Turkish J. Ineq., 10 (1) (2026), Pages 91 –97 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

ASYMPTOTIC INEQUALITIES FOR THE FUNCTION $x \mapsto (1+\frac{a}{x})^x + \left(1-\frac{b}{x}\right)^{-x}$ USING THE EXPONENTIAL AND RATIONAL FUNCTIONS

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Abstract. For any $a, b \in \mathbb{R}$ and $x > 1$ some good approximations to the function $x \mapsto (1+\frac{a}{x})^x + (1-\frac{b}{x})^{-x}$ are presented.

1. Introduction

The function $(1+\frac{1}{x})^x$ has been studied by many authors, see e.g. [3–10,12–14]. This function appeared also recently by authors M. Aoki, Y. Nishizawa, R. Suzuki and G. Takamori. In their contribution [1, Theorem 1], the authors presented the double inequality

$$T(e,x) := \frac{e(2x^e-1)}{x^e-1} < A(1,1,x) < \frac{e(2x^2-1)}{x^2-1} =: T(2,x)\,, \qquad \text{for} \quad x>1, \tag{1.1}$$

where

$$A(a,b,x) := \left(1+\frac{a}{x}\right)^x + \left(1-\frac{b}{x}\right)^{-x}. \tag{1.2}$$

Figure 1 illustrates[1] the inequalities (1.1) by showing the graphs of the functions in question. The derivation of these relations was not so simple and the obtained bounds are not very close to the approximated function $A(1,1,x)$. These facts irritate us to derive more direct, more accurate and more general approximations.

Key words and phrases. Approximation, estimate, expansion, exponential function, inequality.
2010 *Mathematics Subject Classification.* Primary: 41A20, 41A60, 41A80. Secondary: 26D07, 33B10.
Received: 25/11/2025 *Accepted*: 19/05/2026.
Cite this article as: V. Lampret, Asymptotic inequalities for the function $x \mapsto (1+a/x)^x + (1-b/x)^{-x}$ using the exponential and rational functions, Turkish Journal of Inequalities, 10(1) (2026), 91-97.

[1]All graphics and several computations in this paper are made using the computer system [11].

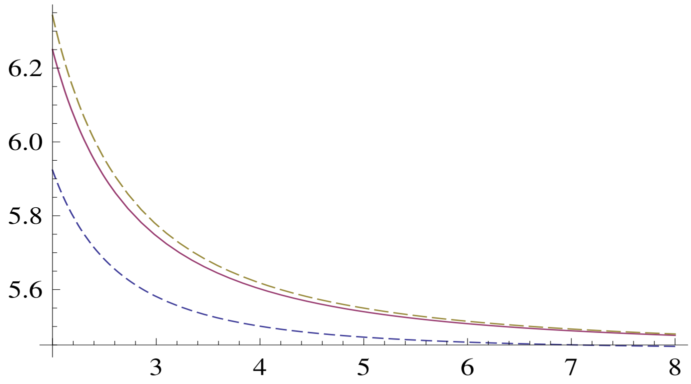

FIGURE 1. The graphs of the functions $T(e,x)$, $A(1,1,x)$ and $T(2,x)$.

2. Background

2.1. **Simple rational approximations to the exponential function.** In 2023, Chris Boucher [2] introduced a very simple approximation to the exponential function. Starting from the elementary approximation $e^x \approx 1+x$, he proceeded as follows:

$$e^x = \left(e^{x/(2n)}e^{x/(2n)}\right)^n = \left(\frac{e^{x/(2n)}}{e^{-x/(2n)}}\right)^n \approx \left(\frac{1+\frac{x}{2n}}{1-\frac{x}{2n}}\right)^n = \left(\frac{2n+x}{2n-x}\right)^n .$$

Lemma 2.1. *For any $c \in \mathbb{R}^+$ and any integer $m \geq 1$, we have*

$$\sum_{j=0}^{m} \frac{x^j}{j!} < e^x < \left(\frac{2c+x}{2c-x}\right)^c, \qquad \text{if} \quad 0 < x < 2c, \tag{2.1}$$

$$\left(\frac{2c+x}{2c-x}\right)^c < e^x < \left(\sum_{j=0}^{m} \frac{|x|^j}{j!}\right)^{-1}, \qquad \text{if} \quad -2c < x < 0. \tag{2.2}$$

Proof. For the function $f(c,x) := \left(\frac{2c+x}{2c-x}\right)^c e^{-x}$, we have $\frac{\partial f}{\partial x}(c,x) = \frac{x^2}{(2c-x)(2c+x)} f(c,x) > 0$, for $2c > x > 0$. Thus, $f(c,x) \geq f(c,0) = 1$, for $0 \leq x < 2c$, i.e. the statement (2.1) is verified considering Taylor's formula of order m. The statement (2.2) is a direct consequence of (2.1). $\square$

2.2. **An approximation of the function $x \mapsto \ln(1+\frac{y}{x})$.** According to the well-known expansion

$$\ln(1-t) = -\sum_{k=1}^{\infty} \frac{t^k}{k} \qquad (-1 \leq t < 1),$$

we have

$$\ln\left(\frac{1+t}{1-t}\right) = \ln(1+t) - \ln(1-t) = -\sum_{k=1}^{\infty} \frac{(-t)^k}{k} + \sum_{k=1}^{\infty} \frac{t^k}{k} = 2\sum_{j=1}^{\infty} \frac{t^{2j-1}}{2j-1} \qquad (|t|<1),$$

where, for an integer $m \geq 1$, we have

$$\sum_{j=m+1}^{\infty} \frac{|t|^{2j-1}}{2j-1} \leq \frac{|t|^{2m+1}}{2m+1} \sum_{i=0}^{\infty} (t^2)^i = \frac{|t|^{2m+1}}{(2m+1)(1-t^2)} \qquad (|t|<1).$$

Therefore, for an integer $m \geq 1$, we estimate

$$2\sum_{j=1}^{m} \frac{t^{2j-1}}{2j-1} < \ln\left(\frac{1+t}{1-t}\right) < 2\sum_{j=1}^{m} \frac{t^{2j-1}}{2j-1} + \frac{2t^{2m+1}}{(2m+1)(1-t^2)} \qquad (0<t<1), \tag{2.3}$$

$$-2\sum_{j=1}^{m} \frac{|t|^{2j-1}}{2j-1} - \frac{2|t|^{2m+1}}{(2m+1)(1-t^2)} < \ln\left(\frac{1+t}{1-t}\right) < -2\sum_{j=1}^{m} \frac{|t|^{2j-1}}{2j-1} \qquad (-1<t<0). \tag{2.4}$$

Now, for numbers $x > a > 0$, $q := \frac{a}{x} \in (0,1)$, $-q := -\frac{a}{x} \in (-1,0)$, $t := \frac{q}{2+q} = \frac{a}{2x+a} \in (0,1)$, and $\tau := -\frac{q}{2-q} = -\frac{a}{2x-a} \in (-1,0)$ we obtain, by (2.3) and (2.4), the following lemma.

Lemma 2.2. *For any numbers $x > a > 0$ and any integers $m, n \geq 1$, we have*

$$S(a,m,x) < \ln\left(1+\frac{a}{x}\right) < S(a,n,x) + R(a,n,x)\,, \tag{2.5}$$

$$-S(-a,n,x) - R(-a,n,x) < \ln\left(1-\frac{a}{x}\right) < -S(-a,m,x)\,, \tag{2.6}$$

with

$$S(y,m,x) := \sum_{j=1}^{m} \tfrac{2}{2j-1}\left(\tfrac{|y|}{2x+y}\right)^{2j-1}, \quad R(y,m,x) := \tfrac{(2x+y)^2}{(4m+2)x(x+y)}\left(\tfrac{|y|}{2x+y}\right)^{2m+1}. \tag{2.7}$$

Setting $m = 2$ and $n = 1$ in Lemma 2.2, we get simple asymptotic estimates, presented in the following example.

Example 2.1. The inequalities

$$\frac{2a}{2x+a}\left(1+\frac{a^2}{3(2x+a)^2}\right) < \ln\left(1+\frac{a}{x}\right) < \frac{2a}{2x+a}\left(1+\frac{a^2}{12x(x+a)}\right), \tag{2.8}$$

$$-\frac{2a}{2x-a}\left(1+\frac{a^2}{12x(x-a)}\right) < \ln\left(1-\frac{a}{x}\right) < -\frac{2a}{2x-a}\left(1+\frac{a^2}{3(2x-a)^2}\right) \tag{2.9}$$

hold for any $x > a > 0$.

2.3. **Approximating the sum of two exponential functions with nearly equal exponents.** Using the finite increment theorem (Taylor's zero-order formula), we have $e^{\pm\delta} = 1 + e^{\pm\vartheta\cdot\delta}\cdot(\pm\delta)$, for every $0 < \delta < 1$ and some $\vartheta = \vartheta(\delta) \in (0,1)$. Therefore we get

$$1+\delta < e^{\delta} < 1 + e^{\delta}\cdot\delta \qquad \text{and} \qquad 1-\delta < e^{-\delta} < 1 + e^{-\delta}(-\delta),$$

that is

$$1+\delta < e^{\delta} < \frac{1}{1-\delta} \qquad \text{and} \qquad 1-\delta < e^{-\delta} < \frac{1}{1+\delta} \qquad (0<\delta<1).$$

Consequently, the next lemma holds.

Lemma 2.3. *We have, for $a \in \mathbb{R}$, $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$, the following estimate:*

$$e^a(2 + (\delta_2 - \delta_1)) < e^{a-\delta_1} + e^{a+\delta_2} < e^a\frac{2-(\delta_2-\delta_1)}{(1+\delta_1))(1-\delta_2)}\,. \tag{2.10}$$

3. Main results – estimations of the function $A(a,b,x)$

By Lemma 2.2 we obtain the following proposition.

Proposition 3.1. *For $x>a>0$, $y>b>0$ and integers $m,n\geq 1$, we have*

$$\exp\Big(x\cdot S(a,m,x)\Big)<\Big(1+\frac{a}{x}\Big)^{x}<\exp\Big(x\cdot S(a,n,x)+x\cdot R(a,n,x)\Big), \tag{3.1}$$

$$\exp\Big(-y\cdot S(-b,n,y)-y\cdot R(-b,n,y)\Big)<\Big(1-\frac{b}{y}\Big)^{y}<\exp\Big(-y\cdot S(-b,m,y)\Big),$$

$$\exp\Big(y\,S(-b,m,y)\Big)<\Big(1-\frac{b}{y}\Big)^{-y}<\exp\Big(y\,S(-y,n,y)+y\,R(-b,n,y)\Big). \tag{3.2}$$

In this way we find, setting $m=2$ and $n=1$ in (3.1) and (3.2), and considering (2.7), the main result given in the next theorem.

Theorem 3.1. *For $a,b\in\mathbb{R}^{+}$ and $x>\max\{a,b\}>0$, we have (see (1.2))*

$$A(a,b,x)>\exp\left(\frac{2ax}{2x+a}\Big(1+\frac{a^2}{3(2x+a)^2}\Big)\right)+\exp\left(\frac{2bx}{2x-b}\Big(1+\frac{b^2}{3(2x-b)^2}\Big)\right) \tag{3.3}$$

$$A(a,b,x)<\exp\left(\frac{2ax}{2x+a}\Big(1+\frac{a^2}{12x(x+a)}\Big)\right)+\exp\left(\frac{2bx}{2x-b}\Big(1+\frac{b^2}{12x(x-b)}\Big)\right). \tag{3.4}$$

Figure 2 illustrates the inequalities (3.3) and (3.4), left for $a=2$ and $b=\frac{1}{2}$, and right for $a=\frac{1}{2}$ and $b=2$.

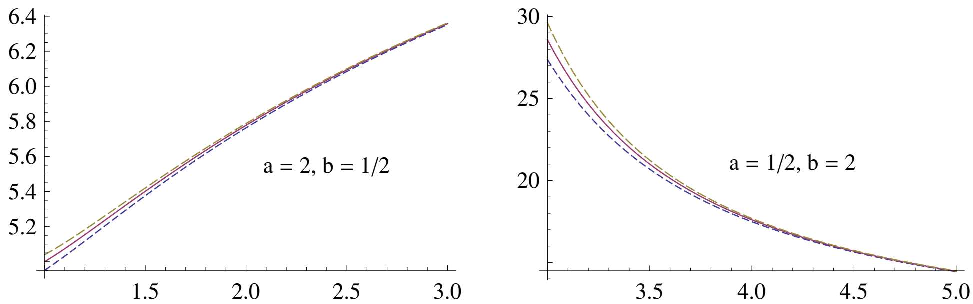

Figure 2. The graphs of the bounds (3.3)–(3.4): left for $a=2$ and $b=\frac{1}{2}$, and right for $a=\frac{1}{2}$ and $b=2$.

Example 3.1. Using Theorem 3.1, we obtain, for $x>1$, the following inequalities:

$$A(1,1,x)>\exp\left(\frac{2x}{2x+1}\Big(1+\frac{1}{12x^2+12x+3}\Big)\right)+\exp\left(\frac{2x}{2x-1}\Big(1+\frac{1}{12x^2-12x+3}\Big)\right) \tag{3.5}$$

$$A(1,1,x)<\exp\left(\frac{2x}{2x+1}\Big(1+\frac{1}{12x^2+12x}\Big)\right)\quad+\quad\exp\left(\frac{2x}{2x-1}\Big(1+\frac{1}{12x^2-12x}\Big)\right). \tag{3.6}$$

Remark 3.1. The accuracy of the approximations of the function $A(a,b,x)$ can be increased by increasing the parameters m and n in the inequalities (3.1) and (3.2).

Corollary 3.1 (rational approximation)**.** *For $x \geq 2$ we have*

$$A^*(x) < A(1,1,x) < A^{**}(x), \tag{3.7}$$

where

$$A^*(x) = 2e\left(1 + \frac{64x^4 - 12x^2 + 3}{3(4x^2-1)^3}\right) \tag{3.8}$$

and

$$A^{**}(x) = 2e\left(1 + \frac{84x^2 - 55}{(12x^2+24x+11)(12x^2-24x+11)}\right). \tag{3.9}$$

Figure 3 illustrates the estimate (3.7) by plotting (left) the graph of the function $A(1,1,x)$, together with the graphs of its lower/upper bounds $A^*(x)$ and $A^{**}(x)$, and by showing the graph of the difference $A^{**}(x) - A^*(x)$ (right).

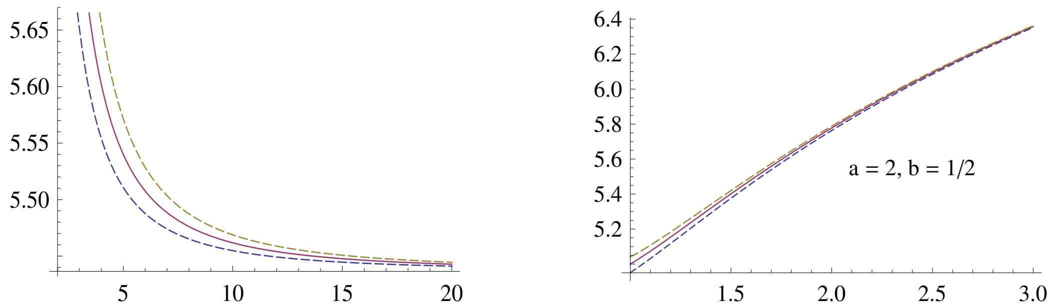

FIGURE 3. Left is illustrated the double inequality (3.7), right is the graph of the function $x \mapsto A^{**}(x) - A^*(x)$.

Proof of Corollary 3.1. According to Theorem 3.1, we have, for $x > 1$,

$$\exp\left(1 - \delta_1(x)\right) + \exp\left(1 + \delta_2(x)\right) < A(1,1,x) \tag{3.10}$$

and

$$A(1,1,x) < \exp\left(1 - \delta_3(x)\right) + \exp\left(1 + \delta_4(x)\right), \tag{3.11}$$

where the functions

$$\delta_1(x) := 1 - \frac{2x}{2x+1}\left(1 + \frac{1}{3(2x+1)^2}\right) = \frac{12x^2+10x+3}{3(2x+1)^3} > 0 \tag{3.12}$$

$$\delta_2(x) := \frac{2x}{2x-1}\left(1 + \frac{1}{3(2x-1)^2}\right) - 1 = \frac{12x^2-10x+3}{3(2x-1)^3} > 0 \tag{3.13}$$

$$\delta_3(x) := 1 - \frac{2x}{2x+1}\left(1 + \frac{1}{12x(x+1)}\right) = \frac{6x+5}{6(x+1)(2x+1)} > 0 \tag{3.14}$$

$$\delta_4(x) := \frac{2x}{2x-1}\left(1 + \frac{1}{12x(x-1)}\right) - 1 = \frac{6x-5}{6(x-1)(2x-1)} > 0 \tag{3.15}$$

are all strictly decreasing in the interval $(1,\infty)$, and $\delta_k(2)<1$, for $k\in\{1,2,3,4\}$. Thus, $0<\delta_k(x)<1$, for $k\in\{1,2,3,4\}$ and $x\ge 2$. Therefore, using (3.10)–(3.15) and Lemma 2.3, we estimate

$$A(1,1,x)>e\Big(2+\big(\delta_2(x)-\delta_1(x)\big)\Big)=2e\left(1+\frac{64x^4-12x^2+3}{3(4x^2-1)^3}\right)$$

$$A(1,1,x)<e\frac{2-\big(\delta_4(x)-\delta_3(x)\big)}{\big(1+\delta_3(x)\big)\big(1-\delta_4(x)\big)}=2e\left(1+\frac{84x^2-55}{\big(12x^2+24x+11\big)\big(12x^2-24x+11\big)}\right),$$

for $x\ge 2$. □

From Theorem 3.1, using Lemma 2.1 with $c=n$, we derive the next theorem.

Theorem 3.2 (rational approximations). *For $a,b>0$, for $x>\max\{a,b+1\}$, and for integers $m\ge 1$ and $n\ge\max\{\frac{a}{2},b(\frac{b}{6}+2)\}$, we have*

$$A(a,b,x)>\sum_{k=0}^{m}\frac{1}{k!}\Big(y_1^k(a,x)+y_2^k(b,x)\Big),\qquad \textit{if}\quad x>a>0, \tag{3.16}$$

$$A(a,b,x)<\left(\frac{2n+z_1(a,x)}{2n-z_1(a,x)}\right)^n+\left(\frac{2n+z_2(b,x)}{2n-z_2(b,x)}\right)^n,\qquad \textit{if}\quad x\ge a+1. \tag{3.17}$$

where

$$y_1(a,x):=\frac{2ax}{2x+a}\Big(1+\frac{a^2}{3(2x+a)^2}\Big)\quad \textit{and}\quad y_2(b,x):=\frac{2bx}{2x-b}\Big(1+\frac{b^2}{3(2x-b)^2}\Big) \tag{3.18}$$

$$z_1(a,x):=\frac{2ax}{2x+a}\Big(1+\frac{a^2}{12x(x+a)}\Big)\quad \textit{and}\quad z_2(b,x):=\frac{2bx}{2x-b}\Big(1+\frac{b^2}{12x(x-b)}\Big). \tag{3.19}$$

Proof. Let all conditions of Theorem 3.2 be met. Then, referring to (3.18), for $a,b>0$ and $x>\max\{a,b\}$, both $y_1(a,x)$ and $y_2(b,x)$ are positive. Thus, (3.3) and (2.1) approve (3.16). Moreover, since the function $x\mapsto z_1(a,x)$ is strictly increasing on $\mathbb{R}^+$ and the function $x\mapsto z_2(b,x)$ is strictly decreasing on (b,∞), we have $z_1(a,x)<z_1(a,\infty)=a\le 2n$ and $z_2(b,x)\le z_2(b,b+1)=\frac{b^3+12b^2+12b}{6(b+2)}<\frac{b^2}{6}+2b\le 2n$, certainly if $x\ge b+1$. Consequently, referring to (3.19), as $z_1(a,x),z_2(b,x)\in(0,2n)$, we validate the inequality (3.17), thanks to (3.4) and (2.1). □

Example 3.2. Using $m=n=4$ in Theorem 3.2, we get the following estimates:

$$A(1,1,x)>\frac{P(x)}{243(4x^2-1)^{12}},\qquad \text{for}\quad x>1, \tag{3.20}$$

where

$$\begin{aligned}P(x)=2\Big(&243-12744x^2+308528x^4-4519968x^6+44433024x^8-305576448x^{10}\\&+1513671168x^{12}-5468246016x^{14}+14401691648x^{16}-27142914048x^{18}\\&+35081551872x^{20}-28226617344x^{22}+11041505280x^{24}\Big),\end{aligned} \tag{3.21}$$

$$A(1,1,x)<\left(\frac{108x^2+156x+49}{84x^2+132x+47}\right)^4+\left(\frac{108x^2-156x+49}{84x^2-132x+47}\right)^4,\qquad \text{for}\quad x\ge 2\,.$$

Figure 4 illustrates the inequalities from Examples 3.1–3.2 by graphing the function $x\mapsto A(1,1,x)$ by a solid line and its bounds using the dashed lines.

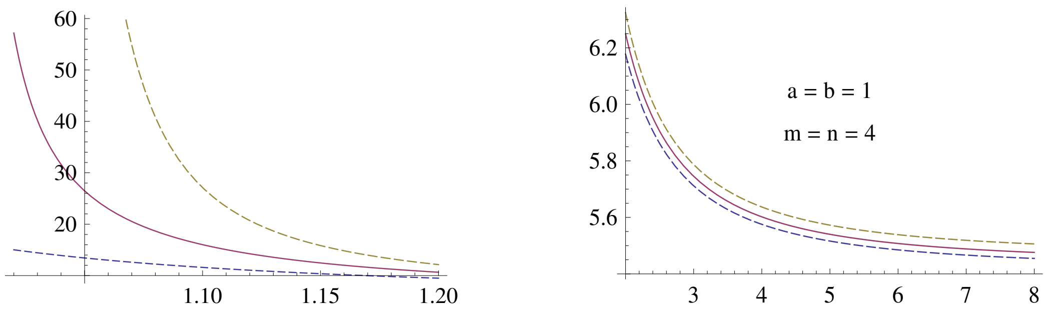

FIGURE 4. Left are the graphs of the bounds (3.5)–(3.6), right are illustrated the inequalities (3.20) and (3.21).

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