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ON BICOMPLEX EXTENSIONS OF SARGENT SEQUENCE SPACES

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Abstract. In this paper, we extend Sargent's sequence spaces to the bicomplex setting and introduce the classes $m_{\mathbb{BC}}(\phi)$ and $n_{\mathbb{BC}}(\phi)$ for admissible weight families $\phi \in \Phi$. Using the Euclidean bicomplex norm, we establish their basic linear and topological properties, including stability under rearrangements and monotonicity with respect to ϕ. In particular, we show that both $m_{\mathbb{BC}}(\phi)$ and $n_{\mathbb{BC}}(\phi)$ are Banach spaces and that $n_{\mathbb{BC}}(\phi)$ is the Köthe–Toeplitz dual of $m_{\mathbb{BC}}(\phi)$. We also derive inclusion criteria linking bicomplex Sargent sequence spaces with Bicomplex Lebesgue sequence spaces. Finally, illustrative examples demonstrate that $m_{\mathbb{BC}}(\phi)$ lies strictly between $\ell^1(\mathbb{BC})$ and $\ell^\infty(\mathbb{BC})$ in general, and that it never coincides with $\ell^p(\mathbb{BC})$ for $1 < p < \infty$.

1. Introduction

Let $\mathbb{N} := \{1, 2, 3, \dots\}$ and define

$$\varphi := \{\, \sigma \subset \mathbb{N} \ :\ \sigma \text{ is a finite set of distinct positive integers} \,\}.$$

For each $s \in \mathbb{N}$, set

$$\varphi_s := \{\, \sigma \in \varphi :\ |\sigma| \le s \,\}.$$

For a sequence $x = (x_n)$, let

$$S(x) := \{\, (x_{\pi(n)})_{n\ge1} :\ \pi \text{ a permutation of } \mathbb{N} \,\}$$

be the set of all rearrangements. Also, define

$$\Phi := \Big\{\, \phi = (\phi_s)_{s\ge1} \ :\ 0 < \phi_s \le \phi_{s+1} < \infty,\ (s+1)\phi_s \ge s\,\phi_{s+1} \text{ for all } s \in \mathbb{N} \Big\}.$$

We also set $\phi_0 := 0$ and $\Delta\phi_s := \phi_s - \phi_{s-1} \ge 0$.

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Sargent's sequence spaces. Sargent defined for a scalar sequence $x = (x_n)$ and $\phi \in \Phi$:

$$m(\phi) := \{\, x : \ \|x\|_{m(\phi)} < \infty \,\},$$

$$\text{where} \qquad \|x\|_{m(\phi)} := \sup_{s\geq 1} \sup_{\sigma\in\varphi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |x_n| \right\},$$

and

$$n(\phi) := \{\, x : \ \|x\|_{n(\phi)} < \infty \,\},$$

$$\text{where} \qquad \|x\|_{n(\phi)} := \sup_{u\in S(x)} \sum_{n=1}^{\infty} |u_n|\, \Delta\phi_n.$$

Sargent gave topological properties and studied matrix transformations between these spaces and classical Lebesgue sequence spaces. Many authors have since investigated these spaces (see [3, 6, 11, 15, 16, 21]).

. The concept of bicomplex numbers, rooted in the pioneering work of Segre in the late nineteenth century [18], has progressively developed into a fertile research area bridging algebra, analysis, and geometry. Early contributions by Scorza [17] and Spampinato [19] established fundamental properties of holomorphic functions in the bicomplex setting, laying the foundation for subsequent generalisations. The monograph by Price [12] further systematised multicomplex spaces and their applications, providing a rigorous entry point into the field.

Over the past three decades, the theory has experienced renewed interest, particularly through a functional-analytic perspective advanced by Alpay, Luna-Elizarrarás, Shapiro, and Struppa [1], and later extended in the comprehensive treatment of bicomplex holomorphic functions by Luna-Elizarrarás et al. [10]. These works emphasise both the structural richness and the analytical challenges posed by bicomplex algebras.

Recent investigations have turned towards applications in sequence spaces, operator theory, and modules. Değirmen and Sağır [5] studied bicomplex BC-modules and their geometric characteristics, while Toksoy and Duyar [20] considered weighted structures in this framework. In parallel, research on bicomplex sequence spaces, such as the Orlicz–lacunary constructions explored by Raj, Esi, and Sharma [13], illustrates the versatility of bicomplex methods in generalising classical functional spaces. Complementary studies have also examined bicomplex Lorentz spaces [7], as well as completeness and transformation properties [14], contributing to the broader understanding of their topological and geometric features.

Beyond pure analysis, bicomplex numbers have found applications in diverse contexts. For example, Kossentini and Gargoubi [9] interpreted them as complexified f-algebras, whereas Kleine, Hanifi, and Henningson [8] employed bicomplex methods in fluid stability analysis. More recently, Alpay, Diki, and Vajiac [2] highlighted the potential of bicomplex-valued neural networks, pointing to new directions at the interface of mathematics and applied science.

Taken together, these contributions underline both the historical depth and the contemporary relevance of bicomplex analysis. They motivate further exploration of its algebraic, analytic, and geometric aspects, as well as its capacity to provide new insights in applied mathematics and related disciplines.

. Let i and j be independent imaginary units such that $i^2 = j^2 = -1$, $ij = ji$, and let $\mathbb{C}$ be the field of complex numbers with imaginary unit i. The set of bicomplex numbers $\mathbb{BC}$ is defined by

$$\mathbb{BC} = \{\zeta = \zeta_1 + j\zeta_2 : \zeta_1, \zeta_2 \in \mathbb{C}\}.$$

The set $\mathbb{BC}$ forms a Banach space and a ring with respect to addition, scalar multiplication, multiplication, and the Euclidean norm. For all $\zeta = \zeta_1 + j\zeta_2,\ t = t_1 + jt_2 \in \mathbb{BC}$ and for all $\lambda \in \mathbb{R}$,

$$\zeta + t = (\zeta_1 + t_1) + j(\zeta_2 + t_2), \qquad \lambda\zeta = \lambda\zeta_1 + j\lambda\zeta_2, \qquad \zeta \times t = (\zeta_1 t_1 - \zeta_2 t_2) + j(\zeta_1 t_2 + \zeta_2 t_1),$$

$$\|\zeta\|_{\mathbb{BC}} = \sqrt{|\zeta_1|^2 + |\zeta_2|^2}.$$

Every bicomplex number can be written as $\zeta = \zeta_1 + j\zeta_2$ with $\zeta_1, \zeta_2 \in \mathbb{C}$. A bicomplex sequence (ζ_n) converges in $\mathbb{BC}$ if and only if its complex components converge coordinatewise. We refer to [12] for the standard topological background.

Lemma 1.1. [4] *(Bicomplex Hölder's Inequality)* Let $1 < p < \infty$ and let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For $\zeta_k, t_k \in \mathbb{BC}$ $(k = 1, 2, \ldots, n)$,

$$\sum_{k=1}^{n} \|\zeta_k t_k\|_{\mathbb{BC}} \leq \sqrt{2} \left(\sum_{k=1}^{n} \|\zeta_k\|_{\mathbb{BC}}^p \right)^{1/p} \left(\sum_{k=1}^{n} \|t_k\|_{\mathbb{BC}}^q \right)^{1/q}.$$

Lemma 1.2. [4] *(Bicomplex Minkowski's Inequality)* Let $1 < p < \infty$ and $\zeta_k, t_k \in \mathbb{BC}$ $(k = 1, 2, \ldots, n)$. Then

$$\left(\sum_{k=1}^{n} \|\zeta_k + t_k\|_{\mathbb{BC}}^p \right)^{1/p} \leq \left(\sum_{k=1}^{n} \|\zeta_k\|_{\mathbb{BC}}^p \right)^{1/p} + \left(\sum_{k=1}^{n} \|t_k\|_{\mathbb{BC}}^p \right)^{1/p}.$$

Definition 1.3. [4] Let $w(\mathbb{BC})$ denote all bicomplex sequences. Define

$$\ell^\infty(\mathbb{BC}) := \left\{ \zeta = (\zeta_k) \in w(\mathbb{BC}) : \sup_{k \in \mathbb{N}} \|\zeta_k\|_{\mathbb{BC}} < \infty \right\},$$

$$\ell^p(\mathbb{BC}) := \left\{ \zeta = (\zeta_k) \in w(\mathbb{BC}) : \sum_{k=1}^{\infty} \|\zeta_k\|_{\mathbb{BC}}^p < \infty \right\}, \quad 0 < p < \infty.$$

Theorem 1.4. [4] $\ell^\infty(\mathbb{BC})$ is a Banach space with the norm

$$\|\zeta\|_{\ell^\infty(\mathbb{BC})} = \sup_{k \in \mathbb{N}} \|\zeta_k\|_{\mathbb{BC}}, \quad \zeta = (\zeta_k) \in \ell^\infty(\mathbb{BC}).$$

Theorem 1.5. [4] $\ell^p(\mathbb{BC})$ is a Banach space for $1 \leq p < \infty$ with the norm

$$\|\zeta\|_{\ell^p(\mathbb{BC})} = \left(\sum_{k=1}^{\infty} \|\zeta_k\|_{\mathbb{BC}}^p \right)^{1/p}, \quad \zeta = (\zeta_k) \in \ell^p(\mathbb{BC}),$$

and a p-Banach space for $0 < p < 1$ with the p-norm

$$\|\zeta\|_{\ell^p(\mathbb{BC})} = \sum_{k=1}^{\infty} \|\zeta_k\|_{\mathbb{BC}}^p, \quad \zeta = (\zeta_k) \in \ell^p(\mathbb{BC}).$$

We extend the classical notion of Sargent's sequence spaces to the bicomplex setting and investigate their structural, topological, and geometric properties. We now define the following sequence spaces. Let $x = (x_n)$ be a sequence in $\mathbb{BC}$. Define

$$\|x\|_{m_{\mathbb{BC}}(\phi)} := \sup_{s\geq 1} \sup_{\sigma\in\varphi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} \|x_n\|_{\mathbb{BC}} \right\},$$

and write $x \in m_{\mathbb{BC}}(\phi)$ whenever this quantity is finite. We also set $x \in n_{\mathbb{BC}}(\phi)$ if and only if

$$\|x\|_{n_{\mathbb{BC}}(\phi)} := \sup_{\pi} \left\{ \sum_{n=1}^{\infty} \|x_{\pi(n)}\|_{\mathbb{BC}} \ \Delta\phi_n \right\} < \infty,$$

where π ranges over all permutations of $\mathbb{N}$ and $\Delta\phi_n := \phi_n - \phi_{n-1}$ (with the convention $\phi_0 := 0$).

2. Results

Throughout this section we write $\Delta\phi_n := \phi_n - \phi_{n-1}$ with the convention $\phi_0 := 0$. For a sequence $x = (x_n)$, let $S(x)$ denote the collection of all rearrangements of x (i.e., sequences of the form $(x_{\pi(n)})$ with π a permutation of $\mathbb{N}$).

We begin by showing that the bicomplex Sargent sequence spaces are complex Banach spaces.

Theorem 2.1. *For every $\phi \in \Phi$, the spaces $m_{BC}(\phi)$ and $n_{BC}(\phi)$ are complex Banach spaces.*

Proof. We first show that $m_{BC}(\phi)$ is a complex vector space. Let $x, y \in m_{\mathbb{BC}}(\phi)$. Then

$$\begin{aligned}
\|x+y\|_{m_{\mathbb{BC}}(\phi)} &= \sup_{s\geq 1} \sup_{\sigma\in\varphi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} \|x_n + y_n\|_{\mathbb{BC}} \right\} \\
&\leq \sup_{s\geq 1} \sup_{\sigma\in\varphi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} \left(\|x_n\|_{\mathbb{BC}} + \|y_n\|_{\mathbb{BC}} \right) \right\} \\
&\leq \|x\|_{m_{\mathbb{BC}}(\phi)} + \|y\|_{m_{\mathbb{BC}}(\phi)} < \infty.
\end{aligned}$$

For any $\alpha \in \mathbb{C}$, $\|\alpha x\|_{m_{\mathbb{BC}}(\phi)} = |\alpha| \, \|x\|_{m_{\mathbb{BC}}(\phi)}$. Hence $m_{\mathbb{BC}}(\phi)$ is a vector space. Similarly, $n_{BC}(\phi)$ is a complex vector space.

Completeness of both spaces follows from the completeness of BC with respect to $\|\cdot\|_{BC}$ together with the definitions of the norms, exactly as in the classical complex case. Therefore, $m_{BC}(\phi)$ and $n_{BC}(\phi)$ are complex Banach spaces. □

The following result shows that the space $m_{BC}(\phi)$ is invariant under rearrangements.

Theorem 2.2. *Let $x \in m_{\mathbb{BC}}(\phi)$ and $u \in S(x)$. Then $u \in m_{\mathbb{BC}}(\phi)$.*

Proof. Let π range over all permutations of $\mathbb{N}$. Then

$$\begin{aligned}\|u\|_{m_{\mathbb{BC}}(\phi)} &= \sup_{s\ge1}\sup_{\sigma\in\varphi_s}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}\|u_n\|_{\mathbb{BC}}\right\}\\ &= \sup_{s\ge1}\sup_{\sigma\in\varphi_s}\left\{\frac{1}{\phi_s}\sum_{n\in\sigma}\|x_{\pi(n)}\|_{\mathbb{BC}}\right\}\\ &= \|x\|_{m_{\mathbb{BC}}(\phi)}.\end{aligned}$$

□

We next record a simple monotonicity property of $m_{BC}(\phi)$.

Proposition 2.1. *Let $x \in m_{\mathbb{BC}}(\phi)$ and suppose $\|u_n\|_{\mathbb{BC}} \le \|x_n\|_{\mathbb{BC}}$ for every n. Then $u = (u_n) \in m_{\mathbb{BC}}(\phi)$ and*

$$\|u\|_{m_{\mathbb{BC}}(\phi)} \le \|x\|_{m_{\mathbb{BC}}(\phi)}.$$

The next observation shows that the sequence $(\Delta\phi_n)$ belongs to $m_{BC}(\phi)$.

Proposition 2.2. *For any $\phi \in \Phi$, one has $\Delta\phi = (\phi_n - \phi_{n-1}) \in m_{\mathbb{BC}}(\phi)$.*

Proof. Since ϕ is real-valued, $\|\Delta\phi_n\|_{\mathbb{BC}} = |\Delta\phi_n|$. The classical Sargent estimate applies verbatim, yielding $\Delta\phi \in m_{\mathbb{BC}}(\phi)$. □

The following estimate will be used in the identification of the Köthe–Toeplitz dual.

Proposition 2.3. *Let $x \in m_{\mathbb{BC}}(\phi)$ and let $\{\|y_{\pi(n)}\|_{\mathbb{BC}}\}$ be a nonincreasing rearrangement of $\{\|y_n\|_{\mathbb{BC}}\}$. Then*

$$\sum_{k=1}^{n}\|y_kx_k\|_{\mathbb{BC}} \le \sqrt{2}\,\|x\|_{m_{\mathbb{BC}}(\phi)}\sum_{k=1}^{n}\|y_k\|_{\mathbb{BC}}\,\Delta\phi_k.$$

Proof. Using $\|zw\|_{\mathbb{BC}} \le \sqrt{2}\,\|z\|_{\mathbb{BC}}\|w\|_{\mathbb{BC}}$ and Abel summation with the nonincreasing rearrangement, we obtain

$$\begin{aligned}\sum_{r=1}^{n}\|y_rx_r\|_{\mathbb{BC}} &\le \sqrt{2}\sum_{r=1}^{n}\|y_r\|_{\mathbb{BC}}\|x_r\|_{\mathbb{BC}}\\ &\le \sqrt{2}\sum_{r=1}^{n-1}(\|y_r\|_{\mathbb{BC}}-\|y_{r+1}\|_{\mathbb{BC}})\sum_{i=1}^{r}\|x_i\|_{\mathbb{BC}} + \sqrt{2}\,\|y_n\|_{\mathbb{BC}}\sum_{i=1}^{n}\|x_i\|_{\mathbb{BC}}\\ &\le \sqrt{2}\,\|x\|_{m_{\mathbb{BC}}(\phi)}\left(\sum_{r=1}^{n-1}(\|y_r\|_{\mathbb{BC}}-\|y_{r+1}\|_{\mathbb{BC}})\phi_r + \|y_n\|_{\mathbb{BC}}\phi_n\right)\\ &= \sqrt{2}\,\|x\|_{m_{\mathbb{BC}}(\phi)}\sum_{r=1}^{n}\|y_r\|_{\mathbb{BC}}\,\Delta\phi_r.\end{aligned}$$

□

The following result identifies the Köthe–Toeplitz dual of the bicomplex Sargent sequence space $m_{BC}(\phi)$.

Theorem 2.3. *Let $\phi \in \Phi$. Then*

$$m_{BC}^{\alpha}(\phi) = n_{BC}(\phi).$$

In other words, $n_{BC}(\phi)$ is the Köthe–Toeplitz dual of $m_{BC}(\phi)$.

Proof. Let $x \in n_{\mathbb{BC}}(\phi)$. For any $\zeta = (\zeta_k) \in m_{\mathbb{BC}}(\phi)$,

$$\begin{aligned}\sum_{k=1}^{\infty} \|x_k \zeta_k\|_{\mathbb{BC}} &\leq \sqrt{2} \sum_{k=1}^{\infty} \|x_k\|_{\mathbb{BC}} \|\zeta_k\|_{\mathbb{BC}} \\ &\leq \sqrt{2}\, \|\zeta\|_{m_{\mathbb{BC}}(\phi)} \sum_{k=1}^{\infty} \|x_{\pi(k)}\|_{\mathbb{BC}} \Delta\phi_k \\ &= \sqrt{2}\, \|\zeta\|_{m_{\mathbb{BC}}(\phi)} \|x\|_{n_{\mathbb{BC}}(\phi)} \\ &< \infty,\end{aligned}$$

where $\{\|x_{\pi(k)}\|_{\mathbb{BC}}\}$ is a nonincreasing rearrangement of $\{\|x_k\|_{\mathbb{BC}}\}$. Hence

$$n_{\mathbb{BC}}(\phi) \subset m_{\mathbb{BC}}^{\alpha}(\phi),$$

where

$$m_{\mathbb{BC}}^{\alpha}(\phi) = \left\{ (x_k) : \sum_{k=1}^{\infty} \|x_k \zeta_k\|_{\mathbb{BC}} < \infty \text{ for all } \zeta = (\zeta_k) \in m_{\mathbb{BC}}(\phi) \right\}.$$

Conversely, let $x \in m_{\mathbb{BC}}^{\alpha}(\phi)$, that is,

$$\sum_{k=1}^{\infty} \|x_k \zeta_k\|_{\mathbb{BC}} < \infty \qquad \text{for every } \zeta = (\zeta_k) \in m_{\mathbb{BC}}(\phi).$$

Since $(\Delta\phi_k) \in m_{\mathbb{BC}}(\phi)$, by choosing $\zeta_k = \Delta\phi_k$ we obtain

$$\sum_{k=1}^{\infty} \|x_{\pi(k)}\|_{\mathbb{BC}} \Delta\phi_k = \sum_{k=1}^{\infty} \|x_{\pi(k)} \Delta\phi_k\|_{\mathbb{BC}} < \infty$$

for every permutation π. Therefore,

$$\sup_{\pi} \sum_{k=1}^{\infty} \|x_{\pi(k)}\|_{\mathbb{BC}} \Delta\phi_k < \infty,$$

so $x \in n_{\mathbb{BC}}(\phi)$. Hence

$$m_{\mathbb{BC}}^{\alpha}(\phi) \subset n_{\mathbb{BC}}(\phi).$$

□

We now characterise the inclusion relation between two bicomplex Sargent sequence spaces.

Theorem 2.4. *Let $\phi, \psi \in \Phi$. Then,*

$$m_{\mathbb{BC}}(\phi) \subset m_{\mathbb{BC}}(\psi) \iff \sup_{s \geq 1} \frac{\phi_s}{\psi_s} < \infty.$$

Proof. ($\Rightarrow$) If $m_{\mathbb{BC}}(\phi) \subset m_{\mathbb{BC}}(\psi)$, then $\Delta\phi \in m_{\mathbb{BC}}(\psi)$, so

$$\phi_s = \sum_{k=1}^{s} \Delta\phi_k \leq \psi_s \, \|\Delta\phi\|_{m_{\mathbb{BC}}(\psi)} \qquad (s \geq 1),$$

whence $\sup_{s\geq 1} \phi_s/\psi_s < \infty$.

($\Leftarrow$) Conversely, set $M := \sup_{s\geq 1} \phi_s/\psi_s < \infty$ and let $x \in m_{\mathbb{BC}}(\phi)$. Then for any s and $\sigma \in \varphi_s$,

$$\frac{1}{\psi_s} \sum_{n\in\sigma} \|x_n\|_{\mathbb{BC}} \leq \frac{M}{\phi_s} \sum_{n\in\sigma} \|x_n\|_{\mathbb{BC}},$$

so $\|x\|_{m_{\mathbb{BC}}(\psi)} \leq M\|x\|_{m_{\mathbb{BC}}(\phi)}$ and hence $x \in m_{\mathbb{BC}}(\psi)$. ☐

Theorem 2.5. *Let $\phi \in \Phi$. Then the following statements hold:*

(i) $\ell^1(\mathbb{BC}) \subseteq m_{\mathbb{BC}}(\phi) \subseteq \ell^\infty(\mathbb{BC})$ *for every* $\phi \in \Phi$.

(ii) $m_{\mathbb{BC}}(\phi) = \ell^1(\mathbb{BC}) \iff \lim_{s\to\infty} \phi_s < \infty$.

(iii) $m_{\mathbb{BC}}(\phi) = \ell^\infty(\mathbb{BC}) \iff \lim_{s\to\infty} \dfrac{\phi_s}{s} > 0$.

Proof. As in Sargent's classical case; the bicomplex norm obeys $\|zw\|_{\mathbb{BC}} \leq \sqrt{2}\,\|z\|_{\mathbb{BC}}\|w\|_{\mathbb{BC}}$, which does not affect these inclusions. ☐

Theorem 2.6. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

(i) $m_{\mathbb{BC}}(\phi) \neq \ell^p(\mathbb{BC})$ *for any* $\phi \in \Phi$.

(ii) $\ell^p(\mathbb{BC}) \subset m_{\mathbb{BC}}(\phi) \iff \sup\limits_{s\geq 1} \dfrac{s^{1/q}}{\phi_s} < \infty$.

(iii) $m_{\mathbb{BC}}(\phi) \subset \ell^p(\mathbb{BC}) \iff \Delta\phi \in \ell^p(\mathbb{BC})$.

(iv) $\displaystyle\bigcup_{\Delta\phi\in\ell^p(\mathbb{BC})} m_{\mathbb{BC}}(\phi) = \ell^p(\mathbb{BC})$.

Proof. We detail (ii). The other parts follow by parallel arguments.

Necessity. Assume $\ell^p(\mathbb{BC}) \subset m_{\mathbb{BC}}(\phi)$. Then for some $M > 0$, $\|x\|_{m_{\mathbb{BC}}(\phi)} \leq M\|x\|_{\ell^p(\mathbb{BC})}$ for all $x \in \ell^p(\mathbb{BC})$. Let χ_σ be the characteristic sequence of $\sigma \in \varphi_s$. Then

$$\frac{s}{\phi_s} \leq \|\chi_\sigma\|_{m_{\mathbb{BC}}(\phi)} \leq M\|\chi_\sigma\|_{\ell^p(\mathbb{BC})} = Ms^{1/p},$$

so $s^{1/q}/\phi_s \leq M$.

Sufficiency. Suppose $\sup_{s\geq 1} s^{1/q}/\phi_s \leq T < \infty$. For $\sigma \in \varphi_s$ and $x \in \ell^p(\mathbb{BC})$, the bicomplex Hölder inequality gives

$$\sum_{n\in\sigma} \|x_n\|_{\mathbb{BC}} \leq \sqrt{2}\Big(\sum_{n\in\sigma} \|x_n\|_{\mathbb{BC}}^p\Big)^{1/p} \Big(\sum_{n\in\sigma} 1^q\Big)^{1/q} \leq \sqrt{2}\,T\,\|x\|_{\ell^p(\mathbb{BC})}\,\phi_s.$$

Taking the supremum in s and σ yields $\|x\|_{m_{\mathbb{BC}}(\phi)} \leq \sqrt{2}\,T\,\|x\|_{\ell^p(\mathbb{BC})}$, hence $\ell^p(\mathbb{BC}) \subset m_{\mathbb{BC}}(\phi)$. ☐

The following examples show that the inclusions

$$\ell^1(BC) \subset m_{BC}(\phi) \subset \ell^\infty(BC)$$

are, in general, strict whenever $\lim_{s\to\infty} \phi_s = \infty$ and

$$\lim_{s\to\infty} \frac{\phi_s}{s} = 0.$$

Example 2.1. Let $\phi_s = s$ for all $s \geq 1$, and define the bicomplex sequence $x = (x_k)$ by

$$x_k = \frac{i + (1+i)j}{k}, \qquad k \geq 1.$$

Then

$$\sum_{k=1}^{\infty} \|x_k\|_{BC} = \sqrt{3} \sum_{k=1}^{\infty} \frac{1}{k} = +\infty,$$

so $x \notin \ell^1(BC)$. On the other hand,

$$\|x\|_{m_{BC}(\phi)} = \sup_{s\geq 1} \sup_{\sigma\in\varphi_s} \left(\frac{1}{s} \sum_{n\in\sigma} \left\| \frac{i + (1+i)j}{n} \right\|_{BC} \right) = \sqrt{3},$$

and hence $x \in m_{BC}(\phi)$. Therefore,

$$\ell^1(BC) \subsetneq m_{BC}(\phi).$$

Example 2.2. Let $\phi_s = \sqrt{s}$ for all $s \geq 1$, and define the bicomplex sequence $x = (x_k)$ by

$$x_k = \frac{2i + ij}{\sqrt{k}}, \qquad k \geq 1.$$

Then

$$\|x_k\|_{BC} = \frac{\sqrt{5}}{\sqrt{k}} \leq \sqrt{5} \qquad (k \geq 1),$$

so $x \in \ell^\infty(BC)$. However,

$$\|x\|_{m_{BC}(\phi)} = \sup_{s\geq 1} \sup_{\sigma\in\varphi_s} \left(\frac{1}{\sqrt{s}} \sum_{n\in\sigma} \frac{\sqrt{5}}{\sqrt{n}} \right) = +\infty,$$

and therefore $x \notin m_{BC}(\phi)$. Hence,

$$m_{BC}(\phi) \subsetneq \ell^\infty(BC).$$

3. Conclusion

In this work, we have extended Sargent's sequence spaces to the bicomplex setting by introducing the classes $m_{\mathbb{BC}}(\phi)$ and $n_{\mathbb{BC}}(\phi)$. Using the Euclidean bicomplex norm, we established their basic structural properties, including stability under rearrangements, monotonicity with respect to the parameter sequence ϕ, and completeness. It was shown that both $m_{BC}(\phi)$ and $n_{BC}(\phi)$ are Banach spaces, and that $n_{BC}(\phi)$ is the Köthe–Toeplitz dual of $m_{BC}(\phi)$.

We also derived precise inclusion criteria between bicomplex Sargent sequence spaces and classical Lebesgue sequence spaces, and proved that $m_{\mathbb{BC}}(\phi)$ lies strictly between $\ell^1(\mathbb{BC})$ and $\ell^\infty(\mathbb{BC})$ in general, but never coincides with $\ell^p(\mathbb{BC})$ for $1 < p < \infty$. Several examples were provided to illustrate the sharpness of these results.

These findings contribute to the development of bicomplex functional analysis and sequence space theory. They provide a foundation for further studies on matrix transformations, duality theory, and operator-related problems in the bicomplex framework. Potential directions for future work include exploring bicomplex analogues of other classical sequence spaces, studying compact operators on these spaces, and investigating applications in applied mathematics and mathematical physics.

References

[1] D. Alpay, M.E. Luna-Elizarrarás, M. Shapiro, D.C. Struppa, *Basics of Functional Analysis with Bicomplex Scalars, and Bicomplex Schur Analysis*, Springer, Basel, 2014. https://doi.org/10.1007/978-3-319-02519-5

[2] D. Alpay, K. Diki, M. Vajiac, *A note on the complex and bicomplex valued neural networks,* Appl. Math. Comput., **445** (2023), 127849. https://doi.org/10.1016/j.amc.2023.127849

[3] Y. Altun, T. Bilgin, *On a new class of sequences related to the ℓ^p space defined by Orlicz function,* Taiwanese J. Math., **13**(4) (2009), 1189–1196.

[4] N. Değirmen, B. Sağır Duyar, *Some geometric properties of bicomplex sequence spaces $\ell_p(BC)$,* Konuralp J. Math., **10**(1) (2022), 44–49.

[5] N. Değirmen, B. Sağır, *On bicomplex BC-modules $l_p^k(BC)$ and some of their geometric properties,* Georgian Math. J., **30**(1) (2023), 65–79. https://doi.org/10.1515/gmj-2022-2085

[6] C. Duyar, O. Oğur, *On a new space $m_2(M, A, \phi, p)$ of double sequences*, J. Funct. Spaces Appl., (2013), Article ID 509613, 8 pp. https://doi.org/10.1155/2013/509613

[7] İ. Eryilmaz, *Characterisation of multiplication operator on bicomplex Lorentz spaces with hyperbolic norm,* Maejo Int. J. Sci. Technol., **19**(1) (2025), 1–16.

[8] V.G. Kleine, A. Hanifi, D.S. Henningson, *Stability of two-dimensional potential flows using bicomplex numbers,* Proc. R. Soc. A, **478** (2022), 2264, 20220127. https://doi.org/10.1098/rspa.2022.0127

[9] S. Kossentini, H. Gargoubi, *Bicomplex numbers as a normal complexified f-algebra,* Commun. Math., **30**(1) (2022), 125–142. https://doi.org/10.2478/cm-2022-0009

[10] M.E. Luna-Elizarrarás, M. Shapiro, D.C. Struppa, A. Vajiac, *Bicomplex Holomorphic Functions: The Algebra, Geometry and Analysis of Bicomplex Numbers*, Birkhäuser, Cham, 2015. https://doi.org/10.1007/978-3-319-24883-9

[11] E. Malkowsky, M. Mursaleen, Matrix transformations between FK-spaces and the sequence spaces $m(\phi)$ and $n(\phi)$, J. Math. Anal. Appl., **196**(2) (1995), 659–665. https://doi.org/10.1006/jmaa.1995.1414

[12] G.B. Price, *An Introduction to Multicomplex Spaces and Functions*, Marcel Dekker, New York, 1991.

[13] K. Raj, A. Esi, C. Sharma, *Orlicz–lacunary bicomplex sequence spaces of difference operators,* Filomat, **37**(8) (2023), 2421–2435. https://doi.org/10.2298/FIL2308421R

[14] N. Sager, B. Sağır, *On completeness of some bicomplex sequence spaces,* Palestine J. Math., **9**(2) (2020), 891–902.

[15] B. Sağır, C. Duyar, O. Oguz, *A new double sequence space $m(F, \phi, p)$ defined by a double sequence of modulus functions,* Int. J. Anal., (2014), Article ID 684512, 4 pp.

[16] W.L.C. Sargent, *Some sequence spaces related to the ℓ^p spaces*, J. London Math. Soc., **35** (1960), 161–171.

[17] G. Scorza, Dragoni sulle funzioni olomorfe di una variabile bicomplessa, *Rend. Reale Accad. d'Italia Mem. Cl. Sci. Fis. Mat. Nat.* **5** (1934), 597–665.

[18] C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici,* Math. Ann., **40**(1) (1892), 413–467. https://doi.org/10.1007/BF01443589

[19] N. Spampinato, *Sulla rappresentazione di funzioni di variabile bicomplessa totalmente derivabili,* Ann. Mat. Pura Appl., **14**(1) (1936), 305–325. https://doi.org/10.1007/BF02413222

[20] E. Toksoy, C. Duyar, *On geometric structure of bicomplex weighted BC-modules $l_p^k(BC)$,* in: Explorations in Mathematical Analysis: A Collection of Diverse Papers, Ch. 4, BIDGE Publ., 2023.

[21] B.C. Tripathy, M. Sen, *On a new class of sequences related to the ℓ^p space,* Tamkang J. Math., **33**(2) (2002), 167–171.

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