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ON NEWTON-MERCER-TYPE INEQUALITY FOR h-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish new Newton–Mercer-type inequalities for h-convex functions by means of Riemann–Liouville fractional integral operators. The obtained results provide fractional extensions of several known Newton–Mercer-type inequalities and are derived by combining the properties of h-convex functions with the recently introduced concept of B-functions. As applications of the main results, a number of new inequalities are obtained for important subclasses of h-convex functions, including convex, s-convex, and P-functions. An illustrative example is also presented to demonstrate the applicability of the established inequalities.

1. INTRODUCTION

Newton-type inequalities arise naturally from three-point Newton–Cotes quadrature formulas and play an important role in numerical integration, approximation theory, and the study of integral inequalities. These inequalities provide estimates for the deviation between weighted combinations of function values and their corresponding integral means. Due to their wide applicability, Newton-type inequalities have attracted considerable attention from several authors in recent years [9].

Several extensions and refinements of Newton-type inequalities have been established under different assumptions on the involved functions. Gao and Shi [9] obtained new Newton-type inequalities for functions whose second derivatives in absolute value are convex. By using (p, q)-calculus, Luangboon et al. [11] derived Simpson- and Newton-type inequalities for convex functions involving post-quantum integral operators. Later, Ali et al. [2] established quantum Newton-type inequalities for quantum differentiable convex functions. Error estimates for Newton-type quadrature formulas in the setting of bounded variation and Lipschitzian mappings were investigated by Erden et al. [8]. Further developments and related results can be found in [1, 6, 10] and the references therein.

Key words and phrases. h-convex functions, Newton-Mercer inequalities, Riemann-Liouville fractional integrals, B-functions, fractional integral inequalities.

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A recent contribution in this direction was given in [15], where the following Newton-type inequality was established for convex functions:

$$\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leq\frac{5(b-a)}{72}\left(|f'(a)|+|f'(b)|\right). \tag{1.1}$$

On the other hand, Varošanec [16] introduced the important class of h-convex functions, which unifies several well-known notions of convexity.

Definition 1.1. Let $h : J \to \mathbb{R}$ be a nonnegative function with $h \not\equiv 0$. A nonnegative function $f : I \to \mathbb{R}$ is said to be h-convex if

$$f(\lambda x+(1-\lambda)y)\leq h(\lambda)f(x)+h(1-\lambda)f(y) \tag{1.2}$$

for all $x, y \in I$ and $\lambda \in (0,1)$. If the inequality (1.2) is reversed, then f is called h-concave.

This notion contains several important classes of functions as special cases:

- $h(\lambda) = \lambda$ gives the class of ordinary convex functions [13];
- $h(\lambda) = 1$ gives the class of P-functions [7, 14];
- $h(\lambda) = \lambda^s$, $s \in (0,1]$, gives the class of s-convex functions [5].

Recently, Benaissa and co-authors [3, 4] introduced the class of B-functions, which will play an essential role in our analysis.

Definition 1.2. Let $a < b$ and let $g : (a,b) \to \mathbb{R}$ be a nonnegative function. Then g is called a B-function if

$$g(x-a)+g(b-x)\leq 2g\left(\frac{a+b}{2}\right) \tag{1.3}$$

for all $x \in (a,b)$. If the inequality (1.3) is reversed, then g is called an A-function. If equality holds in (1.3), then g is called an AB-function.

The following consequence will be useful throughout the paper.

Corollary 1.1. *Let $h : (0,1) \to \mathbb{R}$ be a nonnegative function. Then h is a B-function if*

$$h(\lambda)+h(1-\lambda)\leq 2h\left(\frac{1}{2}\right),\qquad \lambda\in(0,1). \tag{1.4}$$

Moreover,

- *$h(\lambda) = \lambda$ and $h(\lambda) = 1$ are AB-functions;*
- *$h(\lambda) = \lambda^s$, $s \in (0,1]$, is a B-function.*

Another important ingredient is the classical Jensen–Mercer inequality [12], which states that if f is convex on $[a,b]$, then

$$f\left(a+b-\sum_{j=1}^{n}\lambda_j z_j\right)\leq f(a)+f(b)-\sum_{j=1}^{n}\lambda_j f(z_j),$$

where $z_j \in [a,b]$, $\lambda_j \in [0,1]$, and $\sum_{j=1}^{n}\lambda_j = 1$.

We also recall the left- and right-sided Riemann–Liouville fractional integral operators of order $\alpha > 0$:

$$\mathfrak{I}^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,\mathrm{d}t,\qquad x>a,$$

and

$$\mathfrak{I}_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\ \mathrm{d}t, \qquad x < b.$$

Motivated by the growing interest in Newton-type inequalities, Mercer-type inequalities, and fractional integral operators, the main objective of this paper is to establish new Newton–Mercer-type inequalities for h-convex functions via Riemann–Liouville fractional integrals. The obtained results generalize several known Newton-type inequalities and yield new inequalities for convex, s-convex, and P-functions as particular cases.

2. Newton-Mercer-type inequalities

The following integral identity plays a fundamental role in the derivation of our main Newton–Mercer-type inequalities. It provides an exact representation of the deviation between a suitable combination of function values and the corresponding Riemann–Liouville fractional integral means.

Lemma 2.1. *Let $\alpha > 0$ and let $x, y \in [a,b]$ with $x < y$. If $f : [a,b] \to \mathbb{R}$ is differentiable and $f' \in L_1([a,b])$, then the following identity holds:*

$$\begin{aligned}
&\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right] \\
&\quad-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\mathfrak{I}_{(a+b-y)^+}^{\alpha} f\left(a+b-x\right)+\mathfrak{I}_{(a+b-x)^-}^{\alpha} f\left(a+b-y\right)\right] \\
&=\frac{(y-x)}{2}\left\{\int_0^{\frac{1}{3}} t^{\alpha}\left[f'\left(a+b-(tx+(1-t)\,y)\right)-f'\left(a+b-((1-t)\,x+ty)\right)\right]dt\right. \\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(t^{\alpha}-\frac{1}{2}\right)\left[f'\left(a+b-(tx+(1-t)\,y)\right)-f'\left(a+b-((1-t)\,x+ty)\right)\right]dt \\
&\quad\left.+\int_{\frac{2}{3}}^{1}\left(t^{\alpha}-1\right)\left[f'\left(a+b-(tx+(1-t)\,y)\right)-f'\left(a+b-((1-t)\,x+ty)\right)\right]dt\right\}.
\end{aligned} \tag{2.1}$$

Proof. Let $x, y \in [a,b]$ with $x < y$. Applying integration by parts to each integral term, we obtain

$$\begin{aligned}
H_1 &= \int_0^{\frac{1}{3}} t^{\alpha} f'\left(a+b-(tx+(1-t)\,y)\right)dt \\
&= \left(\frac{1}{y-x}\right) t^{\alpha} f\left(a+b-(tx+(1-t)\,y)\right)\Big|_0^{\frac{1}{3}} \\
&\quad-\frac{\alpha}{y-x}\int_0^{\frac{1}{3}} t^{\alpha-1} f\left(a+b-(tx+(1-t)\,y)\right)dt \\
&= \frac{1}{y-x}\left(\frac{1}{3}\right)^{\alpha} f\left(a+b-\frac{x+2y}{3}\right) \\
&\quad-\frac{\alpha}{(y-x)^{\alpha+1}}\int_{a+b-y}^{a+b-\frac{x+2y}{3}} \left(\tau-(a+b-y)\right)^{\alpha-1} f(\tau)\,d\tau,
\end{aligned}$$

$$\begin{aligned}
H_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left(t^\alpha - \frac{1}{2}\right) f'\left(a+b-\left(tx+\left(1-t\right)y\right)\right) dt \\
&= \left(\frac{1}{y-x}\right)\left(t^\alpha - \frac{1}{2}\right) f\left(a+b-\left(tx+\left(1-t\right)y\right)\right)\Big|_{\frac{1}{3}}^{\frac{2}{3}} \\
&\quad -\frac{\alpha}{y-x}\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1} f\left(a+b-\left(tx+\left(1-t\right)y\right)\right) dt \\
&= \frac{1}{y-x}\left[\left(\left(\frac{2}{3}\right)^\alpha - \frac{1}{2}\right) f\left(a+b-\frac{2x+y}{3}\right) - \left(\left(\frac{1}{3}\right)^\alpha - \frac{1}{2}\right) f\left(a+b-\frac{x+2y}{3}\right)\right] \\
&\quad -\frac{\alpha}{(y-x)^{\alpha+1}}\int_{a+b-\frac{x+2y}{3}}^{a+b-\frac{2x+y}{3}} \left(\tau-\left(a+b-y\right)\right)^{\alpha-1} f\left(\tau\right) d\tau,
\end{aligned}$$

and

$$\begin{aligned}
H_3 &= \int_{\frac{2}{3}}^{1} \left(t^\alpha - 1\right) f'\left(a+b-\left(tx+\left(1-t\right)y\right)\right) dt \\
&= \left(\frac{1}{y-x}\right)\left(t^\alpha - 1\right) f\left(a+b-\left(tx+\left(1-t\right)y\right)\right)\Big|_{\frac{2}{3}}^{1} \\
&\quad -\frac{\alpha}{y-x}\int_{\frac{2}{3}}^{1} t^{\alpha-1} f\left(a+b-\left(tx+\left(1-t\right)y\right)\right) dt \\
&= \frac{1}{y-x}\left[\left(1-\left(\frac{2}{3}\right)^\alpha\right) f\left(a+b-\frac{2x+y}{3}\right)\right] \\
&\quad -\frac{\alpha}{(y-x)^{\alpha+1}}\int_{a+b-\frac{2x+y}{3}}^{a+b-x} \left(\tau-\left(a+b-y\right)\right)^{\alpha-1} f\left(\tau\right) d\tau,
\end{aligned}$$

therefore

$$\begin{aligned}
H_1+H_2+H_3 &= \frac{1}{y-x}\left[\frac{1}{2} f\left(a+b-\frac{2x+y}{3}\right) + \frac{1}{2} f\left(a+b-\frac{x+2y}{3}\right)\right] \\
&\quad -\frac{\alpha}{(y-x)^{\alpha+1}}\int_{a+b-y}^{a+b-x} \left(\tau-\left(a+b-y\right)\right)^{\alpha-1} f\left(\tau\right) d\tau \\
&= \frac{1}{2\left(y-x\right)}\left[f\left(a+b-\frac{2x+y}{3}\right) + f\left(a+b-\frac{x+2y}{3}\right)\right] \\
&\quad -\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha+1}}\,\mathfrak{I}^{\alpha}_{(a+b-x)^-} f(a+b-y).
\end{aligned}$$

Similarly,

$$\begin{aligned}
H_4 &= \int_0^{\frac{1}{3}} t^{\alpha} f'(a+b-((1-t)x+ty))\,dt \\
&= -\left(\frac{1}{y-x}\right) t^{\alpha} f(a+b-((1-t)x+ty))\Big|_0^{\frac{1}{3}} \\
&\quad +\frac{\alpha}{y-x}\int_0^{\frac{1}{3}} t^{\alpha-1} f(a+b-((1-t)x+ty))\,dt \\
&= -\frac{1}{y-x}\left(\frac{1}{3}\right)^{\alpha} f\left(a+b-\frac{2x+y}{3}\right) \\
&\quad +\frac{\alpha}{(y-x)^{\alpha+1}}\int_{a+b-\frac{2x+y}{3}}^{a+b-x} ((a+b-x)-\tau)^{\alpha-1} f(\tau)\,d\tau,
\end{aligned}$$

$$\begin{aligned}
H_5 &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left(t^{\alpha}-\frac{1}{2}\right) f'(a+b-((1-t)x+ty))\,dt \\
&= -\left(\frac{1}{y-x}\right)\left(t^{\alpha}-\frac{1}{2}\right) f(a+b-((1-t)x+ty))\Big|_{\frac{1}{3}}^{\frac{2}{3}} \\
&\quad +\frac{\alpha}{y-x}\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1} f(a+b-((1-t)x+ty))\,dt \\
&= -\frac{1}{y-x}\left[\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{1}{2}\right) f\left(a+b-\frac{x+2y}{3}\right)-\left(\left(\frac{1}{3}\right)^{\alpha}-\frac{1}{2}\right) f\left(a+b-\frac{2x+y}{3}\right)\right] \\
&\quad +\frac{\alpha}{(y-x)^{\alpha+1}}\int_{a+b-\frac{x+2y}{3}}^{a+b-\frac{2x+y}{3}} ((a+b-x)-\tau)^{\alpha-1} f(\tau)\,d\tau,
\end{aligned}$$

and

$$\begin{aligned}
H_6 &= \int_{\frac{2}{3}}^{1} \left(t^{\alpha}-1\right) f'(a+b-((1-t)x+ty))\,dt \\
&= -\left(\frac{1}{y-x}\right)\left(t^{\alpha}-1\right) f(a+b-((1-t)x+ty))\Big|_{\frac{2}{3}}^{1} \\
&\quad +\frac{\alpha}{y-x}\int_{\frac{2}{3}}^{1} t^{\alpha-1} f(a+b-((1-t)x+ty))\,dt \\
&= -\frac{1}{y-x}\left[\left(1-\left(\frac{2}{3}\right)^{\alpha}\right) f\left(a+b-\frac{x+2y}{3}\right)\right] \\
&\quad +\frac{\alpha}{(y-x)^{\alpha+1}}\int_{a+b-y}^{a+b-\frac{x+2y}{3}} ((a+b-x)-\tau)^{\alpha-1} f(\tau)\,d\tau,
\end{aligned}$$

Using the identity

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha),$$

the above integral expression can be written in terms of the Riemann–Liouville fractional integral operator, yielding

$$\begin{aligned} H_4+H_5+H_6 &= -\frac{1}{y-x}\left[\frac{1}{2}f\left(a+b-\frac{2x+y}{3}\right)+\frac{1}{2}f\left(a+b-\frac{x+2y}{3}\right)\right] \\ &\quad +\frac{\alpha}{(y-x)^{\alpha+1}}\int_{a+b-y}^{a+b-x}((a+b-x)-\tau)^{\alpha-1}f(\tau)\,d\tau \\ &= -\frac{1}{2(y-x)}\left[f\left(a+b-\frac{2x+y}{3}\right)+f\left(a+b-\frac{x+2y}{3}\right)\right] \\ &\quad +\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha+1}}\mathfrak{I}^{\alpha}_{(a+b-y)^+}f(a+b-x). \end{aligned}$$

Consequently,

$$\begin{aligned} &(H_1+H_2+H_3)-(H_4+H_5+H_6) \\ &= \frac{1}{y-x}\left[f\left(a+b-\frac{2x+y}{3}\right)+f\left(a+b-\frac{x+2y}{3}\right)\right] \\ &\quad -\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha+1}}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^+}f(a+b-x)+\mathfrak{I}^{\alpha}_{(a+b-x)^-}f(a+b-y)\right]. \end{aligned}$$

Substituting the expressions obtained for $H_1+H_2+H_3$ and $H_4+H_5+H_6$ into the last identity yields (2.1). This completes the proof. $\square$

The following theorem provides our first Newton–Mercer-type inequality involving Riemann–Liouville fractional integral operators for h-convex functions.

Theorem 2.1. *Let h be a B-function on $(0,1)$ and assume that the assumptions of Lemma 2.1 hold. If $|f'|$ is an h-convex function on $[a,b]$, then the following Newton-Mercer-type inequality for Riemann-Liouville fractional operators holds*

$$\begin{aligned} &\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right. \\ &\left.-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^+}f(a+b-x)+\mathfrak{I}^{\alpha}_{(a+b-x)^-}f(a+b-y)\right]\right| \qquad (2.2) \\ &\le (y-x)\,h\left(\tfrac{1}{2}\right)K_{\alpha}\left(|f'(a+b-y)|+|f'(a+b-x)|\right) \end{aligned}$$

where

$$K_{\alpha}=\frac{1+2^{\alpha+1}+(\alpha-2)3^{\alpha}}{(\alpha+1)\,3^{\alpha+1}}+C_{\alpha}, \qquad (2.3)$$

and

$$C_\alpha = \begin{cases} \frac{2^{\alpha+1}-1}{(\alpha+1)\,3^{\alpha+1}} - \frac{1}{6}\,, & \alpha \in (0, \frac{\ln 2}{\ln 3}] \\ \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} \left(\frac{\alpha}{\alpha+1}\right) + \frac{2^{\alpha+1}+1}{(\alpha+1)\,3^{\alpha+1}} - \frac{1}{2}\,, & \alpha \in (\frac{\ln 2}{\ln 3}, \frac{\ln 2}{\ln 3 - \ln 2}] \\ \frac{1}{6} - \frac{2^{\alpha+1}-1}{(\alpha+1)\,3^{\alpha+1}}\,, & \alpha > \frac{\ln 2}{\ln 3-\ln 2}. \end{cases} \tag{2.4}$$

Proof. Taking the absolute value on both sides of identity (2.1) and applying the triangle inequality, we obtain

$$\begin{aligned} &\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right. \\ &\qquad \left. - \frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\left[\mathfrak{I}^\alpha_{(a+b-y)^+} f(a+b-x) + \mathfrak{I}^\alpha_{(a+b-x)^-} f(a+b-y)\right]\right| \\ &\le \frac{y-x}{2}\left\{\int_0^{\frac{1}{3}} t^\alpha \Big[|f'(t(a+b-x)+(1-t)(a+b-y))| \right. \\ &\qquad\qquad + |f'((1-t)(a+b-x)+t(a+b-y))|\Big]dt \\ &\quad + \int_{\frac{1}{3}}^{\frac{2}{3}} \left|t^\alpha - \frac{1}{2}\right| \Big[|f'(t(a+b-x)+(1-t)(a+b-y))| \\ &\qquad\qquad + |f'((1-t)(a+b-x)+t(a+b-y))|\Big]dt \\ &\quad + \int_{\frac{2}{3}}^{1} |t^\alpha - 1| \Big[|f'(t(a+b-x)+(1-t)(a+b-y))| \\ &\qquad\qquad \left. + |f'((1-t)(a+b-x)+t(a+b-y))|\Big]dt \right\}. \end{aligned}$$

Using the h-convexity of $|f'|$, we obtain

$$\begin{aligned} &\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right. \\ &\qquad \left. - \frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\left[\mathfrak{I}^\alpha_{(a+b-y)^+} f\left(a+b-x\right) + \mathfrak{I}^\alpha_{(a+b-x)^-} f\left(a+b-y\right)\right]\right| \\ &\le \frac{(y-x)}{2}\left\{\int_0^{\frac{1}{3}} t^\alpha \left[h(t)+h(1-t)\right]\left(\left|f'\left(a+b-y\right)\right| + \left|f'\left(a+b-x\right)\right|\right)dt\right. \end{aligned}$$

$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|[h(t)+h(1-t)]\left(\left|f'\left(a+b-y\right)\right|+\left|f'\left(a+b-x\right)\right|\right)dt$$

$$\left.+\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-1\right|[h(t)+h(1-t)]\left(\left|f'\left(a+b-y\right)\right|+\left|f'\left(a+b-x\right)\right|\right)dt\right\},$$

since h is a B-function, applying inequality (1.4), provides

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right.\\&\left.-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-x\right)+\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-y\right)\right]\right|\\&\le(y-x)\,h\left(\tfrac{1}{2}\right)\left(\left|f'\left(a+b-y\right)\right|+\left|f'\left(a+b-x\right)\right|\right)\\&\times\left\{\int_{0}^{\frac{1}{3}}t^{\alpha}dt+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|dt+\int_{\frac{2}{3}}^{1}\left(1-t^{\alpha}\right)dt\right\}.\end{aligned}$$

Considering that

$$\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|dt=C_{\alpha}.$$

Substituting the above estimates into the previous inequality yields (2.2). This completes the proof. □

By taking $x=a$ and $y=b$ in Theorem 2.1, we establish the following Newton-type inequality via Riemann-Liouville fractional operators for h-convex function.

Corollary 2.1. *Let h be a B-function on $(0,1)$ and assume that the assumptions of Lemma 2.1 hold. If $|f'|$ is a h-convex mapping on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{a^{+}}f\left(b\right)+\mathfrak{I}^{\alpha}_{b^{-}}f\left(a\right)\right]\right|\\&\le(b-a)\,h\left(\tfrac{1}{2}\right)\,K_{\alpha}\,\left(|f'(a)|+|f'(b)|\right)\end{aligned}\tag{2.5}$$

where K_{α} is defined by (2.3).

For $\alpha=1$, a direct computation from (2.3) and (2.4) gives

$$K_{1}=\frac{5}{36}.$$

Hence, Theorem 2.1 reduces to the following Newton–Mercer-type inequality involving the classical Riemann integral.

Corollary 2.2. *Let h be a B-function on $(0,1)$ and assume that the assumptions of Lemma 2.1 hold. If $|f'|$ is a h-convex mapping on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|\\&\leq\frac{5\,(y-x)}{36}h\left(\frac{1}{2}\right)\left(\left|f'\left(a+b-y\right)\right|+\left|f'\left(a+b-x\right)\right|\right).\end{aligned}\tag{2.6}$$

Next, consider some particular cases on h-convexity.

(a) Put $h(t)=t^s$ with $s\in(0,\,1]$ in Theorem 2.1 , we deduce the following result.

Corollary 2.3. *Assume α and f are defined according to Theorem 2.1. If $|f'|$ is a s-convex function on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right.\\&\left.-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^+}f\left(a+b-x\right)+\mathfrak{I}^{\alpha}_{(a+b-x)^-}f\left(a+b-y\right)\right]\right|\\&\leq(y-x)\left(\frac{1}{2}\right)^{s}K_{\alpha}\left(\left|f'\left(a+b-y\right)\right|+\left|f'\left(a+b-x\right)\right|\right).\end{aligned}\tag{2.7}$$

where K_α is defined by (2.3). For $\alpha=1$, we get

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|\\&\leq\frac{5\,(y-x)}{72}\left(\frac{1}{2}\right)^{s-1}\left(\left|f'\left(a+b-y\right)\right|+\left|f'\left(a+b-x\right)\right|\right).\end{aligned}\tag{2.8}$$

It is worth noting that inequality (2.8) extends the Newton-type inequality (1.1). Indeed, by taking $s=1$, $x=a$, and $y=b$, one immediately recovers (1.1).

(b) Assuming $h(\lambda)=1$ in Theorem 2.1 yields the following new result for the class P-function. This result can also be obtained as the limiting case $s\to0^+$ of Corollary 2.3.

Corollary 2.4. *Assume α and f are defined according to Theorem 2.1. If $|f'|$ is a P-function on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right.\\&\left.-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^+}f\left(a+b-x\right)+\mathfrak{I}^{\alpha}_{(a+b-x)^-}f\left(a+b-y\right)\right]\right|\\&\leq(y-x)\,K_{\alpha}\left(\left|f'\left(a+b-y\right)\right|+\left|f'\left(a+b-x\right)\right|\right).\end{aligned}\tag{2.9}$$

Taking $\alpha = 1$ gives

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|\\&\leq\frac{5\,(y-x)}{36}\left(\left|f'\left(a+b-y\right)\right|+\left|f'\left(a+b-x\right)\right|\right).\end{aligned}\tag{2.10}$$

To derive a second family of Newton–Mercer-type inequalities, we shall employ Hölder's integral inequality together with the following elementary estimate: Let $A, B \geq 0$ and $\eta > 0$:

$$A^\eta + B^\eta \leq \max(1, 2^{1-\eta}) \cdot (A+B)^\eta.$$

Theorem 2.2. *Let h be a B-function on $(0,1)$, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and assume that α, f are defined as in Lemma 2.1. If $|f'|^p$ is a h-convex mapping on $[a,b]$, the following Newton-Mercer-type inequality holds.*

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right.\\&\left.-\frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\left[\mathfrak{I}^\alpha_{(a+b-y)^+}f\left(a+b-x\right)+\mathfrak{I}^\alpha_{(a+b-x)^-}f\left(a+b-y\right)\right]\right|\\&\leq (y-x)\,K(\alpha,\,q)\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[\frac{|f'(a+b-y)|^p+|f'(a+b-x)|^p}{3}\right]^{\frac{1}{p}}.\end{aligned}\tag{2.11}$$

Where

$$K(\alpha,\,q)=\left(\int_0^{\frac{1}{3}}t^{q\,\alpha}\,dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|^q\,dt\right)^{\frac{1}{q}}+\left(\int_{\frac{2}{3}}^{1}(1-t^\alpha)^q\,dt\right)^{\frac{1}{q}}.\tag{2.12}$$

Proof. Taking the absolute value on both sides of identity (2.1) and applying the triangle inequality, we obtain:

$$\begin{aligned}&\frac{2}{y-x}\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right.\\&\qquad\left.-\frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\left[\mathfrak{I}^\alpha_{(a+b-y)^+}f(a+b-x)+\mathfrak{I}^\alpha_{(a+b-x)^-}f(a+b-y)\right]\right|\\&\leq I_1+I_2+I_3,\end{aligned}$$

where

$$\begin{aligned}I_1&=\int_0^{1/3}t^\alpha\Big(|f'(u_t)|+|f'(v_t)|\Big)\,\mathrm{d}t,\\I_2&=\int_{1/3}^{2/3}\left|t^\alpha-\frac{1}{2}\right|\Big(|f'(u_t)|+|f'(v_t)|\Big)\,\mathrm{d}t,\\I_3&=\int_{2/3}^{1}(1-t^\alpha)\Big(|f'(u_t)|+|f'(v_t)|\Big)\,\mathrm{d}t,\end{aligned}$$

with the parameterized segments defined as:

$$u_t=t(a+b-x)+(1-t)(a+b-y),$$

and

$$v_t = (1-t)(a+b-x) + t(a+b-y).$$

Applying Hölder's integral inequality to each component part yields:

$$I_j \leq A_j \left(\int_{E_j} \Big(|f'(u_t)| + |f'(v_t)| \Big)^p \, \mathrm{d}t \right)^{1/p}, \qquad (j = 1, 2, 3),$$

where the corresponding integral coefficients are given by:

$$A_1 = \left(\int_0^{1/3} t^{q\alpha} \, \mathrm{d}t \right)^{1/q},$$

$$A_2 = \left(\int_{1/3}^{2/3} \left| t^\alpha - \frac{1}{2} \right|^q \, \mathrm{d}t \right)^{1/q},$$

$$A_3 = \left(\int_{2/3}^{1} (1 - t^\alpha)^q \, \mathrm{d}t \right)^{1/q}.$$

By employing the well-known algebraic inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \geq 0$, we observe that:

$$\Big(|f'(u_t)| + |f'(v_t)| \Big)^p \leq 2^{p-1} \Big(|f'(u_t)|^p + |f'(v_t)|^p \Big).$$

Since $|f'|^p$ is assumed to be an h-convex function on $[a, b]$, we can evaluate it at the points u_t and v_t to get:

$$|f'(u_t)|^p \leq h(t)|f'(a+b-x)|^p + h(1-t)|f'(a+b-y)|^p,$$

and

$$|f'(v_t)|^p \leq h(1-t)|f'(a+b-x)|^p + h(t)|f'(a+b-y)|^p.$$

Adding these two structural inequalities together directly leads to:

$$|f'(u_t)|^p + |f'(v_t)|^p \leq [h(t) + h(1-t)] \Big(|f'(a+b-x)|^p + |f'(a+b-y)|^p \Big).$$

Furthermore, since h is assumed to be a B-function belonging to the class $B(0, 1)$, Corollary 1.1 immediately implies that:

$$h(t) + h(1-t) \leq 2h\left(\frac{1}{2}\right).$$

Consequently, we establish the refined upper bound:

$$|f'(u_t)|^p + |f'(v_t)|^p \leq 2h\left(\frac{1}{2}\right) \Big(|f'(a+b-x)|^p + |f'(a+b-y)|^p \Big).$$

Substituting this estimate back into the inequality chain for I_j, multiplying both sides by $\frac{y-x}{2}$, and carefully simplifying the numerical factors, we obtain:

$$\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f(a+b-x)+\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f(a+b-y)\right]\right|$$
$$\leq (y-x)\left(h\left(\frac{1}{2}\right)\right)^{1/p}\left[\frac{|f'(a+b-y)|^{p}+|f'(a+b-x)|^{p}}{3}\right]^{1/p}\times\Big\{A_1+A_2+A_3\Big\}.$$

Finally, using the predefined explicit constant $K(\alpha,q)=A_1+A_2+A_3$ as detailed in (2.12), we arrive at the desired inequality (2.11). This completes the proof. □

Considering $\alpha=1$ in the above Theorem 2.2, we get

$$K(1,\,q)=\frac{5}{2}\left(\frac{1}{q+1}\left(\frac{1}{3}\right)^{q+1}\right)^{\frac{1}{q}}.$$

Hence the following Newton-Mercer-type inequality via Riemann integral for h-convex functions hold.

Corollary 2.5. *Let h be a B-function on $(0,1)$, $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and assume that the assumptions of Lemma 2.1 hold. If $|f'|^{p}$ is a h-convex mapping on $[a,b]$, we get the following Newton-Mercer-type inequality*

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|\\&\leq\frac{5(y-x)}{2}\left(\frac{1}{q+1}\left(\frac{1}{3}\right)^{q+1}\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[\frac{|f'(a+b-y)|^{p}+|f'(a+b-x)|^{p}}{3}\right]^{\frac{1}{p}}.\end{aligned}\tag{2.13}$$

Now, some special cases on h-convex function are established.

(a) Given $h(\lambda)=\lambda^{s}$ with $s\in(0,\,1]$ in Theorem 2.2, we deduce the following result.

Corollary 2.6. *Assume α and f are defined according to Theorem 2.2. If $|f'|^{p}$ is a s-convex function on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right.\\&\left.-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\,(a+b-x)+\,\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\,(a+b-y)\right]\right|\\&\leq (y-x)\,K(\alpha,\,q)\,\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[\frac{|f'(a+b-y)|^{p}+|f'(a+b-x)|^{p}}{3}\right]^{\frac{1}{p}},\end{aligned}\tag{2.14}$$

$K(\alpha, q)$ is defined as (2.12), and for $\alpha = 1$ we obtain

$$\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(t)dt\right| \leq \frac{5(y-x)}{2}\left(\frac{1}{q+1}\left(\frac{1}{3}\right)^{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[\frac{|f'(a+b-y)|^p+|f'(a+b-x)|^p}{3}\right]^{\frac{1}{p}}. \tag{2.15}$$

Remark 2.1. Putting $s = 1$, $x = a$ and $y = b$ in (2.15) gives the following. For $p, q > 1$ where $\frac{1}{p}+\frac{1}{q} = 1$, if $|f'|^p$ is a convex function on $[a, b]$, then

$$\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{1}{b-a}\int_{a}^{b} f(t)dt\right| \leq \frac{5(b-a)}{2}\left(\frac{1}{q+1}\left(\frac{1}{3}\right)^{q+1}\right)^{\frac{1}{q}}\left[\frac{|f'(a)|^p+|f'(b)|^p}{6}\right]^{\frac{1}{p}}. \tag{2.16}$$

(b) Setting $h(\lambda) = 1$ in Theorem 2.2 produces the following new result about the class P-function.

Corollary 2.7. *Assume α and f are defined according to Theorem 2.2. If $|f'|^p$ is a P-function on $[a, b]$, then*

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]\right.\\ &\left.-\frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^+}f\left(a+b-x\right)+\mathfrak{I}^{\alpha}_{(a+b-x)^-}f\left(a+b-y\right)\right]\right|\\ &\leq (y-x)\,K(\alpha, q)\left[\frac{|f'(a+b-y)|^p+|f'(a+b-x)|^p}{3}\right]^{\frac{1}{p}},\end{aligned} \tag{2.17}$$

setting $\alpha = 1$ gives

$$\left|\frac{1}{2}\left[f\left(a+b-\frac{x+2y}{3}\right)+f\left(a+b-\frac{2x+y}{3}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(t)dt\right| \leq \frac{5(y-x)}{2}\left(\frac{1}{q+1}\left(\frac{1}{3}\right)^{q+1}\right)^{\frac{1}{q}}\left[\frac{|f'(a+b-y)|^p+|f'(a+b-x)|^p}{3}\right]^{\frac{1}{p}}. \tag{2.18}$$

3. Numerical Example

In this section, we provide a numerical example to illustrate the validity of the fractional boundaries established in Corollary 2.6. We consider different fractional orders, namely $\alpha = \frac{1}{2}$, $\alpha = \frac{3}{4}$, and $\alpha = 1$, to confirm the behavior of the obtained estimates.

Example 3.1. Let us consider the function $f(t) = t^2$ defined on the interval $[a, b] = [0, 1]$. Setting the evaluation points as $x = 0$ and $y = 1$, we have $f'(t) = 2t$. Since $|f'(t)| = 2t$ is convex on $[0,1]$, the mapping $|f'|$ is h-convex with $h(\lambda)=\lambda$ (which corresponds to $s=1$). Thus, the assumptions of Corollary 2.6 are satisfied.

Applying inequality (2.6) with $a=0, b=1$, $h(1/2)=1/2$, $|f'(0)|=0$, and $|f'(1)|=2$, the fractional framework reduces to the relation:

$$\mathrm{LHS}(\alpha)\leq \mathrm{RHS}(\alpha), \tag{3.1}$$

where

$$\mathrm{LHS}(\alpha)=\left|\frac{1}{2}\left[f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\mathfrak{I}^{\alpha}_{0+}f\,(1)+\mathfrak{I}^{\alpha}_{1-}f\,(0)\right]\right|,$$

and $\mathrm{RHS}(\alpha)=K_\alpha$. For $f(t)=t^2$, the node term yields $\frac{1}{2}[f(1/3)+f(2/3)]=\frac{5}{18}$. By using the definition of the Riemann-Liouville fractional integrals, we find:

$$\mathfrak{I}^{\alpha}_{0+}f\,(1)=\frac{1}{\Gamma(\alpha)}\int_0^1(1-t)^{\alpha-1}t^2\,\mathrm{d}t=\frac{2}{\Gamma(\alpha+3)},$$

$$\mathfrak{I}^{\alpha}_{1-}f\,(0)=\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)^2\,\mathrm{d}t=\frac{2}{\Gamma(\alpha+3)}.$$

Using the property $\Gamma(\tau+1)=\tau\Gamma(\tau)$, the left-hand side simplifies to the following algebraic form for any $\alpha>0$:

$$\mathrm{LHS}(\alpha)=\left|\frac{5}{18}-\frac{2}{(\alpha+1)(\alpha+2)}\right|.$$

Next, we evaluate (3.1) numerically across different values of α using equations (2.3) and (2.4):

- For $\alpha=1/2$: Since $\alpha\in(0,\frac{\ln 2}{\ln 3}]$, we find $C_{1/2}=\frac{2\sqrt{2}-1}{4.5\sqrt{3}}-\frac{1}{6}\approx 0.0673$ and $K_{1/2}\approx 0.3228$. Thus,
$$\mathrm{LHS}\left(\frac{1}{2}\right)=\left|\frac{5}{18}-\frac{8}{15}\right|=\frac{23}{90}\approx 0.2556\leq 0.3228=\mathrm{RHS}\left(\frac{1}{2}\right).$$
- For $\alpha=3/4$: Since $\alpha\in(\frac{\ln 2}{\ln 3},\frac{\ln 2}{\ln 3-\ln 2}]$, we get $C_{3/4}\approx 0.0435$ and $K_{3/4}\approx 0.2112$. Thus,
$$\mathrm{LHS}\left(\frac{3}{4}\right)=\left|\frac{5}{18}-\frac{32}{77}\right|=\frac{191}{1386}\approx 0.1378\leq 0.2112=\mathrm{RHS}\left(\frac{3}{4}\right).$$
- For $\alpha=1$: Since $\alpha>\frac{\ln 2}{\ln 3-\ln 2}$, we obtain $C_1=\frac{1}{36}\approx 0.0278$ and $K_1=\frac{5}{36}\approx 0.1389$. Thus,
$$\mathrm{LHS}(1)=\left|\frac{5}{18}-\frac{1}{3}\right|=\frac{1}{18}\approx 0.0556\leq 0.1389=\mathrm{RHS}(1).$$

The computed values and their corresponding efficiency ratios (LHS/RHS) are summarized in Table 1.

TABLE 1. Numerical verification of inequality (3.1) for $f(t)=t^2$

Order (α)	LHS(α)	RHS(α)	Ratio (LHS/RHS)
1/2	0.2556	0.3228	0.7918
3/4	0.1378	0.2112	0.6525
1	0.0556	0.1389	0.4000

The numerical results confirm the validity of the obtained fractional Newton–Mercer-type inequalities and illustrate the effectiveness of the derived upper bounds.

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