Turkish J. Ineq., 10 (1) (2026), Pages 112 –130 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

NON-NEWTONIAN PARTIAL METRIC SPACES AND RELEVANT COMMON FIXED POINT APPROXIMATION FOR INTEGRAL TYPE $F-$ CONTRACTIONS

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Abstract. In this paper, using the framework of non-Newtonian calculus, we introduce non-Newtonian partial metric spaces to provide a quantitative framework based on non-Newtonian calculus. Then we reformulate two classical fixed point results, Banach's contraction principle and Caristi's fixed point theorem, in the setting of non-Newtonian partial metric spaces. Finally, we establish several common fixed point theorems for integral-type F-contraction mappings on non-Newtonian partial metric spaces.

1. Introduction

Several researchers have focused their efforts on metric fixed point theory. Many researchers have improved, unified, extended, and generalized Banach's theorem. Because of its applications, fixed point theory has become a significant tool for solving many nonlinear problems in science and technology. For more details on this topic, we refer readers to a few noteworthy papers ([2, 6, 9, 14, 16–18]). S.G. Matthews [7] proposed the concept of a partial metric space in 1992, with the purpose of providing a quantifiable computational model appropriate for programming verification. For more information, see ([7, 11]).

This paper develops a unified framework that combines two generalizations of metric geometry: partial metrics and non-Newtonian metrics. These structures modify key properties of distance, such as self-distance and scaling. Their interaction is therefore nontrivial. The resulting connection conditions are multiplicative and integral in nature and cannot be reduced to classical metric or partial metric contractions by simple transformations. This justifies the need for an intrinsic treatment and leads to fixed point results specific to this

Key words and phrases. non-Newtonian partial metric spaces, non-Newtonian integral type $F-$contraction, fixed point theorems.

2010 *Mathematics Subject Classification.* Primary: 47H09, 47H10 Secondary: 54E50, 37C25.

Received: 08/12/2025 *Accepted*: 19/05/2026.

Cite this article as: N. Sharma, H. Dutta, Non-Newtonian partial metric spaces and relevant common fixed point approximation for integral type $F-$ contractions, Turkish Journal of Inequalities, 10(1) (2026), 112-130.

setting. This framework explains the mechanism of convergence (contraction) while explicitly accounting for two aspects: the self-relation of each object and the relative change in their magnitudes. It also works well with certain product-type and integral inequalities that usually appear when you repeatedly apply a process that either steadily reduces the error or keeps any growth under control.

Another important feature of the proposed framework is that the induced metric transformation does not preserve the specific contractive structures considered in this work, particularly integral-type F-contractions on non-Newtonian partial metric spaces. This shows that the obtained results are not simple restatements of known theorems, but instead rely intrinsically on the combined framework. As a consequence, the proposed methodology offers a more versatile analytical setting, allowing both partial information and multiplicative behaviour to be incorporated into fixed point analysis, and thus extending the applicability of contraction principles beyond the classical context. Such models arise naturally in numerical analysis and the applied sciences, where convergence is often governed by relative error or proportional change rather than absolute difference. The proposed framework is therefore well suited to the analysis of multiplicative dynamical systems. To begin, we adopt Wilson's [15] classical terminology (The term semimetric space (halb-metrischer Raume) is likely due to Karl Menger [8]).

Definition 1.1. Let U be a set and let $\partial : U \times U \to \mathbb{R}$ be a mapping satisfying for each $\alpha, \beta \in U$:

(a) $\partial(\alpha,\beta) \geq 0$, and $\partial(\alpha,\beta) = 0 \iff \alpha = \beta$;
(b) $\partial(\alpha,\beta) = \partial(\beta,\alpha)$.

Then the pair (U, ∂) is called a semimetric space.

Matthews proved a partial metric version of the renowned Banach fixed point theorem, which has become a relevant and significant quantitative fixed point approach for capturing the meaning of recursive denotational specifications in programming languages. This is a type of distance space in which the triangle inequality is strengthened but Wilson's Axiom I (see (a)) is relaxed. As a direct consequence, these spaces are neither metric nor semimetric. The purpose of this paper is to introduce non-Newtonian calculus in the context of partial metric spaces to researchers in analysis and to demonstrate its effectiveness. We begin by describing several topological properties of the multiplicative metric space relevant to this study. Furthermore, we introduce the notion of contraction mappings in non-Newtonian partial metric spaces and establish several fixed point theorems in the setting of complete non-Newtonian partial metric spaces. In this work, we investigate the fundamental properties and lemmas, together with their proofs, that are essential for deriving fixed point results within the framework of non-Newtonian partial metric spaces. We now present the basic properties of non-Newtonian partial metric spaces.

A non-Newtonian partial metric on a nonempty set N is a mapping $\mathfrak{n}_\mathfrak{p} : N \times N \to \mathbb{R}^+$ satisfying the following conditions:

(a) $\alpha = \beta \iff \mathfrak{n}_\mathfrak{p}(\alpha, \alpha) = \mathfrak{n}_\mathfrak{p}(\alpha, \beta) = \mathfrak{n}_\mathfrak{p}(\beta, \beta)$;
(b) $\mathfrak{n}_\mathfrak{p}(\alpha, \alpha) \leq \mathfrak{n}_\mathfrak{p}(\alpha, \beta)$;
(c) $\mathfrak{n}_\mathfrak{p}(\alpha, \beta) = \mathfrak{n}_\mathfrak{p}(\beta, \alpha)$;
(d) non-Newtonian triangle inequality-

$$\mathfrak{n}_\mathfrak{p}(\alpha, \gamma) \leq \frac{\mathfrak{n}_\mathfrak{p}(\alpha, \beta) \times \mathfrak{n}_\mathfrak{p}(\beta, \gamma)}{\mathfrak{n}_\mathfrak{p}(\beta, \beta)}.$$

Condition (d) plays the same role for multiplication that the triangle inequality plays for addition, and it is formulated so that it matches how quantities scale in non-Newtonian calculus. The following propositions are significant to the rest of the discussion.

(a) Every non-Newtonian partial metric $\mathfrak{n}_\mathfrak{p}$ on N induces a topology τ on N, with a basis consisting of open balls of the form

$$\mathcal{O}_{\mathfrak{n}_\mathfrak{p}}(\alpha; \varepsilon) = \Big\{\beta \in N; \ \ \mathfrak{n}_\mathfrak{p}(\alpha, \beta) < \varepsilon \mathfrak{n}_\mathfrak{p}(\alpha, \alpha)\Big\}, \ \ \varepsilon > 1.$$

It is worth noting that the above definition can also describe an open set. Moreover, in the case where $\alpha \in O_{\mathfrak{n}_\mathfrak{p}}(\alpha; \varepsilon)$ for all $\varepsilon > 0$, the following holds: for any two arbitrarily chosen distinct points in N, there exists a ball that contains one of them but not the other. To prove the same, suppose $\alpha, \beta \in N$ with $\alpha \neq \beta$. Then by (a) and (b), either $\mathfrak{n}_\mathfrak{p}(\alpha, \alpha) < \mathfrak{n}_\mathfrak{p}(\alpha, \beta)$ or $\mathfrak{n}_\mathfrak{p}(\beta, \beta) < \mathfrak{n}_\mathfrak{p}(\alpha, \beta)$. Let $\mathfrak{n}_\mathfrak{p}(\alpha, \alpha) < \mathfrak{n}_\mathfrak{p}(\alpha, \beta)$ and

$$\varepsilon = \sqrt{\frac{\mathfrak{n}_\mathfrak{p}(\alpha, \beta)}{\mathfrak{n}_\mathfrak{p}(\alpha, \alpha)}}.$$

Then $\mathfrak{n}_\mathfrak{p}(\alpha, \beta) > \varepsilon \times \mathfrak{n}_\mathfrak{p}(\alpha, \alpha)$, so $\alpha \in \mathcal{O}_{\mathfrak{n}_\mathfrak{p}}(\alpha; \varepsilon)$ and

$$\beta \notin \mathcal{O}_{\mathfrak{n}_\mathfrak{p}}(\alpha; \varepsilon).$$

It is observed that a sequence $\{\alpha_\delta\} \in (N, \mathfrak{n}_\mathfrak{p})$ is said to converge to a point $\alpha \in N$ with respect to $\tau(\mathfrak{n}_\mathfrak{p}) \iff \mathfrak{n}_\mathfrak{p}(\alpha, \alpha) = \lim_{\delta \to \infty} \mathfrak{n}_\mathfrak{p}(\alpha, \alpha_\delta)$; symbolically write

$$\alpha_\delta \xrightarrow{\tau(\mathfrak{n}_\mathfrak{p})} \alpha.$$

(b) If $\mathfrak{n}_\mathfrak{p}$ is a partial metric on N, then the function $\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}} : N \ \times \ N \to \mathbb{R}^+$ given by

$$\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha, \beta) = \frac{(\mathfrak{n}_\mathfrak{p}(\alpha, \beta))^2}{\mathfrak{n}_\mathfrak{p}(\alpha, \alpha)\mathfrak{n}_\mathfrak{p}(\beta, \beta)}, \ \ \alpha, \ \beta \in N$$

defines a multiplicative metric on N. Since, $\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha, \beta) \geq 0 \ \forall \ \alpha, \ \beta \in N$. This transformation normalizes the self-distance and yields a multiplicative metric structure on N. In particular, convergence and completeness in $(N, \mathfrak{n}_{\mathfrak{p}_\mathfrak{q}})$ can be used to analyze the sequences in $(N, \mathfrak{n}_\mathfrak{p})$. Furthermore, for $\alpha, \ \beta, \ \gamma \in N$, i $\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha,\beta)=1 \iff \alpha=\beta$. To prove the same, let $\alpha=\beta$. Then,

$$\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha,\alpha)=\frac{(\mathfrak{n}_\mathfrak{p}(\alpha,\alpha))^2}{\mathfrak{n}_\mathfrak{p}(\alpha,\alpha)\mathfrak{n}_\mathfrak{p}(\alpha,\alpha)}=1,\ \alpha\in N.$$

Now, let $\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha,\beta)=1$, we have

$$\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha,\beta)=\frac{(\mathfrak{n}_\mathfrak{p}(\alpha,\beta))^2}{\mathfrak{n}_\mathfrak{p}(\alpha,\alpha)\mathfrak{n}_\mathfrak{p}(\beta,\beta)}=1.$$

If $\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha,\beta)=1$, then by (N_1) we conclude that $\alpha=\beta$. By (N_2),

$$\mathfrak{n}_\mathfrak{p}(\alpha,\alpha)\le\mathfrak{n}_\mathfrak{p}(\alpha,\beta)\quad\text{and}\quad\mathfrak{n}_\mathfrak{p}(\beta,\beta)\le\mathfrak{n}_\mathfrak{p}(\alpha,\beta).$$

Hence,

$$(\mathfrak{n}_\mathfrak{p}(\alpha,\beta))^2\ge\mathfrak{n}_\mathfrak{p}(\alpha,\alpha)\mathfrak{n}_\mathfrak{p}(\beta,\beta).$$

Therefore,

$$\mathfrak{n}_{\mathfrak{p}_q}(\alpha,\beta)=\frac{(\mathfrak{n}_\mathfrak{p}(\alpha,\beta))^2}{\mathfrak{n}_\mathfrak{p}(\alpha,\alpha)\mathfrak{n}_\mathfrak{p}(\beta,\beta)}\ge1.$$

Moreover,

$$\mathfrak{n}_{\mathfrak{p}_q}(\alpha,\alpha)=1.$$

Hence,

$$\mathfrak{n}_{\mathfrak{p}_q}(\alpha,\beta)=1\iff\alpha=\beta.$$

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$$\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha,\beta)=\frac{(\mathfrak{n}_\mathfrak{p}(\alpha,\beta))^2}{\mathfrak{n}_\mathfrak{p}(\alpha,\alpha)\mathfrak{n}_\mathfrak{p}(\beta,\beta)}$$

by and

$$\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\beta,\alpha)=\frac{(\mathfrak{n}_\mathfrak{p}(\beta,\alpha))^2}{\mathfrak{n}_\mathfrak{p}(\beta,\beta)\mathfrak{n}_\mathfrak{p}(\alpha,\alpha)}.$$

By condition (c) of non-Newtonian partial metric, $\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\beta,\alpha)=\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha,\beta)$.

iii Using condition (d) of non-Newtonian partial metric,

$$\mathfrak{n}_\mathfrak{p}(\alpha,\gamma)\le\frac{\mathfrak{n}_\mathfrak{p}(\alpha,\beta)\mathfrak{n}_\mathfrak{p}(\beta,\gamma)}{\mathfrak{n}_\mathfrak{p}(\beta,\beta)}.$$

Squaring both sides, we obtain

$$(\mathfrak{n}_\mathfrak{p}(\alpha,\gamma))^2\le\frac{(\mathfrak{n}_\mathfrak{p}(\alpha,\beta))^2(\mathfrak{n}_\mathfrak{p}(\beta,\gamma))^2}{(\mathfrak{n}_\mathfrak{p}(\beta,\beta))^2}.$$

Now, dividing by

$$\mathfrak{n}_\mathfrak{p}(\alpha,\alpha)\mathfrak{n}_\mathfrak{p}(\gamma,\gamma),$$

we have

$$\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha,\gamma)\le\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\alpha,\beta)\times\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\beta,\gamma).$$

iv A sequence $\{\alpha_\delta\}$ is a Cauchy sequence in $(N,\mathfrak{n}_\mathfrak{p})\iff\{\alpha_\delta\}$ is a Cauchy sequence in the metric space $(N,\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}})$.

v A non-Newtonian partial metric space $(N, \mathfrak{n}_{\mathfrak{p}})$ is complete iff every Cauchy sequence in $(N, \mathfrak{n}_{\mathfrak{p}})$ is convergent to an element of N.

vi $(N, \mathfrak{n}_{\mathfrak{p}})$ is complete iff $(N, \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}})$ is complete. Furthermore, for a sequence $\{\alpha_\delta\} \in (N, \mathfrak{n}_{\mathfrak{p}})$ and $\alpha \in N$, $\lim_{\delta\to\infty} \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}(\alpha, \alpha_\delta) = 1$ iff $\mathfrak{n}_{\mathfrak{p}}(\alpha, \alpha) = \lim_{\delta\to\infty} \mathfrak{n}_{\mathfrak{p}}(\alpha, \alpha_\delta) = \lim_{\delta,\delta_2\to\infty} \mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \alpha_{\delta_2}) = 1$

Remark 1.1. The mapping $\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}$ acts as a link between the multiplicative partial structure and the framework of classical metric analysis. Nevertheless, the contractive conditions considered in this work are expressed directly in terms of $\mathfrak{n}_{\mathfrak{p}}$ and cannot be reduced to conditions involving $\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}$.

Example 1.1. Let $N_{\mathfrak{s}} = \{[\mathfrak{u}, \mathfrak{t}] : \mathfrak{u}, \mathfrak{t} \in \mathbb{R}^+,\ \mathfrak{u} \neq \mathfrak{t}\}$. For any two elements $[\mathfrak{u}_1, \mathfrak{t}_1],\ [\mathfrak{u}_2, \mathfrak{t}_2] \in N_{\mathfrak{s}}$, define

$$\mathfrak{n}_{\mathfrak{p}}([\mathfrak{u}_1, \mathfrak{t}_1], [\mathfrak{u}_2, \mathfrak{t}_2]) = \max\left\{\frac{\mathfrak{u}_1}{\mathfrak{u}_2}, \frac{\mathfrak{u}_2}{\mathfrak{u}_1}, \frac{\mathfrak{t}_1}{\mathfrak{t}_2}, \frac{\mathfrak{t}_2}{\mathfrak{t}_1}\right\}.$$

Then $(N_{\mathfrak{s}}, \mathfrak{n}_{\mathfrak{p}})$ is a non-Newtonian partial metric space.

Example 1.2. Let $N = \mathbb{R}^+$ and define $\mathfrak{n}_{\mathfrak{p}} : N \times N \to \mathbb{R}^+$ by

$$\mathfrak{n}_{\mathfrak{p}}(\alpha, \beta) = \max\left\{\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right\}.$$

Then $(N, \mathfrak{n}_{\mathfrak{p}})$ is a non-Newtonian partial metric space. Consider the mapping $\wp : N \to N$ given by

$$\wp(\alpha) = \sqrt{\alpha}.$$

For all $\alpha, \beta \in N$, we obtain

$$\mathfrak{n}_{\mathfrak{p}}(\wp(\alpha), \wp(\beta)) = \max\left\{\sqrt{\frac{\alpha}{\beta}}, \sqrt{\frac{\beta}{\alpha}}\right\} = \left(\mathfrak{n}_{\mathfrak{p}}(\alpha, \beta)\right)^{1+\frac{1}{2}}.$$

Consequently,

$$\mathfrak{n}_{\mathfrak{p}}(\wp(\alpha), \wp(\beta)) \leq \left(\mathfrak{n}_{\mathfrak{p}}(\alpha, \beta)\right)^{\xi}, \quad \text{where } \xi = \tfrac{1}{2} \in [0, 1).$$

Hence, the mapping $\wp$ is a non-Newtonian contraction and, by Theorem 2.1, it has a unique fixed point in N.

This example illustrates that the non-Newtonian partial metric framework provides a natural setting for multiplicative contraction behavior beyond the classical additive metric framework.

2. non-Newtonian Partial Metric Contraction

This section introduces the non-Newtonian partial metric structure and establishes auxiliary properties required for the main results.

Theorem 2.1. *Let $(N, \mathfrak{n}_{\mathfrak{p}})$ be a complete non-Newtonian partial metric space and let arbitrarily chosen $\xi \in [0,1)$, $\wp : N \to N$ satisfy*

$$\mathfrak{n}_{\mathfrak{p}}(\wp(\alpha), \wp(\beta)) \leq (\mathfrak{n}_{\mathfrak{p}}(\alpha, \beta))^{\xi}$$

$\forall\ \alpha, \beta \in N$. Then there exists a unique $\hat{\alpha} \in N$ such that $\wp\hat{\alpha} = \hat{\alpha}$ and $\mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \hat{\alpha}) = 1$.

Proof. Let $\omega \in N$. Then $\forall\ \delta, \delta_1 \in \mathbb{N}$,

$$\begin{aligned}\mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+\delta_1+1}(\omega), \wp^{\delta}(\omega)) &\leq \frac{\mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+\delta_1+1}(\omega), \wp^{\delta+\delta_1}(\omega)) \times \mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+\delta_1}(\omega), \wp^{\delta}(\omega))}{\mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+\delta_1}(\omega), \wp^{\delta+\delta_1}(\omega))} \\ &\leq (\mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega))^{\xi^{(\delta+\delta_1)}} \times \mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+\delta_1}(\omega), \wp^{\delta}(\omega)).\end{aligned}$$

Then, $\forall\ \delta, \delta_1 \in \mathbb{N}$,

$$\begin{aligned}\mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+\delta_1+1}(\omega), \wp^{\delta}(\omega)) &\leq (\mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega))^{\xi^{(\delta+\delta_1)}} \times (\mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega))^{\xi^{(\delta+\delta_1-1)}} \\ &\quad \times \mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+\delta_1-1}(\omega), \wp^{\delta}(\omega)) \\ &\leq (\mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega))^{\xi^{(\delta+\delta_1)}} \times (\mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega))^{\xi^{(\delta+\delta_1-1)}} \\ &\quad \times \left((\mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega))^{\xi^{(\delta+\delta_1-2)}} \times \mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+\delta_1-2}(\omega), \wp^{\delta}(\omega))\right).\end{aligned}$$

Inductively, we have

$$\begin{aligned}\mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+\delta_1+1}(\omega), \wp^{\delta}(\omega)) &\leq (\mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega))^{\xi^{\delta+\delta_1}} \times \mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega) \times \mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+\delta_1-1}(\omega), \wp^{\delta}(\omega)) \\ &\leq \dots \\ &\leq \mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega)^{\xi^{\delta+\delta_1}+\xi^{\delta+\delta_1-1}+\xi^{\delta+\delta_1-2}+\ldots+\xi^{\delta}} \times \mathfrak{n}_{\mathfrak{p}}(\wp^{\delta}(\omega), \wp^{\delta}(\omega)) \\ &\leq \mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega)^{\xi^{\delta+\delta_1}+\xi^{\delta+\delta_1-1}+\xi^{\delta+\delta_1-2}+\ldots+\xi^{\delta}} \times \mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \wp(\omega))^{\xi^{\delta}} \\ &\leq \left(\mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega)^{\left(\frac{1-\xi^{\delta_1+1}}{1-\xi}\right)} \times \mathfrak{n}_{\mathfrak{p}}(\omega, \omega)\right)^{\xi^{\delta}} \qquad (2.1) \\ &\leq \left(\mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega)^{\left(\frac{1}{1-\xi}\right)} \times \mathfrak{n}_{\mathfrak{p}}(\omega, \omega)\right)^{\xi^{\delta}}.\end{aligned}$$

For $\delta_2 > \delta$, we have

$$\mathfrak{n}_{\mathfrak{p}}(\wp^{\delta_2}(\omega), \wp^{\delta}(\omega)) \leq \left(\mathfrak{n}_{\mathfrak{p}}(\wp(\omega), \omega)^{\left(\frac{1}{1-\xi}\right)} \times \mathfrak{n}_{\mathfrak{p}}(\omega, \omega)\right)^{\xi^{\delta}}$$

which implies that

$$\mathfrak{n}_{\mathfrak{p}}(\wp^{\delta_2}(\omega), \wp^{\delta}(\omega)) \to 1 \quad \text{as} \quad \delta, \delta_2 \to \infty.$$

Thus, $\forall\ \delta_2 \in \mathbb{N}$,

$$\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}(\wp^{\delta_2}(\omega), \wp^{\delta}(\omega)) \leq (\mathfrak{n}_{\mathfrak{p}}(\wp^{\delta_2}(\omega), \wp^{\delta}(\omega)))^2 \to 1 \quad \text{as} \quad \delta \to \infty.$$

Hence, the sequence $\{\wp^{\delta}(\omega)\}$ is Cauchy in $(N, \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}})$. Since the space $(N, \mathfrak{n}_{\mathfrak{p}})$ is complete, hence $(N, \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}})$ is complete. Therefore, $\{\wp^{\delta}(\omega)\}$ converges to $\hat{\alpha} \in N$. Therefore,

$$\mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \hat{\alpha}) = \lim_{\delta \to \infty} \mathfrak{n}_{\mathfrak{p}}(\wp^{\delta}(\omega), \hat{\alpha}) = \lim_{\delta, \delta_2 \to \infty} \mathfrak{n}_{\mathfrak{p}}(\wp^{\delta}(\omega), \wp^{\delta_2}(\omega)) = 1.$$

In addition

$$\begin{aligned}\mathfrak{n}_{\mathfrak{p}}(\wp(\hat{\alpha}), \hat{\alpha}) &\leq \frac{\mathfrak{n}_{\mathfrak{p}}(\wp(\hat{\alpha}), \wp^{\delta+1}(\omega)) \times \mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+1}(\omega), \hat{\alpha})}{\mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+1}(\omega), \wp^{\delta+1}(\omega))} \\ &\leq (\mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \wp^{\delta}(\omega)))^{\xi} \times \mathfrak{n}_{\mathfrak{p}}(\wp^{\delta+1}(\omega), \hat{\alpha}).\end{aligned}$$

For $\delta \to \infty$,

$$\mathfrak{n}_{\mathfrak{p}}(\wp(\hat{\alpha}), \hat{\alpha}) = 1.$$

Hence,

$$\mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \hat{\alpha}) \leq \mathfrak{n}_{\mathfrak{p}}(\wp(\hat{\alpha}), \hat{\alpha}) = 1,$$

and using condition (b) of non-Newtonian partial metric,

$$\mathfrak{n}_{\mathfrak{p}}(\wp(\hat{\alpha}), \wp(\hat{\alpha})) \leq \mathfrak{n}_{\mathfrak{p}}(\wp(\hat{\alpha}), \hat{\alpha}) = 1.$$

Hence,

$$\mathfrak{n}_{\mathfrak{p}}(\wp(\hat{\alpha}), \hat{\alpha}) = 1,$$

and therefore, by condition (a) of non-Newtonian partial metric,

$$\wp(\hat{\alpha}) = \hat{\alpha}.$$

To prove uniqueness, let $\hat{\beta} \in N$ be another fixed point (i.e., $\hat{\beta} = \wp(\hat{\beta})$). Then

$$\mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \hat{\beta}) = \mathfrak{n}_{\mathfrak{p}}(\wp(\hat{\alpha}), \wp(\hat{\beta})) \leq (\mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \hat{\beta}))^{\xi}.$$

Since $\xi \in [0, 1)$, it follows that

$$\mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \hat{\beta}) = 1,$$

and hence $\hat{\alpha} = \hat{\beta}$. □

3. Caristi's Theorem in non-Newtonian partial metric spaces

We now develop contraction principles in the above framework, beginning with a Banach-type result adapted to the non-Newtonian partial setting. Next, we propose the following two alternatives in order to provide an adequate conception of a Caristi mapping within the framework of partial metric spaces.

(a) A self-mapping $\wp$ of a partial metric space $(N, \mathfrak{n}_{\mathfrak{p}})$ is called a $\mathfrak{n}_{\mathfrak{p}}-$Caristi mapping on N if there is a function $\Phi : N \to [0, \infty)$ which is lower semi-continuous for $(N, \mathfrak{n}_{\mathfrak{p}})$ and satisfies

$$\mathfrak{n}_{\mathfrak{p}}(\alpha, \wp\alpha) \leq \frac{\Phi(\alpha)}{\Phi(\wp\alpha)},$$

$\forall\ \alpha \in N$.

(b) A self-mapping $\wp$ of a partial metric space $(N, \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}})$ is called a $\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}-$Caristi mapping on N if there is a function $\Phi : N \to [0, \infty)$ which is lower semicontinuous for $(N, \mathfrak{n}_{\mathfrak{p}})$ and satisfies

$$\mathfrak{n}_{\mathfrak{p}}(\alpha, \wp\alpha) \leq \frac{\Phi(\alpha)}{\Phi(\wp\alpha)},$$

$\forall\ \alpha \in N$.

It is obvious that every $\mathfrak{n}_{\mathfrak{p}}-$*Caristi mapping* is also $\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}-$*Caristi mapping* but not vice versa, i.e. $\alpha_\delta \xrightarrow{\tau(\mathfrak{n}_{\mathfrak{p}})} \alpha$ and $\Phi(\alpha_\delta) \longrightarrow \overset{\circ}{r}$ implies $\Phi(\alpha_\delta) \leq \overset{\circ}{r}$. Note that $\alpha_\delta \xrightarrow{\tau(\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}})} \alpha$ iff $\alpha_\delta \xrightarrow{\tau(\mathfrak{n}_{\mathfrak{p}})} \alpha$. That is, a sequence converges in $(N, \mathfrak{n}_{\mathfrak{p}_q})$ if and only if it converges in $(N, \mathfrak{n}_{\mathfrak{p}})$. To prove the same, let $\alpha_\delta \xrightarrow{\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}} \alpha$ and $\epsilon > 1$. Then there exists $\delta_0 \in \mathbb{N}$ in such a way that for $\delta \geq \delta_0$

$$\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}(\alpha_\delta, \alpha) = \frac{(\mathfrak{n}_{\mathfrak{p}})^2(\alpha, \alpha_\delta)}{\mathfrak{n}_{\mathfrak{p}}(\alpha, \alpha) \times \mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \alpha_\delta)} \leq \epsilon.$$

Using condition (b) of non-Newtonian partial metric,

$$\frac{\mathfrak{n}_{\mathfrak{p}}(\alpha, \alpha_\delta)}{\mathfrak{n}_{\mathfrak{p}}(\alpha, \alpha)} \leq \frac{(\mathfrak{n}_{\mathfrak{p}}(\alpha, \alpha_\delta))^2}{\mathfrak{n}_{\mathfrak{p}}(\alpha, \alpha) \times \mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \alpha_\delta)}.$$

It follows that for $\delta \geq \delta_0$, $\alpha_\delta \in \mathcal{O}(\alpha; \varepsilon)$, that is

$$\alpha_\delta \xrightarrow{\tau(\mathfrak{n}_{\mathfrak{p}})} \alpha.$$

Definition 3.1. For $\delta \in \mathbb{N}$, a sequence $\{\alpha_\delta\} \in (N, \mathfrak{n}_{\mathfrak{p}})$ is said to be $1-$ Cauchy if

$$\lim_{\delta, \delta_1 \to \infty} \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}(\alpha_\delta, \alpha_{\delta_1}) = 1.$$

The space $(N, \mathfrak{n}_{\mathfrak{p}})$ is said to be $1-$complete if every $1-$Cauchy sequence in N is convergent to $\alpha \in N$ with respect to $\tau(\mathfrak{n}_{\mathfrak{p}})$ and $\mathfrak{n}_{\mathfrak{p}}(\alpha, \alpha) = 1$. It is obvious that every complete $(N, \mathfrak{n}_{\mathfrak{p}})$ is $1-$complete but not vice versa.

Metric completeness is defined by the fixed point property of *Caristi* maps. Specifically, for metric space $(N, \mathfrak{n}_{\mathfrak{p}})$, a mapping $\wp : N \to N$ is known as *Caristi mapping* if $\exists\ \Phi : N \to \mathbb{R}^+$ which is bounded below and lower semicontinuous mapping for which

$$\mathfrak{n}_{\mathfrak{p}}(\alpha, \wp(\alpha)) \leq \frac{\Phi(\alpha)}{\Phi(\wp(\alpha))}$$

$\forall\ \alpha \in N$.

Lemma 3.1. *Let a non-Newtonian partial metric space* $(N, \mathfrak{n}_{\mathfrak{p}})$ *and the function* $\mathfrak{n}_{\varkappa} : N \to \mathbb{R}^+\ \forall \varkappa \in N$ *defined by* $\mathfrak{n}_{\varkappa}(\beta) = \mathfrak{n}_{\mathfrak{p}}(\varkappa, \beta)$ *is lower semicontinuous for* $(N, \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}})$.

Proof. Let $\lim_{\delta\to\infty} \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}(\beta_\delta, \beta) = 1$. Also, it is given that

$$\mathfrak{n}_{\mathfrak{p}}(\beta_\delta, \beta) = \frac{(\mathfrak{n}_{\mathfrak{p}}(\beta_\delta, \beta))^2}{\mathfrak{n}_{\mathfrak{p}}(\beta_\delta, \beta_\delta) \times \mathfrak{n}_{\mathfrak{p}}(\beta, \beta)}$$

and $\mathfrak{n}_{\mathfrak{p}}(\beta, \beta) \leq \mathfrak{n}_{\mathfrak{p}}(\beta, \beta_\delta)$, we have

$$\begin{aligned}
\mathfrak{n}_{\varkappa}(\beta) &\leq \frac{\mathfrak{n}_{\varkappa}(\beta_\delta) \times \mathfrak{n}_{\mathfrak{p}}(\beta_\delta, \beta)}{\mathfrak{n}_{\mathfrak{p}}(\beta_\delta, \beta_\delta)} \\
&= \frac{\mathfrak{n}_{\varkappa}(\beta_\delta) \times \mathfrak{n}_{\mathfrak{p}}(\beta_\delta, \beta)}{\mathfrak{n}_{\mathfrak{p}}(\beta_\delta, \beta) \times \mathfrak{n}_{\mathfrak{p}}(\beta, \beta)} \\
&\leq \mathfrak{n}_{\varkappa}(\beta_\delta) \times \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}(\beta_\delta, \beta).
\end{aligned}$$

It is given that $\mathfrak{n}_{\mathfrak{p}}(\beta, \beta) \leq \mathfrak{n}_{\mathfrak{p}}(\beta, \beta_\delta)$, hence we have $\mathfrak{n}_{\varkappa}(\beta) \leq \liminf_{\delta\to\infty} \mathfrak{n}_{\varkappa}(\beta_\delta)$. $\square$

The main result is as follows:

Theorem 3.1. *If $(N, \mathfrak{n}_{\mathfrak{p}})$ is $1-$complete, then every $\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}-$Caristi mapping has a fixed point.*

Proof. Let $(N, \mathfrak{n}_{\mathfrak{p}})$ is $1-$complete, and $\wp$ is $\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}-$Caristi mapping defined on N. Then there exists $\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}-$ lower semicontinuous function $\Phi : N \to \mathbb{R}^+$, which implies $\mathfrak{n}_{\mathfrak{p}}(\alpha, \wp(\alpha)) \leq \frac{\Phi(\alpha)}{\Phi(\wp(\alpha))}$ $\forall\ \alpha \in N$. Let us define a set $B_{\varkappa} := \{\beta \in N\ :\ \mathfrak{n}_{\mathfrak{p}}(\alpha, \beta) \leq \frac{\Phi(\alpha)}{\Phi(\beta)}\}$. Since, $\wp(\alpha) \in B_{\varkappa}$ implies that the set $B_{\varkappa}$ is nonempty. Furthermore, the set $B_{\varkappa}$ is closed in $(N, \mathfrak{n}_{\mathfrak{p}})$ because Φ is lower semi-continuous. Now, let $\alpha_0 \in N$ is fixed and $\exists\ \alpha_1 \in B_{\varkappa_0}$ such that $\Phi(\alpha_1) < 1 + \frac{1}{2}\inf_{\beta \in B_{\varkappa_0}} \Phi(\beta)$. Now, it is obvious that $B_{\varkappa_1} \subseteq B_{\varkappa_0}$. Now, $\forall\ \varkappa \in B_{\varkappa_1}$, we have

$$\begin{aligned} \mathfrak{n}_{\mathfrak{p}}(\alpha_1, \alpha) &\leq \frac{\Phi(\alpha_1)}{\Phi(\alpha)} \\ &\leq 1 + \frac{1}{2}\frac{\inf_{\beta \in B_{\varkappa_0}} \Phi(\beta)}{\Phi(\alpha)} \\ &\leq 1 + \frac{1}{2}. \end{aligned} \tag{3.1}$$

Inductively, for closed subsets of $(N, \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}})$, we can construct a sequence $\{\alpha_\delta\} \in N$ such that which is an associated nested sequence and denoted by $\{B_{\varkappa_\delta}\}$ satisfies

(a) $B_{\varkappa_{\delta+1}} \subseteq B_{\varkappa_\delta}$ and $\alpha_{\delta+1} \in B_{\varkappa_\delta}\ \forall\ \delta \in \mathbb{N}$;
(b) $\mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \alpha_{\delta'}) \leq 1 + \frac{1}{2^\delta}\ \forall\ \delta' \geq \delta$.

As, $\mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \alpha_\delta) \leq \mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \alpha_{\delta+1})$. By using this condition with (a) and (b), we have

$$\mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \alpha_{\delta_1}) \leq 1 + \frac{1}{2^\delta} \quad \forall \quad \delta_1 > \delta.$$

It follows that $\lim_{\delta,\delta_1 \to \infty} \mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \alpha_{\delta_1}) = 1$. Moreover, the family of sets $\{B_{\varkappa_{\delta+1}}\} \subseteq B_{\varkappa_\delta}$, for $\delta' > \delta$, we have

$$\mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}(\alpha_\delta, \alpha_{\delta'}) \leq 1 + \frac{1}{2^\delta},$$

which implies that

$$\lim_{\delta,\delta' \to \infty} \mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \alpha_{\delta'}) = 1.$$

Therefore the sequence $\{\alpha_\delta\}$ is a $1-$Cauchy sequence in $(N, \mathfrak{n}_{\mathfrak{p}})$. By the $1-$ completeness of the space, there exists $\hat{\alpha} \in N$ such that $\lim_{\delta \to \infty} \mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \alpha_\delta) = \mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \hat{\alpha}) = 1$ and hence $\lim_{\delta \to \infty} \mathfrak{n}_{\mathfrak{p}_{\mathfrak{q}}}(\hat{\alpha}, \alpha_\delta) = 1$. It follows that, $\hat{\alpha} \in \bigcap_{\delta=1}^{\infty} B_{\varkappa_\delta}\ \forall \delta \in \mathbb{N}$. To prove that $\hat{\alpha} = \wp(\hat{\alpha})$, we have

$$\begin{aligned} \mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \wp(\hat{\alpha})) &\leq \mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \hat{\alpha}) \times \mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \wp(\hat{\alpha})) \\ &\leq \frac{\Phi(\alpha_\delta) \times \Phi(\hat{\alpha})}{\Phi(\hat{\alpha}) \times \Phi(\wp(\hat{\alpha}))}, \end{aligned} \tag{3.2}$$

$\forall\ \delta \in \mathbb{N}$, it follows that $\wp(\hat{\alpha}) \in \bigcap_{\delta=1}^{\infty} B_{\varkappa_\delta}$, now using $(\mathfrak{c}_2)$ $\mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \wp(\hat{\alpha})) < 1 + \frac{1}{2^\delta}\ \forall\ \delta \in \mathbb{N}$. Also, by the multiplicative triangle inequality,

$$\mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \wp(\hat{\alpha})) \leq \frac{\mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \alpha_\delta) \times \mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \wp(\hat{\alpha}))}{\mathfrak{n}_{\mathfrak{p}}(\alpha_\delta, \alpha_\delta)}.$$

Taking limit as $\delta \to \infty$ and using $\lim_{\delta\to\infty} \mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \alpha_\delta) = 1$, we get $\mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \wp(\hat{\alpha})) = 1$, it results $\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\hat{\alpha}, \wp(\hat{\alpha})) = 1$. Also, it is given that $\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\hat{\alpha}, \wp(\hat{\alpha})) \leq \mathfrak{n}_{\mathfrak{p}}(\hat{\alpha}, \wp(\hat{\alpha}))$, It follows that $\hat{\alpha} = \wp\hat{\alpha}$. On the contrary, let there exists non-convergent $1-$Cauchy sequence $\{\alpha_\delta\} \in (N, \mathfrak{n}_{\mathfrak{p}})$. Let $\{\beta_\delta\}$ is a sub sequence of $\{\alpha_\delta\}$ for which $\mathfrak{n}_{\mathfrak{p}}(\beta_\delta, \beta_{\delta+1}) < 1 + \frac{1}{2^\delta}\ \forall\ \delta \in \mathbb{N}$. Let $B := \{\beta_\delta :$ such that $\delta \in \mathbb{N}\}$, and let $\wp : N \to N$ for which $\Phi(\alpha) = \mathfrak{n}_{\mathfrak{p}}(\alpha, \beta_0)$ for $\alpha \in N - B$ and $\wp(\beta_\delta) = \beta_{\delta+1}$. Since, B is closed in $(N, \mathfrak{n}_{\mathfrak{p}})$, let $\Phi : N \to \mathbb{R}^+$ such that $\Phi(\alpha) = \frac{\mathfrak{n}_{\mathfrak{p}}(\alpha,\beta_0)}{2}$ and $\Phi(\beta_\delta) \leq \Phi(\alpha)$ for all arbitrarily chosen element $\alpha \in N - B$. Now using Lemma 3.1, it can easily be deduced that the function Φ is lower semi continuous in $(N, \mathfrak{n}_{\mathfrak{p}_\mathfrak{q}})$. Furthermore, for $\alpha \in N - B$

$$\mathfrak{n}_{\mathfrak{p}}(\alpha, \wp\alpha) = \mathfrak{n}_{\mathfrak{p}}(\alpha, \beta_0) = \frac{\Phi(\alpha)}{\Phi(\wp(\alpha))},$$

$\forall\ \beta_\delta \in B$,

$$\begin{aligned} \mathfrak{n}_{\mathfrak{p}}(\beta_\delta, \wp(\beta_\delta)) = \mathfrak{n}_{\mathfrak{p}}(\beta_\delta, \beta_{\delta+1}) &< 1 + \frac{1}{2^{\delta+1}} \\ &= \frac{\Phi(\beta_\delta)}{\Phi(\beta_{\delta+1})} \\ &= \frac{\Phi(\beta_\delta)}{\Phi(\wp(\beta_\delta))}. \end{aligned} \tag{3.3}$$

Thus, $\wp$ is a Caristi-type mapping that has no fixed point.

□

4. non-Newtonian $F-$ Contraction

In metric and generalized metric spaces, there are numerous generalizations of the Banach contraction principle. Branciari defined one of them as integral type contraction [4]. Wardowski [14] on the other hand, as a generalisation of the Banach contraction principle, conceptualised $F-$contraction in metric spaces. For an extensive relevant literature, refer to ([1,2,4,6–8,10–13,15,18]). Now, we establish the common fixed point theorem for generalized integral type $F-$ contraction in non-Newtonian partial metric spaces.

Definition 4.1. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

(a) F is strictly increasing;
(b) For each sequence $\{\alpha_\delta\}_{\delta\in\mathbb{N}} \subset \mathbb{R}^+$,

$$\lim_{\delta\to\infty} \alpha_\delta = 1 \iff \lim_{\delta\to\infty} F(\alpha_\delta) = -\infty.$$

(c) There exists $\kappa \in (0, 1)$ such that

$$\lim_{\alpha\to 1^+} (\alpha - 1)^\kappa F(\alpha) = 0.$$

A mapping $\wp : N \to N$ is said to be an non-Newtonian $F-$contraction if $\exists\ \mathring{\tau} > 1$ such that

$$\forall\ \alpha, \beta \in N,\ \mathfrak{n}_{\mathfrak{p}}(\wp\alpha, \wp\beta) > 1 \text{ implies } \mathring{\tau} \times F(\mathfrak{n}_{\mathfrak{p}}(\wp\alpha, \wp\beta)) \leq F(\mathfrak{n}_{\mathfrak{p}}(\alpha, \beta)).$$

A family of $F-$Contraction mappings are represented by $\mathfrak{F}$.

Definition 4.2. [5] The mappings $\wp_1,\ \wp_2 : N \to N$ are said to be weakly compatible if $\wp_1$ and $\wp_2$ commute at each coincidence point; i.e., $\wp_1\alpha = \wp_2\alpha$ for some $\alpha \in N$.

Theorem 4.1. *Let $(N, \mathfrak{n}_\mathfrak{p})$ is non-Newtonian partial metric space and $\wp_1,\ \wp_2,\ \wp_3,\ \wp_4 : N \to N$ satisfying $\wp_1(N) \subseteq \wp_4(N)$ and $\wp_2(N) \subseteq \wp_3(N)$. Let $F \in \mathfrak{F}$ and $\overset{\circ}{\tau}$ be such that $\forall$ $\alpha, \beta \in N$ satisfies $\mathfrak{n}_\mathfrak{p}(\wp_1\alpha, \wp_2\beta) > 1$*

$$\overset{\circ}{\tau} \times F\Big(\int_0^{\mathfrak{n}_\mathfrak{p}(\wp_1\alpha,\wp_2\beta)} \Psi(\mu)\, d\mu\Big) \le F\Big(\int_0^{L(\alpha,\beta)} \Psi(\mu)\, d\mu\Big) \tag{N_{I_1}}$$

where $L(\alpha, \beta)$ is

$$L(\alpha,\beta) = \max\Big\{\mathfrak{n}_\mathfrak{p}(\wp_3\alpha, \wp_4\beta),\ \mathfrak{n}_\mathfrak{p}(\wp_1\alpha, \wp_3\alpha),\ \mathfrak{n}_\mathfrak{p}(\wp_2\beta, \wp_4\beta),\ \sqrt{\mathfrak{n}_\mathfrak{p}(\wp_3\alpha, \wp_2\beta)\, \mathfrak{n}_\mathfrak{p}(\wp_1\alpha, \wp_4\beta)}\Big\} \tag{N_{I_2}}$$

and $\Psi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable, nonnegative and $\forall\ \overset{\circ}{\mu} > 1$

$$\int_0^{\overset{\circ}{\mu}} \Psi(\mu)^{d\mu} > 1. \tag{N_{I_3}}$$

If the following conditions

(a) *$\wp_1(N),\ \wp_2(N),\ \wp_3(N),\ \wp_4(N)$ are closed,*
(b) *F is continuous,*
(c) *$\{\wp_1, \wp_3\}$ and $\{\wp_2, \wp_4\}$ are weakly compatible,*

hold, then the pairs $\wp_1,\ \wp_2,\ \wp_3,$ and $\wp_4$ have a unique common fixed point.

Proof. Let $\alpha_0 \in N$, then a sequence $\{\alpha_\delta\}$ is defined as $\beta_{2\delta+1} = \wp_1\alpha_{2\delta} = \wp_4\alpha_{2\delta+1}$ and $\beta_{2\delta+2} = \wp_2\alpha_{2\delta+1} = \wp_3\alpha_{2\delta+1}$,
Step I. Prove that $\mathfrak{n}_\mathfrak{p}(\beta_\delta, \beta_{\delta+1})$ is a null sequence. Using (N_{I_1})

$$\begin{aligned}\overset{\circ}{\tau} \times F\Big(\int_0^{\mathfrak{n}_\mathfrak{p}(\beta_{2\delta+1},\beta_{2\delta+2})} \Psi(\mu)^{d\mu}\Big) &= \overset{\circ}{\tau} \times F\Big(\int_0^{\mathfrak{n}_\mathfrak{p}(\wp_1\alpha_{2\delta},\wp_2\alpha_{2\delta+1})} \Psi(\mu)^{d\mu}\Big) \\ &\le F\Big(\int_0^{L(\alpha_{2\delta},\alpha_{2\delta+1})} \Psi(\mu)^{d\mu}\Big).\end{aligned} \tag{4.1}$$

Also

$$\begin{aligned} L(\alpha_{2\delta}, \alpha_{2\delta+1}) &= max \left\{ \begin{array}{c} \mathfrak{n}_{\mathfrak{p}}(\wp_3\alpha_{2\delta}, \wp_1\alpha_{2\delta+1}), \\ \mathfrak{n}_{\mathfrak{p}}(\wp_4\alpha_{2\delta}, \wp_2\alpha_{2\delta}), \mathfrak{n}_{\mathfrak{p}}(\wp_3\alpha_{2\delta+1}, \wp_4\alpha_{2\delta+1}), \\ \sqrt{\mathfrak{n}_{\mathfrak{p}}(\wp_4\alpha_{2\delta}, \wp_1\alpha_{2\delta+2}) \times \mathfrak{n}_{\mathfrak{p}}(\wp_3\alpha_{2\delta}, \wp_2\alpha_{2\delta+1})} \end{array} \right\} \\ &= max \left\{ \begin{array}{c} \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta}, \beta_{2\delta+1}), \\ \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+2}), \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta}, \beta_{2\delta+1}), \\ \sqrt{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+1}) \times \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta}, \beta_{2\delta+2})} \end{array} \right\} \\ &\leq max \left\{ \begin{array}{c} \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta}, \beta_{2\delta+1}), \\ \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+2}), \\ \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+1}), \\ \sqrt{\frac{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+1}) \times \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta}, \beta_{2\delta+1}) \times \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+2})}{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+1})}}. \end{array} \right\} \\ &\leq max \left\{ \ \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta}, \beta_{2\delta+1}), \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+2})) \ \right\}. \end{aligned}$$

If the maximum value of $L(\alpha_{2\delta}, \alpha_{2\delta+1})$ is $\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+2})$, then from (4.1)

$$\mathring{\tau} \times F\left(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+2})} \Psi(\mu)^{d\mu} \right) \leq F\left(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+2})} \Psi(\mu)^{d\mu} \right), \tag{4.2}$$

as it is given that $\mathring{\tau} > 1$, then

$$max \left\{ \ \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta}, \beta_{2\delta+1}), \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+2}) \ \right\} = \mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta}, \beta_{2\delta+1}).$$

Now, by equation (4.2)

$$F\left(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1}, \beta_{2\delta+2})} \Psi(\mu)^{d\mu} \right) \leq (\mathring{\tau})^{-1} \left(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta}, \beta_{2\delta+1})} \Psi(\mu)^{d\mu} \right). \tag{4.3}$$

Inductively, we have

$$F\left(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta}, \beta_{2\delta+1})} \Psi(\mu)^{d\mu} \right) \leq (\mathring{\tau})^{-1} \left(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta-1}, \beta_{2\delta})} \Psi(\mu)^{d\mu} \right). \tag{4.4}$$

By using equations (4.3) and (4.4), we have

$$\begin{aligned}
F\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta+1},\beta_{2\delta+2})}\Psi(\mu)^{d\mu}\Big) &\leq (\mathring{\tau})^{-1}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta},\beta_{2\delta+1})}\Psi(\mu)^{d\mu}\Big)\\
&\leq (\mathring{\tau})^{-2}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta-1},\beta_{2\delta})}\Psi(\mu)^{d\mu}\Big)\\
&\leq (\mathring{\tau})^{-3}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta-2},\beta_{2\delta-1})}\Psi(\mu)^{d\mu}\Big)\\
&\leq (\mathring{\tau})^{-4}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta-3},\beta_{2\delta-2})}\Psi(\mu)^{d\mu}\Big)\\
&\leq (\mathring{\tau})^{-5}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta-4},\beta_{2\delta-3})}\Psi(\mu)^{d\mu}\Big)\\
&\leq (\mathring{\tau})^{-6}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta-5},\beta_{2\delta-4})}\Psi(\mu)^{d\mu}\Big)\\
&\leq \cdot\ \ \cdot\ \cdot\\
&\leq (\mathring{\tau})^{-(2\delta)}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_1,\beta_2)}\Psi(\mu)^{d\mu}\Big)\\
&\leq (\mathring{\tau})^{-(2\delta+1)}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_0,\beta_1)}\Psi(\mu)^{d\mu}\Big).
\end{aligned}\tag{4.5}$$

Also,

$$\begin{aligned}
F\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta},\beta_{2\delta+1})}\Psi(\mu)^{d\mu}\Big) &\leq (\mathring{\tau})^{-1}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta-1},\beta_{2\delta})}\Psi(\mu)^{d\mu}\Big)\\
&\leq (\mathring{\tau})^{-2}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta-2},\beta_{2\delta-1})}\Psi(\mu)^{d\mu}\Big)\\
&\leq (\mathring{\tau})^{-3}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta-3},\beta_{2\delta-2})}\Psi(\mu)^{d\mu}\Big)\\
&\leq (\mathring{\tau})^{-4}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_{2\delta-4},\beta_{2\delta-3})}\Psi(\mu)^{d\mu}\Big)\\
&\leq \cdot\ \ \cdot\ \cdot\\
&\leq (\mathring{\tau})^{-2\delta}\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_0,\beta_1)}\Psi(\mu)^{d\mu}\Big).
\end{aligned}\tag{4.6}$$

It follows that, $\lim_{\delta\to\infty} F\Big(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\beta_\delta,\beta_{\delta+1})}\Psi(\mu)^{d\mu}\Big) = -\infty$, we have

$$\lim_{\delta\to\infty}\mathfrak{n}_{\mathfrak{p}}(\beta_\delta,\beta_{\delta+1}) = 1 \tag{4.7}$$

Step II. To prove that the sequence β_δ is $\mathfrak{n}_{\mathfrak{p}}-$Cauchy sequence, conditions (b) and (c) of Definition 4.1 are very useful for which $\exists\ \kappa\in(0,1)$ such that

$$\lim_{\delta\to\infty}(\mathfrak{n}_{\mathfrak{p}}(\beta_\delta,\beta_{\delta+1}))^{\kappa}F(\mathfrak{n}_{\mathfrak{p}}(\beta_\delta,\beta_{\delta+1})) = 1. \tag{4.8}$$

Now, on using (4.5) and (4.6), we have

$$\left[\frac{F\left(\int_0^{\mathfrak{n}_\mathfrak{p}(\beta_{2\delta+1},\beta_{2\delta+2})}\Psi(\mu)^{d\mu}\right)}{F\left(\int_0^{\mathfrak{n}_\mathfrak{p}(\beta_0,\beta_1)}\Psi(\mu)^{d\mu}\right)}\right]^{(\mathfrak{n}_\mathfrak{p}(\beta_{2\delta+1},\beta_{2\delta+2}))^\kappa} \leq \left[(\mathring{\tau})^{-(2\delta+1)}\right]^{(\mathfrak{n}_\mathfrak{p}(\beta_{2\delta+1},\beta_{2\delta+2}))^\kappa} \leq 1. \quad (4.9)$$

Also,

$$\left[\frac{F\left(\int_0^{\mathfrak{n}_\mathfrak{p}(\beta_{2\delta},\beta_{2\delta+1})}\Psi(\mu)^{d\mu}\right)}{F\left(\int_0^{\mathfrak{n}_\mathfrak{p}(\beta_0,\beta_1)}\Psi(\mu)^{d\mu}\right)}\right]^{(\mathfrak{n}_\mathfrak{p}(\beta_{2\delta},\beta_{2\delta+1}))^\kappa} \leq \left[(\mathring{\tau})^{-(2\delta)}\right]^{(\mathfrak{n}_\mathfrak{p}(\beta_{2\delta},\beta_{2\delta+1}))^\kappa} \leq 1. \quad (4.10)$$

Using inequality (4.8) together with inequalities (4.9) and (4.10), we have

$$\lim_{\delta\to\infty} \delta(\mathfrak{n}_\mathfrak{p}(\beta_{2\delta},\beta_{2\delta+1}))^\kappa = 1.$$

Hence, $\exists\ \delta_1 \in \mathbb{N}$ such that $\delta(\mathfrak{n}_\mathfrak{p}(\beta_{2\delta},\beta_{2\delta+1}))^\kappa < 1\ \forall\ \delta > \delta_1$, otherwise

$$\mathfrak{n}_\mathfrak{p}(\beta_{2\delta},\beta_{2\delta+1}) < \frac{1}{\delta^{\frac{1}{\kappa}}}.$$

Let $\delta_2, \delta \in \mathbb{N}$ with the condition $\delta_2 > \delta > \delta_1$ and using condition (d) of non-Newtonian partial metric, we have

$$\begin{aligned}
\mathfrak{n}_\mathfrak{p}(\beta_\delta,\beta_{\delta_2}) &= \frac{\mathfrak{n}_\mathfrak{p}(\beta_\delta,\beta_{\delta+1})\mathfrak{n}_\mathfrak{p}(\beta_{\delta+1},\beta_{\delta+2}).\ .\ .\mathfrak{n}_\mathfrak{p}(\beta_{\delta_2-1},\beta_{\delta_2})}{[\mathfrak{n}_\mathfrak{p}(\beta_{\delta_1+1},\beta_{\delta_1+1})\mathfrak{n}_\mathfrak{p}(\beta_{\delta+2},\beta_{\delta+2}).\ .\ .\mathfrak{n}_\mathfrak{p}(\beta_{\delta_2-1},\beta_{\delta_2-1})]} \\
&\leq \mathfrak{n}_\mathfrak{p}(\beta_\delta,\beta_{\delta+1})\mathfrak{n}_\mathfrak{p}(\beta_{\delta+1},\beta_{\delta+2}).\ .\ .\mathfrak{n}_\mathfrak{p}(\beta_{\delta_2-1},\beta_{\delta_2}) \\
&= \prod_{i=\delta}^{\delta_2-1} \mathfrak{n}_\mathfrak{p}(\beta_i,\beta_{i+1}) \\
&= \prod_{i=\delta}^{\infty} \mathfrak{n}_\mathfrak{p}(\beta_i,\beta_{i+1}) \\
&= \prod_{i=\delta}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}.
\end{aligned}$$

Also, it is given that $\kappa \in (0,1)$, the series $\prod_{i=\delta}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ is convergent, so

$$\lim_{\delta,\delta_2\to\infty} \mathfrak{n}_\mathfrak{p}(\beta_\delta,\beta_{\delta_2}) = 1,$$

which confirms that the sequence $\{\beta_\delta\}$ is Cauchy in $(N, \mathfrak{n}_{\mathfrak{p}_\mathfrak{q}})$. Also the completeness of $(N, \mathfrak{n}_\mathfrak{p})$ confirms the completeness of $(N, \mathfrak{n}_{\mathfrak{p}_\mathfrak{q}})$. Let there exist a point $\mathring{u} \in N$ such that $\lim_{\delta\to\infty} \mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\beta_\delta, \mathring{u}) = 1$. Furthermore

$$\mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\mathring{u},\mathring{u}) = \lim_{\delta\to\infty} \mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\beta_\delta,\mathring{u}) = \lim_{\delta,\delta_2\to\infty} \mathfrak{n}_{\mathfrak{p}_\mathfrak{q}}(\beta_\delta,\beta_{\delta_2}) = 1.$$

As the sequence $\{\beta_\delta\}$ is convergent to $\mathring{u}$, hence $\wp_1^{2\delta}$, $\wp_2^{2\delta+1}$, $\wp_3^{2\delta+2}$, $\wp_4^{2\delta+1}$ are also convergent to $\mathring{u}$.

Step III. Now, it is to prove that $\wp_1$, $\wp_2$, $\wp_3$, and $\wp_4$ have a coincidence point. Let $\wp_4(N)$ is closed, then there exists a point $\mathring{v} \in N$ such that $\wp_4\mathring{v} = \mathring{u}$. To show that $\wp_2\mathring{v} = \mathring{u}$, we have

$$\mathring{\tau} \times F\bigg(\int_0^{\mathfrak{n}_\mathfrak{p}(\wp_1\alpha_{2\delta},\wp_2\mathring{v})} \Psi(\mu)^{d\mu}\bigg) \leq F\bigg(\int_0^{L(\alpha_{2\delta},\mathring{v})} \Psi(\mu)^{d\mu}\bigg) \tag{4.11}$$

where

$$L(\alpha_{2\delta},\mathring{v}) = max\left\{\begin{array}{c}\mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_1\alpha_{2\delta}),\\ \mathfrak{n}_\mathfrak{p}(\wp_4\mathring{v},\wp_3\mathring{v}),\ \mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_4\mathring{v}),\\ \sqrt{\mathfrak{n}_\mathfrak{p}(\wp_4\mathring{v},\wp_1\alpha_{2\delta}) \times \mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_2\mathring{v})}\end{array}\right\} \tag{4.12}$$

$$L(\alpha_{2\delta},\mathring{v}) = max\left\{\begin{array}{c}\mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_1\alpha_{2\delta}),\\ \mathfrak{n}_\mathfrak{p}(\mathring{u},\wp_3\mathring{v}),\ \mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\mathring{u}),\\ \sqrt{\mathfrak{n}_\mathfrak{p}(\mathring{u},\wp_1\alpha_{2\delta}) \times \mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_2\mathring{v})}\end{array}\right\} \tag{4.13}$$

For $\delta \to \infty$,

$$\mathring{\tau} \times F\bigg(\int_0^{\mathfrak{n}_\mathfrak{p}(\mathring{u},\wp_2\mathring{v})} \Psi(\mu)^{d\mu}\bigg) \leq F\bigg(\int_0^{\mathfrak{n}_\mathfrak{p}(\mathring{u},\wp_2\mathring{v})} \Psi(\mu)^{d\mu}\bigg) \tag{4.14}$$

which is a contradiction with $\mathring{\tau} > 1$, hence $\wp_2\mathring{v} = \mathring{u}$ and hence $\wp_2\mathring{u} = \wp_2\wp_4\mathring{v} = \wp_4\wp_2\mathring{v} = \mathring{u}$. Now, it is to show that $\wp_2\mathring{u} = \mathring{u}$.

$$\mathring{\tau} \times F\bigg(\int_0^{\mathfrak{n}_\mathfrak{p}(\wp_1\alpha_{2\delta},\wp_2\mathring{u})} \Psi(\mu)^{d\mu}\bigg) \leq F\bigg(\int_0^{L(\alpha_{2\delta},\mathring{u})} \Psi(\mu)^{d\mu}\bigg) \tag{N_{I_1}}$$

where

$$L(\alpha_{2\delta},\mathring{u}) = max\left\{\begin{array}{c}\mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_1\alpha_{2\delta}),\\ \mathfrak{n}_\mathfrak{p}(\wp_4\mathring{u},\wp_2\mathring{u}),\ \mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_4\mathring{u}),\\ \sqrt{\mathfrak{n}_\mathfrak{p}(\wp_4\mathring{u},\wp_1\alpha_{2\alpha}) \times \mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_2\mathring{u})}\end{array}\right\} \tag{4.15}$$

$$L(\alpha_{2\delta},\mathring{u}) = max\left\{\begin{array}{c}\mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_1\alpha_{2\delta}),\\ \mathfrak{n}_\mathfrak{p}(\wp_2\mathring{u},\wp_2\mathring{u}),\ \mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_2\mathring{u}),\\ \sqrt{\mathfrak{n}_\mathfrak{p}(\wp_2\mathring{u},\wp_1\alpha_{2\alpha}) \times \mathfrak{n}_\mathfrak{p}(\wp_3\alpha_{2\delta},\wp_2\mathring{u})}\end{array}\right\} \tag{4.16}$$

Since F is continuous, for $\delta \to \infty$

$$\mathring{\tau} \times F\bigg(\int_0^{\mathfrak{n}_\mathfrak{p}(\mathring{u},\wp_2\mathring{u})} \Psi(\mu)^{d\mu}\bigg) \leq F\bigg(\int_0^{\mathfrak{n}_\mathfrak{p}(\mathring{u},\wp_2\mathring{u})} \Psi(\mu)^{d\mu}\bigg), \tag{4.17}$$

which gives a contradiction, hence $\mathfrak{n}_\mathfrak{p}(\mathring{u}), \wp_2\mathring{u}) = 1$ (i.e., $\mathring{u}$ is a fixed point of $\wp_2$ and $\wp_4$.) Now, it is to prove that $\mathring{u}$ is a fixed point of $\wp_1$ and $\wp_3$. It given that $\wp_2(N) \subseteq \wp_3(N)$, $\exists$ $\mathring{w} \in N$ such that $\wp_2\mathring{u} = \wp_3\mathring{u}$. Consider $\wp_1\mathring{w} \neq \wp_3\mathring{w}$, results

$$\mathring{\tau} \times F\bigg(\int_0^{\mathfrak{n}_\mathfrak{p}(\wp_1\mathring{w},\wp_2\mathring{u})} \Psi(\mu)^{d\mu}\bigg) \leq F\bigg(\int_0^{L(\mathring{w},\mathring{u})} \Psi(\mu)^{d\mu}\bigg) \tag{4.18}$$

where

$$L(\mathring{w},\mathring{u}) = max\left\{\begin{array}{c}\mathfrak{n}_{\mathfrak{p}}(\wp_3\mathring{w},\wp_1\mathring{w}),\ \mathfrak{n}_{\mathfrak{p}}(\wp_4\mathring{u},\wp_2\mathring{u}),\ \mathfrak{n}_{\mathfrak{p}}(\wp_3\mathring{w},\wp_4\mathring{u}),\\ \sqrt{\mathfrak{n}_{\mathfrak{p}}(\wp_4\mathring{u},\wp_1\mathring{w})\times\mathfrak{n}_{\mathfrak{p}}(\wp_3\mathring{w},\wp_2\mathring{u})}\end{array}\right\} \tag{4.19}$$

which is a contradiction. Hence $\wp_1\mathring{w} = \wp_2\mathring{u} = \wp_3\mathring{w}$. Because of weak compatibility of $\wp_1$ and $\wp_3$, $\wp_3\wp_1\mathring{u} = \wp_1\wp_3\mathring{w} = \wp_1\mathring{u}$. Now, it is to prove that $\wp_1\mathring{u} = \mathring{u}$. Using ($N_{I_1}$)

$$L(\mathring{u},\mathring{u}) = max\left\{\begin{array}{c}\mathfrak{n}_{\mathfrak{p}}(\wp_3\mathring{u},\wp_1\mathring{u}),\ \mathfrak{n}_{\mathfrak{p}}(\wp_4\mathring{u},\wp_2\mathring{u}),\ \mathfrak{n}_{\mathfrak{p}}(\wp_3\mathring{u},\wp_4\mathring{u}),\\ \sqrt{\mathfrak{n}_{\mathfrak{p}}(\wp_4\mathring{u},\wp_1\mathring{u})\times\mathfrak{n}_{\mathfrak{p}}(\wp_3\mathring{u},\wp_2\mathring{u})}\end{array}\right\} \tag{4.20}$$

$$L(\mathring{u},\mathring{u}) = max\left\{\begin{array}{c}\mathfrak{n}_{\mathfrak{p}}(\wp_1\mathring{u},\wp_1\mathring{u}),\ \mathfrak{n}_{\mathfrak{p}}(\mathring{u},\mathring{u}),\ \mathfrak{n}_{\mathfrak{p}}(\wp_1\mathring{u},\mathring{u}),\\ \sqrt{\mathfrak{n}_{\mathfrak{p}}(\mathring{u},\wp_1\mathring{u})\times\mathfrak{n}_{\mathfrak{p}}(\wp_1\mathring{u},\mathring{u})}\end{array}\right\} \tag{4.21}$$

hence we have that $L(\mathring{u},\mathring{u}) = \mathfrak{n}_{\mathfrak{p}}(\wp_1\mathring{u},\mathring{u})$. Also,

$$\mathring{\tau}\times F\bigg(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\wp_1\mathring{u},\mathring{u})}\Psi(\mu)^{d\mu}\bigg) \leq F\bigg(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\wp_1\mathring{u},\mathring{u})}\Psi(\mu)^{d\mu}\bigg) \tag{4.22}$$

which proves that $\wp_1\mathring{u} = \wp_3\mathring{u} = \mathring{u}$, and hence $\mathring{u}$ is a fixed point which is common to $\wp_1$, $\wp_2$, $\wp_3$, and $\wp_4$. To prove the uniqueness of $\mathring{u}$ let $\mathring{s}$ is another common fixed point of $\wp_1$, $\wp_2$, $\wp_3$, and $\wp_4$, then

$$\mathring{\tau}\times F\bigg(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\mathring{u},\mathring{s})}\Psi(\mu)^{d\mu}\bigg) = \mathring{\tau}\times F\bigg(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\wp_1\mathring{u},\wp_1\mathring{s})}\Psi(\mu)^{d\mu}\bigg) \leq F\bigg(\int_0^{L(\mathring{u},\mathring{s})}\Psi(\mu)^{d\mu}\bigg) \tag{4.23}$$

where $L(\alpha,\beta)$ is

$$L(\mathring{u},\mathring{s}) = max\left\{\begin{array}{c}\mathfrak{n}_{\mathfrak{p}}(\wp_3\mathring{u},\wp_1\mathring{u}),\mathfrak{n}_{\mathfrak{p}}(\wp_4\mathring{s},\wp_2\mathring{s}),\ \mathfrak{n}_{\mathfrak{p}}(\wp_3\mathring{u},\wp_4\mathring{s}),\\ \sqrt{\mathfrak{n}_{\mathfrak{p}}(\wp_4\mathring{s},\wp_1\mathring{u})\times\mathfrak{n}_{\mathfrak{p}}(\wp_3\mathring{u},\wp_2\mathring{s})}\end{array}\right\} \tag{4.24}$$

$$L(\mathring{u},\mathring{s}) = max\left\{\begin{array}{c}\mathfrak{n}_{\mathfrak{p}}(\mathring{u},\mathring{u}),\mathfrak{n}_{\mathfrak{p}}(\mathring{s},\mathring{s}),\ \mathfrak{n}_{\mathfrak{p}}(\mathring{u},\mathring{s}),\\ \sqrt{\mathfrak{n}_{\mathfrak{p}}(\mathring{s},\mathring{u})\times\mathfrak{n}_{\mathfrak{p}}(\mathring{u},\mathring{s})}\end{array}\right\}, \tag{4.25}$$

which gives

$$\mathring{\tau}\times F\bigg(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\mathring{u},\mathring{s})}\Psi(\mu)^{d\mu}\bigg) \leq F\bigg(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\mathring{u},\mathring{s})}\Psi(\mu)^{d\mu}\bigg) \tag{4.26}$$

which gives a contradiction and hence $\mathring{u} = \mathring{s}$. □

Corollary 4.1. *Let $(N,\mathfrak{n}_{\mathfrak{p}})$ is non-Newtonian partial metric space and $\wp_1,\ \wp_2,\ \wp_3,\ \wp_4 : N \to N$ satisfying $\wp_1(N) \subseteq \wp_4(N)$ and $\wp_2(N) \subseteq \wp_3(N)$. Let $F \in \mathfrak{F}$ and $\mathring{\tau}$ such that $\forall\, \alpha,\beta \in N$ satisfying $\mathfrak{n}_{\mathfrak{p}}(\wp_1\alpha,\wp_2\beta) > 1$*

$$\mathring{\tau}\times F\bigg(\int_0^{\mathfrak{n}_{\mathfrak{p}}(\wp_1\alpha,\wp_2\beta)}\Psi(\mu)^{d\mu}\bigg) \leq F\bigg(\int_0^{L(\alpha,\beta)}\Psi(\mu)^{d\mu}\bigg) \tag{4.27}$$

where $L(\alpha,\beta)$ is

$$L(\alpha,\beta) = max\left\{\begin{array}{c}\mathfrak{n}_{\mathfrak{p}}(\alpha,\beta),\mathfrak{n}_{\mathfrak{p}}(\wp_1\alpha,\alpha),\ \mathfrak{n}_{\mathfrak{p}}(\wp_2\beta,\beta),\\ \sqrt{\mathfrak{n}_{\mathfrak{p}}(\alpha,\wp_2\beta)\times\mathfrak{n}_{\mathfrak{p}}(\wp_1\alpha,\beta)}\end{array}\right\} \tag{4.28}$$

and $\Psi : [0, \infty) \to [0, \infty)$ *is a Lebesgue integrable mapping which is summable, nonnegative and* $\forall \overset{\circ}{\mu} > 1$

$$\int_0^{\overset{\circ}{\mu}} \Psi(\mu)^{d\mu} > 1. \tag{4.29}$$

If the following conditions

(a) $\wp_1(N)$, *or* $\wp_2(N)$ *are closed,*
(b) F *is continuous,*

hold, then the pair $\wp_1$ *and* $\wp_2$ *has a unique common fixed point.*

Corollary 4.2. *Let* $(N, \mathfrak{n}_\mathfrak{p})$ *is non-Newtonian partial metric space and* $\wp_1, \wp_2, \wp_3, \wp_4 : N \to N$ *satisfying* $\wp_1(N) \subseteq \wp_4(N)$ *and* $\wp_2(N) \subseteq \wp_3(N)$. *Let* $F \in \mathfrak{F}$ *and* $\overset{\circ}{\tau}$ *such that* $\forall \alpha, \beta \in N$ *satisfying* $\mathfrak{n}_\mathfrak{p}(\wp_1\alpha, \wp_2\beta) > 1$

$$\overset{\circ}{\tau} \times F\left(\int_0^{\mathfrak{n}_\mathfrak{p}(\wp_1\alpha, \wp_2\beta)} \Psi(\mu)^{d\mu}\right) \leq F\left(\int_0^{\mathfrak{n}_\mathfrak{p}(\alpha,\beta)} \Psi(\mu)^{d\mu}\right) \tag{4.30}$$

and $\Psi : [0, \infty) \to [0, \infty)$ *is a Lebesgue integrable mapping which is summable, nonnegative and* $\forall \overset{\circ}{\mu} > 1$

$$\int_0^{\overset{\circ}{\mu}} \Psi(\mu)^{d\mu} > 1, \tag{4.31}$$

If the following condition

(a) $\wp_1(N)$, *or* $\wp_2(N)$ *is closed,*
(b) F *is continuous,*

holds, then the pair $\wp_1$ *and* $\wp_2$ *has a unique common fixed point.*

5. Application: Multiplicative Error Reduction Models

In many numerical and applied settings, it is more natural to track how large the error is "relatively" (in percentage or multiplicative terms) rather than in absolute size. This happens, for example, in growth processes, iterative numerical methods, and systems where quantities evolve by scaling rather than by addition. In such cases, the usual (additive) metric structures are not always the most appropriate tools, because they quantify absolute differences. To illustrate how the proposed framework addresses this, consider the space $N = \mathbb{R}^+$ endowed with the non-Newtonian partial metric

$$\mathfrak{n}_\mathfrak{p}(\alpha, \beta) = \max\left\{\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right\}.$$

This quantity measures the relative discrepancy between two positive numbers: it is equal to 1 if and only if $\alpha = \beta$, and it increases as the two values differ multiplicatively from each other. Hence, it is well suited to settings where relative error is the key quantity. Now define an iterative scheme by

$$\wp(\alpha) = \alpha^\theta, \quad \theta \in (0, 1).$$

For any $\alpha, \beta \in N$, we have

$$\mathfrak{n}_{\mathfrak{p}}(\wp(\alpha), \wp(\beta)) = \max\left\{\left(\frac{\alpha}{\beta}\right)^{\theta}, \left(\frac{\beta}{\alpha}\right)^{\theta}\right\} = \left(\mathfrak{n}_{\mathfrak{p}}(\alpha, \beta)\right)^{\theta}.$$

Thus, the map $\wp$ satisfies the non-Newtonian contraction condition

$$\mathfrak{n}_{\mathfrak{p}}(\wp(\alpha), \wp(\beta)) \leq \left(\mathfrak{n}_{\mathfrak{p}}(\alpha, \beta)\right)^{\xi},$$

with $\xi = \theta \in (0, 1)$. By Theorem 2.1, this ensures that the iterative process generated by $\wp$ converges to a unique fixed point.

This application shows that the non-Newtonian partial metric framework naturally captures convergence driven by multiplicative error reduction. Such behavior is not adequately modeled by classical additive metrics, highlighting the practical usefulness of the developed theory in applications where relative, rather than absolute, errors govern the dynamics.

6. Conclusion

Inspired by two well-known concepts, the non-Newtonian calculus of Bashirov et al. [3] and the theory of partial metric spaces introduced by Matthews [7], we introduce in this work the notion of *non-Newtonian partial metric spaces*. Furthermore, we investigate integral-type F-contraction mappings within the framework of non-Newtonian partial metric spaces. Extending the partial metric version of the celebrated Banach fixed point theorem established by Matthews, we generalize his results to this new setting. Our findings are mathematically distinct from the existing results in the literature and, in particular, extend and complement several known fixed point theorems. In future work, we aim to investigate Nadler-type theorems for fixed point results of single-valued and multivalued mappings in non-Newtonian partial metric spaces, which cannot be derived from the corresponding results in classical metric spaces. Within this framework, we will analyze families of contractive mappings to identify novel applications. Furthermore, we plan to weaken existing contractive conditions and to develop more general formulations by employing admissible mappings, auxiliary functions, and suitable algebraic structures in the context of non-Newtonian partial metric spaces. The proposed framework provides a natural analytical context for studying problems governed by multiplicative dynamics and relative error structures.

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