K$ for all $x \in P -_l P$. Therefore, $P \preceq^p P$.

(ii) Let $P, R, S \in \mathcal{C}(\mathbb{R}^n)$ and $P \preceq^p R$ and $R \preceq^p S$. We have $R -_l P \subseteq K$ and $S -_l R \subseteq K$. $x_1 \in K$ for all $x_1 \in R -_l P$, and $x_2 \in K$ for all $x_2 \in S -_l R$. Then,

$$\{x_1 : x_1 + P \subseteq R + K\} \subseteq K \tag{3.1}$$

and

$$\{x_2 : x_2 + R \subseteq S + K\} \subseteq K. \tag{3.2}$$

Assume that $P \preceq^p S$ is not satisfied. That is, $\{x : x + P \subseteq S + K\} \nsubseteq K$. Then, there exist an x such that $x + P \subseteq S + K$ and $x \notin K$. We have

$$x + P \subseteq x - x_1 + R + K \subseteq x - x_1 - x_2 + S + K \subseteq x + S + K \tag{3.3}$$

where $x_1, x_2 \in K$ from (3.1) and (3.2). Because $x + P \subseteq x + S + K$ from (3.3) and $x + P \subseteq S + K$, x must be belong to cone K. That is, $x \in K$. This contradicts the assumption. Therefore, $P \preceq^p S$.

(iii) The proof can be obtained similar to (ii).

□

The order relation $\preceq^p$ isn't antisymmetric and partial order relation on $\mathcal{C}(\mathbb{R}^n)$. For example, let $K = R^2_+$, $P = conv\{(1,1),(1,2),(2,2),(2,1)\}$ and $R = conv\{(1,1),(1.5,1.5),(2,1)\}$ be given. Then, we get $P \preceq^p R \Leftrightarrow R -_l P \subseteq K$ and $\{x : x + P \subseteq R + K\} = K \subseteq K$. Additionally $R \preceq^p P \Leftrightarrow P -_l R \subseteq K$ and $\{x : x + R \subseteq P + K\} = K \subseteq K$. $P \preceq^p R$ and $R \preceq^p P$. But $P \neq R$. Hence, $\preceq^p$ isn't antisymmetric.

The order relation $\preceq^p$ isn't symmetric on $\mathcal{C}(\mathbb{R}^n)$. For example, let $K = R^2_+$, $P = \{(1,1)\}$ and $R = \{(2,2)\}$. When $P \preceq^p R$, $R \npreceq^p P$. Then $\preceq^p$ isn't symmetric.

Corollary 3.1. *The order relation $\preceq^p$ is a pre-order relation.*

Lemma 3.1. *Assume that $P, R, S \in \mathcal{C}(\mathbb{R}^n)$. Then the following implications hold:*

(i) $P \preceq^p R \Leftrightarrow \{0_{\mathbb{R}^n}\} \preceq^p R -_l P$,
(ii) $P \preceq^p R \Leftrightarrow P + K \preceq^p R + K$.

Proof. (i)

$$\begin{aligned} P \preceq^p R \Leftrightarrow R -_l P \subseteq K &\Rightarrow (R -_l P) + K \subseteq K + K = K \\ &\Rightarrow (R -_l P) + K \subseteq K \\ &\Rightarrow (R -_l P) -_l 0_{\mathbb{R}^n} \subseteq K \\ &\Rightarrow \{0_{\mathbb{R}^n}\} \preceq^p B -_l P \end{aligned}$$

$$\begin{aligned} \{0_{\mathbb{R}^n}\} \preceq^p R -_l P &\Rightarrow (R -_l P) -_l \{0_{\mathbb{R}^n}\} \subseteq K \\ &\Rightarrow (R -_l P) + K \subseteq K \\ &\Rightarrow (R -_l P) \subseteq K \\ &\Rightarrow P \preceq^p R \end{aligned}$$

(ii) It can be obtained easily.

□

Proposition 3.4. *Let a and b two vectors, then $a \preceq b \Leftrightarrow \{a\} \preceq^p \{b\}$.*

Proof. If $a \preceq b$, then $b - a \in K$.

$$\begin{aligned} \{b\} -_l \{a\} &= \{x : x + a \in b + K\} \\ &= \{x : x \in b - a + K\} \\ &= \{x : x \in K + K = K\} \subseteq K. \end{aligned}$$

So, $\{a\} \preceq^p \{b\}$. Conversely,

$$\begin{aligned} \{a\} \preceq^p \{b\} &\Rightarrow \{b\} -_l \{a\} \subseteq K \\ &\Rightarrow \{x : x + a \in b + K\} \subseteq K \\ &\Rightarrow \{x : x \in b - a + K\} \subseteq K. \end{aligned}$$

Since $b - a \in \{x : x \in b - a + K\} \subseteq K$, then $b - a \in K$. Therefore, $a \preceq b$. □

Then, the relation $\preceq^p$ is an extension of the vector order relation.

Lemma 3.2. *p-order relation implies m_1-order relation. That is,*

$$P \preceq^p R \Rightarrow P \preceq^{m_1} R + K$$

and

$$P \preceq^p R \Rightarrow P \preceq^{m_1} R$$

for $P, R \in \mathcal{P}(\mathbb{R}^n)$ and $R -_l P \neq \emptyset$.

Proof.

$$\begin{aligned} P \preceq^p R &\Rightarrow R -_l P \subseteq K \\ &\Rightarrow (R + K) -_m P \subseteq K \\ &\Rightarrow [(R + K) -_m P] \cap K \neq \emptyset \\ &\Rightarrow P \preceq^{m_1} R + K \end{aligned}$$

and

$$\begin{aligned} P \preceq^p R &\Rightarrow R -_l P \subseteq K \\ &\Rightarrow (R + K) -_m P \subseteq K \\ &\Rightarrow (R -_m P) \cap K \neq \emptyset \\ &\Rightarrow P \preceq^{m_1} R. \end{aligned}$$

□

The other direction may not be correct. For example, let $P = \{(0,0)\}, R = \bar{\mathbf{B}}((0,0),1)$ and $K = \mathbb{R}^2_+$ in space $\mathbb{R}^2$. It can be easily prove that $P \preceq^{m_1} R$ and $P \npreceq^p R$.

Let $P, R \in \mathcal{P}(\mathbb{R}^n)$. $P \preceq^l R$ doesn't imply $P \preceq^p R$. For example, if we take $P = \{(x,y) : x \leq 0, y \leq 0\}$ and $R = \{(x,y) : x = y, y \leq 0\}$ and $K = \{(x,y) : x, y \geq 0\} \subseteq \mathbb{R}^2$, then $P \preceq^l R \Leftrightarrow R \subseteq P + K$ holds. $P \preceq^p R \Leftrightarrow R -_l P \subseteq K$ doesn't hold. Therefore $P \npreceq^p R$.

Let $P, R \in \mathcal{P}(\mathbb{R}^n)$. $P \preceq^p R$ doesn't imply $P \preceq^l R$. For example, if we take $P = \{(x,y) : y = 1/x, x > 0\}$ and $R = \{(x,y) : x \geq 0, y = 0\}$ and $K = \mathbb{R}^2_+$, then $P \preceq^p R \Leftrightarrow R -_l P \subseteq K$ is hold, but $R \nsubseteq P + K$. That is $P \npreceq^l R$.

Let $P, R \in \mathcal{C}(\mathbb{R}^n)$. $P \preceq^p R$ doesn't imply $P \preceq^l R$. For example, $P = \bar{\mathbf{B}}((1,1),1)$, $R = conv\{(0,0),(0,2),(2,2),(2,0)\}$ and $K = \mathbb{R}^2_+$ in $\mathbb{R}^2$. It can be easily seen that $P \preceq^p R$ and $P \npreceq^l R$.

Proposition 3.5. *Let $P, R \in \mathcal{C}(\mathbb{R}^n)$ be given. If $P \preceq^l R$ implies $P \preceq^p R$.*

Proof. Since $P \preceq^l R$, we have

$$R \subseteq P + K. \tag{3.4}$$

We will show that $P \preceq^p R$. Let's take $x \in R -_l P$. Then, $x + P \subseteq R + K$. From (3.4), we get

$$x + P \subseteq R + K \subseteq P + K + K = P + K.$$

So, $x \in K$. Therefore,

$$\{x : x + P \subseteq R + K\} \subseteq K$$

implies $P \preceq^p R$. □

Proposition 3.6. *Let $P, R \in \mathcal{P}(\mathbb{R}^n)$, B bounded and $R -_l P \neq \emptyset$. If $P \preceq^p R$, then $P \preceq^{\ell_1} R$.*

Proof. $P \preceq^p R \Leftrightarrow R -_l P \subseteq K$. For all $x \in R -_l P$, $x \in K$. Hence $(R -_l P) \cap K \neq \emptyset$. We obtain $P \preceq^{\ell_1} R$. □

Let $P, R \in \mathcal{P}(\mathbb{R}^n)$. $P \preceq^{\ell_1} R$ doesn't imply $P \preceq^p R$. For example, if we take $P = conv\{(3,1),(3,2)\}$, $R = conv\{(1,1),(3,1)\}$ and $K = \mathbb{R}^2_+$. Since $R -_l P = \{x : x + P \subseteq R + K\} = [(-2,\infty) \times (0,\infty)]$, $(R -_l P) \cap K \neq \emptyset$. Then $P \preceq^{\ell_1} R$ is satisfied, but $R -_l P = [(-2,\infty)\text{Ö}(0,\infty)] \nsubseteq K$. $P \npreceq^p R$ is obtained.

Definition 3.2. Let $\mathbb{R}^n$ be an ordered real vector space with cone K, $\mathcal{A}$ be a family of nonempty subsets of $\mathbb{R}^n$ and $\bar{P} \in \mathcal{A}$. Then,

(i) if $P \preceq^p \bar{P}$ implies $\bar{P} \preceq^p P$ for some $P \in \mathcal{A}$, then $\bar{P}$ is called a minimal of $\mathcal{A}$ according to the $\preceq^p$ relation,

(ii) if $\bar{P} \preceq^p P$ implies $P \preceq^p \bar{P}$ for some $P \in \mathcal{A}$, then $\bar{P}$ is called a maximal of $\mathcal{A}$ according to the $\preceq^p$ relation.

Let S be a nonempty set, $F : S \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. When we consider the set-valued optimizations according to set optimization using p–relation, the optimization problem is denoted as:

$$(p - SOP) \begin{cases} \min(\max) F(x) \\ x \in S. \end{cases}$$

In addition, let $\mathcal{F}(S)$ be the set family of F, that is $\mathcal{F}(S) = \{F(x) : x \in S\}$. By combining the definitions given for sets and set-valued optimization, we can state the following relationship: If $F(x) \preceq^p F(x_0)$ for some $x \in S$ implies $F(x_0) \preceq^p F(x)$, then x_0 is a minimal solution of $(p - SOP)$ according to the $\preceq^p$ relation. Similarly, if $F(x_0) \preceq^p F(x)$ for some $x \in S$ implies $F(x) \preceq^p F(x_0)$, then x_0 is called a maximal solution of $(p - SOP)$ according to the $\preceq^p$ relation. The following examples demonstrate how to solve a set-valued optimization based on the $\preceq^p$ order relation.

Example 3.1. Let $\mathcal{N} = \{0, 1, 2, \dots\}, K = \mathbb{R}^2_+$, set-valued map $F : N \rightrightarrows \mathbb{R}^2$ is defined as $F(x) = P + \{(x, x)\}$, where $P = conv\{(0, 1), (1, 1)\} \cup conv\{(1, 0), (1, 1)\}$. Let's consider the following set-valued optimization

$$(p - SOP) \begin{cases} \min F(x) \\ x \in \mathcal{N}. \end{cases}$$

$x_0 = 0$ is a solution of the problem because there is no x such that $F(x) \preceq^p F(0)$.

Example 3.2. Let's set valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x) = [-|x|, -2|x|] \times [-|x|, -2|x|]$$

for all $x \in \mathbb{R}$ and consider the following set-valued optimization

$$(p - SOP) \begin{cases} \max F(x) \\ x \in \mathbb{R}. \end{cases}$$

$x_0 = 0$ is a solution of the problem because there is no x such that $F(0) \preceq^p F(x)$.

References

[1] Q.H. Ansari, N. Hussain, P.K. Sharma, *Scalarization for Set Optimization in Vector Spaces. In International Conference on Recent Trends in Convex Optimization*, Theory, Algorithms and Applications, Springer Nature Singapore, Singapore, 2025, 247–277.

[2] Q.H. Ansari, J.-C. Yao, *Recent developments in vector optimization*, Springer-Verlag, Berlin, 2012.

[3] C. Gutiérrez, B. Jiménez, E. Miglierina, E. Molho, *Scalarization in set optimization with solid and nonsolid orderin cones*, J. Glob. Optim., **62**(3) (2015), 525–552.

[4] E. Hernández, L. Rodríguez-Marín, *Nonconvex scalarization in set optimization with set-valued maps*, J. Math. Anal. Appl., **325** (2007), 1–18.

[5] J. Jahn, *Directional derivatives in set optimization with the set less order relation*, Taiwanese J. Math., **19**(3) (2015), 737–757.

[6] J. Jahn, *Vectorization in set optimization*, J. Optim. Theory Appl., **4**(3) (2013), 163–170.

[7] J. Jahn, T.X.D. Ha, *New order relations in set optimization*, J. Optim. Theory Appl., **148** (2011), 209–236.

[8] E. Karaman, *A New Pre-Order Relation for Set Optimization using l-difference*, Communications in Advanced Mathematical Sciences, **167**(3) (2021), 783–795.

[9] E. Karaman, *Gömme Fonksiyonu Kullanılarak Küme Optimizasyonuna Göre Verilen Küme Değerli Optimizasyon Problemlerinin Optimallik Koşulları*, Süleyman Demirel Üniversitesi Fen Edebiyat Fakültesi Fen Dergisi, **14**(1) (2019), 105–111.

[10] E. Karaman, İ. Atasever Güvenç, M. Soyertem, *Optimality conditions in set-valued optimization problems with respect to a partial order relation by using subdifferentials*, Optimization, **70**(3) (2021), 613–630.

[11] E. Karaman, İ. Atasever Güvenç, M. Soyertem, D. Tozkan, M. Küçük, Y. Küçük, *A vectorization for nonconvex set-valued optimization*, Turkish J. Math., **42**(4) (2018), 1815–1832.

[12] E. Karaman, M. Soyertem, İ. Atasever Güvenç, *Optimality conditions in set-valued optimization problem with respect to a partial order relation via directional derivative*, Taiwanese J. Math., **24**(3) (2020), 709–722.

[13] E. Karaman, M. Soyertem, İ. Atasever Güvenç, D. Tozkan, M. Küçük, Y. Küçük, *Partial order relations on family of sets and scalarizations for set optimization*, Positivity, **22** (2018), 783–802.
[14] A. A. Khan, C. Tammer and C. Zalinescu, *Set-Valued Optimization: An Introduction with Applications*, Springer, Berlin, 2015.
[15] C.S. Khusboo and Lalitha, *Scalarizations for a set optimization problem using generalized oriented distance function*, Positivity, **23**(5) (2019), 1195-1213.
[16] D. Kuroiwa, *On derivatives of set-valued maps in set optimization*, J. Nonlinear Convex Anal., **1611** (2008), 105-111.
[17] D. Kuroiwa, *The natural criteria in set-valued optimization*, RIMS Kokyuroku, **1031** (1998), 85–90.
[18] D. Kuroiwa, T. Tanaka, T.X.D. Ha, *On cone convexity of set-valued maps*, Nonlinear Anal-Theor., **30**(3) (1997), 1487–1496.
[19] D. Pallaschke, R. Urbanski, *Pairs of Compact Convex Sets*, Kluwer Academic Publishers, Dordrecht, 2002.
[20] M. Pilecka, *Optimality conditions in set-valued programming using the set criterion*, Thecnical University of Freiberg, Preprint, **2014** (2014).
[21] R. Schneider, *Convex Bodies: The Brunn-Minkowski theory*, Cambridge University Press, Cambridge, 1993.
[22] Y. Zhai, G. Yu, T. Tang, W. Han, *Optimal conditions in uncertain set-valued optimization problems via second-order subdifferential involving Minkowski difference with application to zero-sum matrix games*, J. Comput. Appl. Math., **464** (2020), 116530.

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