Turkish J. Ineq., 10 (1) (2026), Pages 131 –141 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

NEW EXTENDED SRIVASTAVA HYPERGEOMETRIC FUNCTIONS AND THEIR VARIOUS INTEGRAL REPRESENTATIONS

ENES ATA[1], HASAN GÜNDÜZ[2], AND ERCAN ÇELİK[3,4]

ABSTRACT. This paper introduces novel and comprehensive extensions of the classical Srivastava hypergeometric functions H_A, H_B, and H_C by incorporating a beta function with a general kernel into their definitions. Through this generalization, a broader class of hypergeometric-type functions is constructed, unifying several well-known special functions under a single analytical framework. In addition, various integral representations associated with these newly defined functions are systematically derived, providing deeper insights into their structural and operational properties. The proposed extensions not only enrich and generalize the theoretical foundation of classical hypergeometric functions but also significantly expand their applicability in diverse fields of mathematical analysis, fractional calculus, and applied sciences. These results are expected to open new avenues for further analytical investigation and for the modeling of complex phenomena arising in mathematical physics, engineering, and related disciplines.

1. INTRODUCTION AND PRELIMINARIES

In mathematics, hypergeometric functions form a broad class encompassing many special functions and reduce to simpler functions in particular or limiting cases. The Gauss hypergeometric function, denoted by ${}_2F_1(\cdot)$, is defined by (see, for example, [1])

$$ {}_2F_1(\zeta_1, \zeta_2; \zeta_3; \rho) = \sum_{n=0}^{\infty} \frac{(\zeta_1)_n(\zeta_2)_n}{(\zeta_3)_n} \frac{\rho^n}{n!}, \quad |\rho| < 1, \tag{1.1} $$

where $(\cdot)_n$ denotes the Pochhammer symbol,

$$ (w)_n = \begin{cases} w(w+1)\dots(w+n-1), & n \in \mathbb{N}, \\ 1, & n = 0. \end{cases} $$

Key words and phrases. Srivastava hypergeometric functions, integral representations, gamma function, beta function with general kernel, Exton's function, Gauss hypergeometric function.

2010 *Mathematics Subject Classification.* Primary: 33C60, 33C70. Secondary: 30E20, 33B15, 33C05.

Received: 28/01/2026 *Accepted*: 11/06/2026.

Cited this article as: E. Ata, H. Gündüz, E. Çelik, New extended Srivastava hypergeometric functions and their various integral representations, Turkish Journal of Inequalities, 10(1) (2026), 131-141.

This function is a fundamental member of the hypergeometric family and satisfies a second-order linear ordinary differential equation. The Pochhammer symbol also obeys the relation

$$(w)_{n+m} = (w)_n (w+n)_m. \tag{1.2}$$

The gamma function, denoted by the symbol $\Gamma(\cdot)$, is a generalized version of the factorial function and is defined in [1] by the following generalized integral representation:

$$\Gamma(w) = \int_0^\infty \tau^{w-1} \exp(-\tau) d\tau, \quad \Re(w) > 0.$$

A profound and fundamental association exists between the Pochhammer symbol and the gamma function. This relationship, which frequently arises in mathematical analysis and the theory of special functions, is formally established in [1] as follows:

$$(w)_n = \frac{\Gamma(w+n)}{\Gamma(w)}, \quad \Re(w) > 0. \tag{1.3}$$

For $n \in \mathbb{Z}_{\geq 0} = \mathbb{N} \cup 0$, the ratio $\frac{\Gamma(w+n)}{\Gamma(w)}$ extends to an entire function of w, since the simple poles of $\Gamma(w+n)$ and $\Gamma(w)$ cancel. The beta function is an important special function in mathematical analysis, especially closely related to the gamma function and probability theory. It is usually denoted by $B(\cdot)$ and is defined in [1] by means of the generalized integral as

$$B(\zeta_1, \zeta_2) = \int_0^1 \tau^{\zeta_1 - 1} (1-\tau)^{\zeta_2 - 1} d\tau, \quad \Re(\zeta_1), \Re(\zeta_2) > 0.$$

By taking the product of two gamma functions and changing variables on the double integral (e.g. switching to polar coordinates), the most important relationship between the beta function and the gamma function is obtained in [1] as follows:

$$B(\zeta_1, \zeta_2) = \frac{\Gamma(\zeta_1)\Gamma(\zeta_2)}{\Gamma(\zeta_1 + \zeta_2)}, \quad \Re(\zeta_1), \Re(\zeta_2) > 0. \tag{1.4}$$

Using the gamma function representations given in Eqs. (1.3) and (1.4), a direct connection between the Pochhammer symbol and the beta function is obtained, which is essential in the analysis of special functions:

$$\frac{(\zeta_1)_n}{(\zeta_2)_n} = \frac{B(\zeta_1 + n, \zeta_2 - \zeta_1)}{B(\zeta_1, \zeta_2 - \zeta_1)}, \quad \Re(\zeta_2) > \Re(\zeta_1) > 0. \tag{1.5}$$

The relation between the Pochhammer symbol and the beta function in Eq. (1.5) has been widely used to define generalized special functions and investigate their properties (see Refs. [2–7, 17]). Similar applications and related analytical studies can also be found in the literature (see Refs. [15, 16]). This is typically achieved by replacing the beta function in the numerator with a generalized beta function. In this work, we introduce new extensions of the Srivastava hypergeometric functions based on a beta function with a general kernel that encompasses various forms appearing in the literature.

Let K denote the general kernel, and let $\mathbf{X} = \mathbf{X}(p, q, \alpha, \beta)$ be a multiparameter variable. For $\Re(p), \Re(q), \Re(\alpha), \Re(\beta), \Re(\zeta_1), \Re(\zeta_2) > 0$, the beta function with general kernel, which is employed in the present work, was originally introduced by Ata and Kıymaz in [8] as follows:

$$ {}^{K}\widehat{B}(\zeta_1,\zeta_2)=\int_0^1 \tau^{\zeta_1-1}(1-\tau)^{\zeta_2-1}K(\tau,\mathbf{X})d\tau. \tag{1.6}$$

Srivastava hypergeometric functions refer to special classes of hypergeometric functions in multivariable and more general forms, introduced and developed by the mathematician Hari Mohan Srivastava. These functions allow classical hypergeometric series to be applied to more complex situations and are often used in mathematical physics, such as quantum mechanics and thermodynamic models; in statistics, such as multivariate distributions and probability density functions; and in fractional calculus, such as applications to fractional differentiation and integration.

Srivastava's triple hypergeometric functions H_A, H_B and H_C, are functions that have an important place in mathematical analysis and represent special classes of multivariate hypergeometric functions and are defined in [21, 22], respectively, as follows:

$$H_A(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5;\Upsilon_1,\Upsilon_2,\Upsilon_3)=\sum_{m,n,k=0}^{\infty}\frac{(\zeta_1)_{m+k}(\zeta_2)_{m+n}(\zeta_3)_{n+k}}{(\zeta_4)_m(\zeta_5)_{n+k}}\frac{\Upsilon_1^m}{m!}\frac{\Upsilon_2^n}{n!}\frac{\Upsilon_3^k}{k!},$$

$$(\eta<1,\mu<1,\xi<(1-\eta)(1-\mu)),$$

$$H_B(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5,\zeta_6;\Upsilon_1,\Upsilon_2,\Upsilon_3)=\sum_{m,n,k=0}^{\infty}\frac{(\zeta_1)_{m+k}(\zeta_2)_{m+n}(\zeta_3)_{n+k}}{(\zeta_4)_m(\zeta_5)_n(\zeta_6)_k}\frac{\Upsilon_1^m}{m!}\frac{\Upsilon_2^n}{n!}\frac{\Upsilon_3^k}{k!},$$

$$\left(\xi+\eta+\mu+2\sqrt{\xi\eta\mu}<1\right),$$

and

$$H_C(\zeta_1,\zeta_2,\zeta_3;\zeta_4;\Upsilon_1,\Upsilon_2,\Upsilon_3)=\sum_{m,n,k=0}^{\infty}\frac{(\zeta_1)_{m+k}(\zeta_2)_{m+n}(\zeta_3)_{n+k}}{(\zeta_4)_{m+n+k}}\frac{\Upsilon_1^m}{m!}\frac{\Upsilon_2^n}{n!}\frac{\Upsilon_3^k}{k!},$$

$$\left(\xi<1,\eta<1,\mu<1,\xi+\eta+\mu-2\sqrt{(1-\xi)(1-\eta)(1-\mu)}<2\right),$$

where for brevity, the co-ordinates are written (ξ,η,μ) instead of $(|\Upsilon_1|,|\Upsilon_2|,|\Upsilon_3|)$. These functions have a rich literature, especially in terms of integral representations, derivative and difference equations, transformation formulae, and the study of special cases, see [9–12, 20].

2. New extended Srivastava hypergeometric functions

In this section, we derive a modified relation employing the beta function with general kernel expressed in Eq. (1.6), in place of the classical beta function that appears in the numerator on the right-hand side of Eq. (1.5). This newly obtained relation is subsequently utilized within the framework of Srivastava hypergeometric functions H_A, H_B, and H_C, leading to the definition of novel extended versions of these functions.

Assume that the co-ordinates are written as (ξ,η,μ) instead of $(|\Upsilon_1|,|\Upsilon_2|,|\Upsilon_3|)$ and let the conditions $\Re(p),\Re(q),\Re(\alpha),\Re(\beta)>0$. We then introduce the new extended forms for the Srivastava hypergeometric functions H_A, H_B, and H_C, respectively, as follows:

$$\begin{aligned}&{}^K\widehat{H}_A(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5;\Upsilon_1,\Upsilon_2,\Upsilon_3) = {}^K H^{(\alpha,\beta)}_{A,p,q}(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5;\Upsilon_1,\Upsilon_2,\Upsilon_3)\\ &:= \sum_{m,n,k=0}^{\infty}\frac{(\zeta_1)_{m+k}(\zeta_2)_{m+n}}{(\zeta_4)_m}\frac{{}^K\widehat{B}(\zeta_3+n+k,\zeta_5-\zeta_3)}{B(\zeta_3,\zeta_5-\zeta_3)}\frac{\Upsilon_1^m}{m!}\frac{\Upsilon_2^n}{n!}\frac{\Upsilon_3^k}{k!},\end{aligned} \tag{2.1}$$

$$(\eta<1,\mu<1,\xi<(1-\eta)(1-\mu)),$$

$$\begin{aligned}&{}^K\widehat{H}_B(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5,\zeta_6;\Upsilon_1,\Upsilon_2,\Upsilon_3) = {}^K H^{(\alpha,\beta)}_{B,p,q}(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5,\zeta_6;\Upsilon_1,\Upsilon_2,\Upsilon_3)\\ &:= \sum_{m,n,k=0}^{\infty}\frac{(\zeta_1+\zeta_2)_{2m+n+k}(\zeta_3)_{n+k}}{(\zeta_4)_m(\zeta_5)_n(\zeta_6)_k}\frac{{}^K\widehat{B}(\zeta_1+m+k,\zeta_2+m+n)}{B(\zeta_1,\zeta_2)}\frac{\Upsilon_1^m}{m!}\frac{\Upsilon_2^n}{n!}\frac{\Upsilon_3^k}{k!},\end{aligned} \tag{2.2}$$

$$\left(\xi+\eta+\mu+2\sqrt{\xi\eta\mu}<1\right),$$

and

$$\begin{aligned}&{}^K\widehat{H}_C(\zeta_1,\zeta_2,\zeta_3;\zeta_4;\Upsilon_1,\Upsilon_2,\Upsilon_3) = {}^K H^{(\alpha,\beta)}_{C,p,q}(\zeta_1,\zeta_2,\zeta_3;\zeta_4;\Upsilon_1,\Upsilon_2,\Upsilon_3)\\ &:= \sum_{m,n,k=0}^{\infty}\frac{(\zeta_2)_{m+n}(\zeta_3)_{n+k}}{(\zeta_4)_n}\frac{{}^K\widehat{B}(\zeta_1+m+k,\zeta_4+n-\zeta_1)}{B(\zeta_1,\zeta_4+n-\zeta_1)}\frac{\Upsilon_1^m}{m!}\frac{\Upsilon_2^n}{n!}\frac{\Upsilon_3^k}{k!},\end{aligned} \tag{2.3}$$

$$\left(\xi<1,\eta<1,\mu<1,\xi+\eta+\mu-2\sqrt{(1-\xi)(1-\eta)(1-\mu)}<2\right).$$

We refer to these as the Srivastava hypergeometric functions ${}^K\widehat{H}_A$, ${}^K\widehat{H}_B$, ${}^K\widehat{H}_C$, respectively. Unless otherwise specified, we assume throughout this section and the remainder of the paper that $\Re(p),\Re(q),\Re(\alpha),\Re(\beta)>0$.

Remark 2.1. It is worth mentioning that the newly introduced classes ${}^K\widehat{H}_A$, ${}^K\widehat{H}_B$, and ${}^K\widehat{H}_C$, provide a unified framework that encompasses several previously studied extensions available in the literature. In particular, by choosing the general kernel function $K(\tau,\mathbf{X})$ appropriately, numerous known results can be recovered as special cases. For instance, the choice the general kernel function $K(\tau,\mathbf{X})=1$ reduces these functions to the classical Srivastava hypergeometric functions. Furthermore, by selecting the general kernel function $K(\tau,\mathbf{X})$ as exponential-type kernels, Mittag–Leffler kernels, or generalized M-series kernels, one can derive various extended Srivastava-type hypergeometric functions investigated in earlier works by different authors (see, for example, [9, 13, 14, 18, 19]). Hence, the present formulation not only generalizes the classical theory but also unifies a broad family of previously reported extensions within a single kernel-based analytical structure.

3. Integral representations

In this section, we derive various integral representations of the new extended Srivastava hypergeometric functions. These representations are obtained using the integral representations of the beta function with general kernel given in Eq. (1.6), which are (see Ref.

[8]):

$$ {}^{K}\widehat{B}(\zeta_1,\zeta_2) = 2\int_0^{\frac{\pi}{2}} (\sin\phi)^{2\zeta_1-1}(\cos\phi)^{2\zeta_2-1}\ K\left((\sin\phi)^2,\mathbf{X}\right)d\phi, \tag{3.1} $$

$$ {}^{K}\widehat{B}(\zeta_1,\zeta_2) = \int_0^{\infty} \frac{u^{\zeta_1-1}}{(1+u)^{\zeta_1+\zeta_2}}\ K\left(\frac{u}{1+u},\mathbf{X}\right)du, \tag{3.2} $$

and

$$ {}^{K}\widehat{B}(\zeta_1,\zeta_2) = (b-a)^{1-\zeta_1-\zeta_2}\int_a^{b} (u-a)^{\zeta_1-1}(b-u)^{\zeta_2-1}\ K\left(\frac{u-a}{b-a},\mathbf{X}\right)du. \tag{3.3} $$

3.1. **Srivastava hypergeometric function ${}^{K}\widehat{H}_A$.**

Theorem 3.1. *The Srivastava hypergeometric function ${}^{K}\widehat{H}_A$ has the following integral representations.*

(i) Let $\Re(\zeta_5) > \Re(\zeta_3) > 0$. Then,

$$ \begin{aligned} &{}^{K}\widehat{H}_A(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5;\Upsilon_1,\Upsilon_2,\Upsilon_3) \\ &\quad = \frac{\Gamma(\zeta_5)}{\Gamma(\zeta_3)\Gamma(\zeta_5-\zeta_3)}\int_0^1 \tau^{\zeta_3-1}(1-\tau)^{\zeta_5-\zeta_3-1}(1-\Upsilon_2\tau)^{-\zeta_2}(1-\Upsilon_3\tau)^{-\zeta_1} \\ &\quad\quad \times K(\tau,\boldsymbol{X})\ {}_2F_1\left(\zeta_1,\zeta_2;\zeta_4;\frac{\Upsilon_1}{(1-\Upsilon_2\tau)(1-\Upsilon_3\tau)}\right)d\tau. \end{aligned} $$

(ii) Let $\Re(\zeta_4) > \Re(\zeta_2) > 0$, $\Re(\zeta_5) > \Re(\zeta_3) > 0$. Then,

$$ \begin{aligned} &{}^{K}\widehat{H}_A(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5;\Upsilon_1,\Upsilon_2,\Upsilon_3) \\ &\quad = \frac{\Gamma(\zeta_4)\Gamma(\zeta_5)}{\Gamma(\zeta_2)\Gamma(\zeta_3)\Gamma(\zeta_4-\zeta_2)\Gamma(\zeta_5-\zeta_3)}\int_0^1\int_0^1 \vartheta^{\zeta_2-1}\tau^{\zeta_3-1}(1-\vartheta)^{\zeta_4-\zeta_2-1} \\ &\quad\quad \times (1-\tau)^{\zeta_5-\zeta_3-1}(1-\Upsilon_2\tau)^{\zeta_1-\zeta_2}\left[(1-\Upsilon_2\tau)(1-\Upsilon_3\tau)-\Upsilon_1\vartheta\right]^{-\zeta_1} \\ &\quad\quad \times K(\vartheta,\boldsymbol{X})K(\tau,\boldsymbol{X})d\vartheta d\tau. \end{aligned} $$

(iii) Let $\Re(\zeta_4) > \Re(\zeta_2) > 0$, $\Re(\zeta_5) > \Re(\zeta_3) > 0$. Then,

$$ \begin{aligned} &{}^{K}\widehat{H}_A(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5;\Upsilon_1,\Upsilon_2,\Upsilon_3) \\ &\quad = \frac{\Gamma(\zeta_4)\Gamma(\zeta_5)}{\Gamma(\zeta_2)\Gamma(\zeta_3)\Gamma(\zeta_4-\zeta_2)\Gamma(\zeta_5-\zeta_3)}\int_0^1\int_0^1 \vartheta^{\zeta_2-1}\tau^{\zeta_3-1}(1-\vartheta)^{\zeta_4-\zeta_2-1} \\ &\quad\quad \times (1-\tau)^{\zeta_5-\zeta_3-1}(1-\Upsilon_2\tau)^{-\zeta_2}(1-\Upsilon_1\vartheta-\Upsilon_3\tau)^{-\zeta_1} \\ &\quad\quad \times \left(1-\frac{\Upsilon_1\Upsilon_2\vartheta\tau}{(1-\Upsilon_2\tau)(1-\Upsilon_1\vartheta-\Upsilon_3\tau)}\right)^{-\zeta_1}\quad K(\vartheta,\boldsymbol{X})K(\tau,\boldsymbol{X})d\vartheta d\tau. \end{aligned} $$

(iv) Let $\Re(\zeta_5) > \Re(\zeta_3) > 0$. *Then,*

$$\begin{aligned} &{}^{K}\widehat{H}_A(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5;\Upsilon_1,\Upsilon_2,\Upsilon_3)\\ &\quad= \frac{2\Gamma(\zeta_5)}{\Gamma(\zeta_3)\Gamma(\zeta_5-\zeta_3)}\int_0^{\frac{\pi}{2}} (\sin\phi)^{2\zeta_3-1}(\cos\phi)^{2\zeta_5-2\zeta_3-1}\left(1-\Upsilon_2(\sin\phi)^2\right)^{-\zeta_2}\\ &\qquad\times\left(1-\Upsilon_3(\sin\phi)^2\right)^{-\zeta_1}\ K\left((\sin\phi)^2,\boldsymbol{X}\right)\\ &\qquad\times {}_2F_1\left(\zeta_1,\zeta_2;\zeta_4;\frac{\Upsilon_1}{(1-\Upsilon_2(\sin\phi)^2)\,(1-\Upsilon_3(\sin\phi)^2)}\right)d\phi. \end{aligned}$$

(v) Let $\Re(\zeta_5) > \Re(\zeta_3) > 0$. *Then,*

$$\begin{aligned} &{}^{K}\widehat{H}_A(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5;\Upsilon_1,\Upsilon_2,\Upsilon_3)\\ &\quad= \frac{\Gamma(\zeta_5)}{\Gamma(\zeta_3)\Gamma(\zeta_5-\zeta_3)}\int_0^{\infty} \vartheta^{\zeta_3-1}(1+\vartheta)^{\zeta_1+\zeta_2-\zeta_5}(1+\vartheta-\Upsilon_2\vartheta)^{-\zeta_2}\\ &\qquad\times(1+\vartheta-\Upsilon_3\vartheta)^{-\zeta_1}\ K\left(\frac{\vartheta}{1+\vartheta},\boldsymbol{X}\right)\\ &\qquad\times {}_2F_1\left(\zeta_1,\zeta_2;\zeta_4;\frac{\Upsilon_1(1+\vartheta)^2}{(1+\vartheta-\Upsilon_2\vartheta)(1+\vartheta-\Upsilon_3\vartheta)}\right)d\vartheta. \end{aligned}$$

(vi) Let $\Re(\zeta_5) > \Re(\zeta_3) > 0$. *Then,*

$$\begin{aligned} &{}^{K}\widehat{H}_A(\zeta_1,\zeta_2,\zeta_3;\zeta_4,\zeta_5;\Upsilon_1,\Upsilon_2,\Upsilon_3)\\ &\quad= \frac{\Gamma(\zeta_5)}{\Gamma(\zeta_3)\Gamma(\zeta_5-\zeta_3)}(v-u)^{1+\zeta_1+\zeta_2-\zeta_5}\int_u^{v} (\vartheta-u)^{\zeta_3-1}(v-\vartheta)^{\zeta_5-\zeta_3-1}\\ &\qquad\times(v-u-\Upsilon_2(\vartheta-u))^{-\zeta_2}(v-u-\Upsilon_3(\vartheta-u))^{-\zeta_1}\ K\left(\frac{\vartheta-u}{v-u},\boldsymbol{X}\right)\\ &\qquad\times {}_2F_1\left(\zeta_1,\zeta_2;\zeta_4;\frac{\Upsilon_1(v-u)^2}{(v-u-\Upsilon_2(\vartheta-u))\,(v-u-\Upsilon_3(\vartheta-u))}\right)d\vartheta. \end{aligned}$$

Proof. The following steps are taken for the proofs.

- For the proof of (i), Eqs. (1.1), (1.4), and (1.6), as well as the binomial theorem, are used in Eq. (2.1), where the binomial theorem is defined as follows (see Ref. [1]):

$$(1-\tau)^{-\zeta} = \sum_{n=0}^{\infty}(\zeta)_n\frac{\tau^n}{n!},\qquad |\tau|<1. \tag{3.4}$$

- For the proofs of (ii) and (iii), Eq. (1.6) is used in Eq. (2.1), and the desired results are obtained by performing the necessary mathematical operations.
- For the proof of (iv), Eq. (3.1) is used in Eq. (2.1), and the proof is completed following the appropriate mathematical steps.
- For the proof of (v), Eq. (3.2) is used in Eq. (2.1), and the proof is completed by carrying out the relevant mathematical operations.

- For the proof of (vi), Eq. (3.3) is used in Eq. (2.1), and the proof is completed once the appropriate steps are followed.

□

3.2. **Srivastava hypergeometric function** ${}^{K}\widehat{H}_B$.

Theorem 3.2. *The Srivastava hypergeometric function* ${}^{K}\widehat{H}_B$ *has the following integral representations.*

(i) Let $\Re(\zeta_1) > 0$, $\Re(\zeta_2) > 0$. *Then,*

$$
\begin{aligned}
&{}^{K}\widehat{H}_B(\zeta_1, \zeta_2, \zeta_3; \zeta_4, \zeta_5, \zeta_6; \Upsilon_1, \Upsilon_2, \Upsilon_3) \\
&= \frac{\Gamma(\zeta_1+\zeta_2)}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \int_0^1 \tau^{\zeta_1-1}(1-\tau)^{\zeta_2-1}\ K(\tau, \boldsymbol{X}) \\
&\quad \times X_4(\zeta_1+\zeta_2, \zeta_3; \zeta_4, \zeta_5, \zeta_6; \Upsilon_1\tau(1-\tau), \Upsilon_2(1-\tau), \Upsilon_3\tau)d\tau,
\end{aligned}
$$

where X_4 *is the Exton's function (see Ref. [21]), which defined as*

$$
X_4(\zeta_1, \zeta_2; \zeta_3, \zeta_4, \zeta_5; \Upsilon_1, \Upsilon_2, \Upsilon_3) = \sum_{m,n,k=0}^{\infty} \frac{(\zeta_1)_{2m+n+k}(\zeta_2)_{n+k}}{(\zeta_3)_m(\zeta_4)_n(\zeta_5)_k} \frac{\Upsilon_1^m}{m!} \frac{\Upsilon_2^n}{n!} \frac{\Upsilon_3^k}{k!}.
$$

(ii) Let $\Re(\zeta_1) > 0$, $\Re(\zeta_2) > 0$. *Then,*

$$
\begin{aligned}
&{}^{K}\widehat{H}_B(\zeta_1, \zeta_2, \zeta_3; \zeta_4, \zeta_5, \zeta_6; \Upsilon_1, \Upsilon_2, \Upsilon_3) \\
&= \frac{2\Gamma(\zeta_1+\zeta_2)}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \int_0^{\frac{\pi}{2}} (\sin\phi)^{2\zeta_1-1}(\cos\phi)^{2\zeta_2-1}\ K\left((\sin\phi)^2, \boldsymbol{X}\right) \\
&\quad \times X_4(\zeta_1+\zeta_2, \zeta_3; \zeta_4, \zeta_5, \zeta_6; \Upsilon_1(\sin\phi)^2(\cos\phi)^2, \Upsilon_2(\cos\phi)^2, \Upsilon_3(\sin\phi)^2)d\phi.
\end{aligned}
$$

(iii) Let $\Re(\zeta_1) > 0$, $\Re(\zeta_2) > 0$. *Then,*

$$
\begin{aligned}
&{}^{K}\widehat{H}_B(\zeta_1, \zeta_2, \zeta_3; \zeta_4, \zeta_5, \zeta_6; \Upsilon_1, \Upsilon_2, \Upsilon_3) \\
&= \frac{\Gamma(\zeta_1+\zeta_2)}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \int_0^{\infty} \frac{\vartheta^{\zeta_1-1}}{(1+\vartheta)^{\zeta_1+\zeta_2}}\ K\left(\frac{\vartheta}{1+\vartheta}, \boldsymbol{X}\right) \\
&\quad \times X_4\left(\zeta_1+\zeta_2, \zeta_3; \zeta_4, \zeta_5, \zeta_6; \frac{\Upsilon_1\vartheta}{(1+\vartheta)^2}, \frac{\Upsilon_2}{1+\vartheta}, \frac{\Upsilon_3\vartheta}{1+\vartheta}\right) d\vartheta.
\end{aligned}
$$

(iv) Let $\Re(\zeta_1) > 0$, $\Re(\zeta_2) > 0$. *Then,*

$$
\begin{aligned}
&{}^{K}\widehat{H}_B(\zeta_1, \zeta_2, \zeta_3; \zeta_4, \zeta_5, \zeta_6; \Upsilon_1, \Upsilon_2, \Upsilon_3) \\
&= \frac{\Gamma(\zeta_1+\zeta_2)}{\Gamma(\zeta_1)\Gamma(\zeta_2)}(v-u)^{1-\zeta_1-\zeta_2} \int_u^v (\vartheta-u)^{\zeta_1-1}(v-\vartheta)^{\zeta_2-1}\ K\left(\frac{\vartheta-u}{v-u}, \boldsymbol{X}\right) \\
&\quad \times X_4\left(\zeta_1+\zeta_2, \zeta_3; \zeta_4, \zeta_5, \zeta_6; \frac{\Upsilon_1(\vartheta-u)(v-\vartheta)}{(v-u)^2}, \frac{\Upsilon_2(v-\vartheta)}{v-u}, \frac{\Upsilon_3(\vartheta-u)}{v-u}\right) d\vartheta.
\end{aligned}
$$

Proof. The following steps are taken for the proofs.

- For the proof of (i), Eq. (1.4) and Eq. (1.6), as well as Exton's function, are used in Eq. (2.2), and the desired result is obtained through the necessary calculations.

- For the proof of (ii), Eq. (1.4) and Eq. (3.1), as well as Exton's function, are used in Eq. (2.2), and the expected result is obtained when the relevant calculations are performed.
- For the proof of (iii), Eq. (1.4) and Eq. (3.2), as well as Exton's function, are used in Eq. (2.2), and the final result is achieved by accurately carrying out the mathematical operations.
- For the proof of (iv), Eq. (1.4) and Eq. (3.3), as well as Exton's function, are used in Eq. (2.2), and the desired result is revealed upon the completion of the calculations.

$\square$

3.3. Srivastava hypergeometric function ${}^K\widehat{H}_C$.

Theorem 3.3. *The Srivastava hypergeometric function* ${}^K\widehat{H}_C$ *has the following integral representations.*

(i) Let $\Re(\zeta_4) > \Re(\zeta_1) > 0$. *Then,*

$$\begin{aligned}
&{}^K\widehat{H}_C(\zeta_1, \zeta_2, \zeta_3; \zeta_4; \Upsilon_1, \Upsilon_2, \Upsilon_3)\\
&\quad= \frac{\Gamma(\zeta_4)}{\Gamma(\zeta_1)\Gamma(\zeta_4-\zeta_1)} \int_0^1 \tau^{\zeta_1-1}(1-\tau)^{\zeta_4-\zeta_1-1}(1-\Upsilon_1\tau)^{-\zeta_2}(1-\Upsilon_3\tau)^{-\zeta_3}\\
&\quad\quad\times K(\tau, \boldsymbol{X})\ {}_2F_1\left(\zeta_2, \zeta_3; \zeta_4-\zeta_1; \frac{\Upsilon_2(1-\tau)}{(1-\Upsilon_1\tau)(1-\Upsilon_3\tau)}\right) d\tau.
\end{aligned}$$

(ii) Let $\Re(\zeta_1) > 0$, $\Re(\zeta_2) > 0$, $\Re(\zeta_4) > 0$, $\Re(\zeta_4-\zeta_1-\zeta_2) > 0$. *Then,*

$$\begin{aligned}
&{}^K\widehat{H}_C(\zeta_1, \zeta_2, \zeta_3; \zeta_4; \Upsilon_1, \Upsilon_2, \Upsilon_3)\\
&\quad= \frac{\Gamma(\zeta_4)}{\Gamma(\zeta_1)\Gamma(\zeta_2)\Gamma(\zeta_4-\zeta_1-\zeta_2)} \int_0^1 \int_0^1 \tau^{\zeta_1-1}\vartheta^{\zeta_2-1}(1-\tau)^{\zeta_4-\zeta_1-1}\\
&\quad\quad\times (1-\vartheta)^{\zeta_4-\zeta_1-\zeta_2-1}(1-\Upsilon_1\tau)^{\zeta_3-\zeta_2}\\
&\quad\quad\times (1-\Upsilon_1\tau-\Upsilon_2\vartheta-\Upsilon_3\tau+\Upsilon_2\tau\vartheta+\Upsilon_1\Upsilon_3\tau^2)^{-\zeta_3}\\
&\quad\quad\times K(\vartheta, \boldsymbol{X})K(\tau, \boldsymbol{X})d\vartheta d\tau.
\end{aligned}$$

(iii) Let $\Re(\zeta_4) > \Re(\zeta_1) > 0$. *Then,*

$$\begin{aligned}
&{}^K\widehat{H}_C(\zeta_1, \zeta_2, \zeta_3; \zeta_4; \Upsilon_1, \Upsilon_2, \Upsilon_3)\\
&\quad= \frac{2\Gamma(\zeta_4)}{\Gamma(\zeta_1)\Gamma(\zeta_4-\zeta_1)} \int_0^{\frac{\pi}{2}} (\sin\phi)^{2\zeta_1-1}(\cos\phi)^{2\zeta_4-2\zeta_1-1}\left(1-\Upsilon_1(\sin\phi)^2\right)^{-\zeta_2}\\
&\quad\quad\times \left(1-\Upsilon_3(\sin\phi)^2\right)^{-\zeta_3}\ K\left((\sin\phi)^2, \boldsymbol{X}\right)\\
&\quad\quad\times {}_2F_1\left(\zeta_2, \zeta_3; \zeta_4-\zeta_1; \frac{\Upsilon_2(\cos\phi)^2}{(1-\Upsilon_1(\sin\phi)^2)(1-\Upsilon_3(\sin\phi)^2)}\right) d\phi.
\end{aligned}$$

(iv) *Let* $\Re(\zeta_4) > \Re(\zeta_1) > 0$. *Then,*

$$\begin{aligned}
&{}^{K}\widehat{H}_C(\zeta_1, \zeta_2, \zeta_3; \zeta_4; \Upsilon_1, \Upsilon_2, \Upsilon_3) \\
&= \frac{\Gamma(\zeta_4)}{\Gamma(\zeta_1)\Gamma(\zeta_4 - \zeta_1)} \int_0^\infty \vartheta^{\zeta_1 - 1}(1+\vartheta)^{\zeta_2+\zeta_3-\zeta_4}(1+\vartheta-\Upsilon_1\vartheta)^{-\zeta_2} \\
&\times (1+\vartheta-\Upsilon_3\vartheta)^{-\zeta_3}\ K\left(\frac{\vartheta}{1+\vartheta}, \boldsymbol{X}\right) \\
&\times {}_2F_1\left(\zeta_2, \zeta_3; \zeta_4-\zeta_1; \frac{\Upsilon_2(1+\vartheta)}{(1+\vartheta-\Upsilon_1\vartheta)(1+\vartheta-\Upsilon_3\vartheta)}\right) d\vartheta.
\end{aligned}$$

(v) *Let* $\Re(\zeta_4) > \Re(\zeta_1) > 0$. *Then,*

$$\begin{aligned}
&{}^{K}\widehat{H}_C(\zeta_1, \zeta_2, \zeta_3; \zeta_4; \Upsilon_1, \Upsilon_2, \Upsilon_3) \\
&= \frac{\Gamma(\zeta_4)}{\Gamma(\zeta_1)\Gamma(\zeta_4 - \zeta_1)} (v-u)^{1+\zeta_2+\zeta_3-\zeta_4} \int_u^v (\vartheta-u)^{\zeta_1-1}(v-\vartheta)^{\zeta_4-\zeta_1-1} \\
&\times \big(v-u-\Upsilon_1(\vartheta-u)\big)^{-\zeta_2}\big(v-u-\Upsilon_3(\vartheta-u)\big)^{-\zeta_3}\ K\left(\frac{\vartheta-u}{v-u}, \boldsymbol{X}\right) \\
&\times {}_2F_1\left(\zeta_2, \zeta_3; \zeta_4-\zeta_1; \frac{\Upsilon_2(v-\vartheta)(v-u)}{\big(v-u-\Upsilon_1(\vartheta-u)\big)\big(v-u-\Upsilon_3(\vartheta-u)\big)}\right) d\vartheta.
\end{aligned}$$

Proof. The following steps are taken for the proofs.

- For the proof of (i), Eq. (1.6) is used in Eq. (2.3) and then the resulting equation is multiplied by $\frac{\Gamma(\zeta_4)\Gamma(\zeta_4-\zeta_1)}{\Gamma(\zeta_4)\Gamma(\zeta_4-\zeta_1)}$ and the desired result is obtained using Eqs. (1.2), (1.3), (1.4), and (3.4).
- For the proof of (ii), Eq. (1.6) is used in Eq. (2.3) and the desired result is obtained by performing the necessary mathematical operations.
- For the proof of (iii), Eq. (3.1) is used in Eq. (2.3) and the expected result is obtained when the relevant calculations are performed.
- For the proof of (iv), Eq. (3.2) is used in Eq. (2.3) and the final result is achieved by accurately carrying out the mathematical operations.
- For the proof of (v), Eq. (3.3) is used in Eq. (2.3) and the desired result is revealed upon the completion of the calculations.

□

4. Conclusion

In this paper, we have introduced new extensions of the Srivastava hypergeometric functions by incorporating a beta function with a general kernel into their definitions. These extensions broaden the existing framework of special functions and provide a more flexible analytical setting. Several integral representations have also been established, contributing to a clearer understanding of the structural properties of the proposed functions.

The results presented here may serve as a foundation for further study of their analytical and functional aspects. Future work may include a detailed investigation of their properties, as well as potential applications in mathematical physics, engineering, and applied analysis. Comparisons with classical and other generalized hypergeometric functions may also help clarify the distinctive features and scope of the present extensions.

Acknowledgements. The authors gratefully acknowledge Bingol University for financial support under the Scientific Research Projects (BAP) program (Project No. BAP-FEF.2026.005). This work was carried out within the Faculty of Arts and Sciences, Department of Mathematics, under the Basic Research and Development Projects (BARGEP) scheme.

References

[1] G.E. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.

[2] E. Ata, I.O. Kiymaz, *A study on certain properties of generalized special functions defined by Fox-Wright function*, Appl. Math. Nonlinear Sci., **5**(1) (2020), 147–162.

[3] E. Ata, I.O. Kiymaz, *Generalized gamma, beta and hypergeometric functions defined by Wright function and applications to fractional differential equations*, Cumhuriyet Sci. J., **43**(4) (2022), 684–695.

[4] E. Ata, I.O. Kiymaz, *New generalized special functions with two generalized M-series at their kernels and solution of fractional PDEs via double Laplace transform*, Comput. Methods Differ. Equ., **12**(1) (2024), 31–43.

[5] E. Ata, I.O. Kiymaz, P. Agarwal, S. Jain, *A study on the properties of new generalized special functions, their integral transformations, and applications to fractional differential equations*, Fractional Differential Equations, Academic Press, 2024.

[6] E. Ata, *M-Lauricella hypergeometric functions: integral representations and solutions of fractional differential equations*, Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat., **72**(2) (2023), 512–529.

[7] E. Ata, I.O. Kiymaz, *On the generating functions of the newly defined generalized hypergeometric functions*, TWMS J. Appl. Eng. Math., **14**(1) (2024), 64–78.

[8] E. Ata, I.O. Kiymaz, *Special functions with general kernel: Properties and applications to fractional partial differential equations*, Int. J. Math. Comput. Eng., **3**(2) (2025), 153–170.

[9] E. Ata, *M-Srivastava hypergeometric functions: integral representations and solution of fractional differential equations*, Miskolc Math. Notes, **25**(1) (2024), 93–108.

[10] J. Choi, A. Hasanov, M. Turaev, *Integral representations for Srivastava's hypergeometric function H_A*, Honam Math. J., **34**(1) (2012), 113–124.

[11] J. Choi, A. Hasanov, M. Turaev, *Integral representations for Srivastava's hypergeometric function H_B*, Pure Appl. Math., **19**(2) (2012), 137–145.

[12] J. Choi, A. Hasanov, M. Turaev, *Integral representations for Srivastava's hypergeometric function H_C*, Honam Math. J., **34**(4) (2012), 473–482.

[13] S.A. Dar, R.B. Paris, *A (p,q)-extension of Srivastava's triple hypergeometric function H_B and its properties*, J. Comput. Appl. Math., **348** (2019), 237–245.

[14] S.A. Dar, M. Kamarujjama, M. Daud, *Some properties of generalized Srivastava's triple hypergeometric function $H_{C,p,q}(\cdot)$*, Int. J. Appl. Comput. Math., **8**(4) (2022), 168.

[15] U. Demirbilek, *Analytical study on the generalized q-deformed Sinh-Gordon (Eleuch) equation*, Comput. Math. Math. Phys., **65**(4) (2025), 825–839.

[16] U. Demirbilek, *On the solitary wave solutions of different versions of fractional 3D-Wazwaz-Benjamin-Bona-Mahony equations*, Istanbul Ticaret Univ. J. Sci., **22**(44) (2023), 340–351.

[17] S. Jain, B.B. Jaimini, M. Buri, P. Agarwal, *On extended k-generalized Mittag-Leffler function and its properties*, Math. Found. Comput., **8**(4) (2025), 472–479.

[18] M. Kamarujjama, A. Yasmin, S.A. Dar, *A few explicit properties involving integral transforms and fractional calculus of Srivastava's triple hypergeometric function $H^{\lambda,l}_{C,p,\alpha}$*, Preprints, (2025), 1–17.

[19] R.K. Parmar, T.K. Pogany, *Extended Srivastava's triple hypergeometric $H_{A,p,q}$ function and related bounding inequalities*, J. Contemp. Math. Anal., **52**(6) (2017), 276–287.

[20] A. Saboor, G. Rahman, Z. Anjum, K.S. Nisar, S. Araci, *A new extension of Srivastava's triple hypergeometric functions and their associated properties*, Analysis, **41**(1) (2021), 13–24.
[21] H.M. Srivastava, P.W. Karlsson, *Multiple Gaussian hypergeometric series*, Halsted Press, New York, 1985.
[22] H.M. Srivastava, H.L. Manocha, *A treatise on generating functions*, Halsted Press Wiley, New York, 1984.

[1]Department of Mathematics,
Bingol University,
Bingol, Turkey
Email address: eata@bingol.edu.tr

[2]Department of Mathematics,
Bingol University,
Bingol, Turkey
Email address: hgunduz@bingol.edu.tr

[3]Department of Applied Mathematics and Informatics,
Kyrgyz-Turkish Manas University,
Bishkek, Kyrgyzstan
Email address: ercan.celik@manas.edu.kg

[4]Department of Mathematics,
Ataturk University,
Erzurum, Turkey
Email address: ercelik@atauni.edu.kg