Turkish J. Ineq., 10 (1) (2026), Pages 57 –66 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

ON A NEW REFINEMENT OF CUSA-HUYGENS INEQUALITY

YOGESH. J. BAGUL[1] AND BARKAT A. BHAYO[2]

Abstract. In this paper, we propose a refinement of the famous Cusa-Huygens inequality as well as its hyperbolic version by establishing the following inequalities:

$$\left(\frac{2+\cos x}{3}\right)e^{-\alpha x^4} < \frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)e^{-\beta x^4}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\left(\frac{2+\cosh x}{3}\right)e^{-\beta x^4} < \frac{\sinh x}{x} < \left(\frac{2+\cosh x}{3}\right)e^{-\gamma x^4}, \quad 0 < x < c,$$

where $\alpha = 16\ln(\pi/3)/\pi^4 \approx 0.0076$, $\beta = 1/180 \approx 0.0056$ and $\gamma = \frac{\ln[c(2+\cosh c)/(3\sinh c)]}{c^4}$ are the best possible constants.

1. Introduction

In the recent decades, the refinements of the inequalities involving trigonometric functions have been studied extremely by numerous authors, e.g., see [8, 9, 12, 28, 29, 32, 33] and the references therein. In these papers, authors were establishing the sharp upper and lower bounds for the function $(\sin x)/x$, and refining some well-known inequalities from the literature. Motivated by these rapid studies, in this paper we make a contribution to the subject by providing a new refinement of the Cusa-Huygens inequality.

The inequalities

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2+\cos x}{3}, \tag{1.1}$$

hold true for $0 < |x| < \pi/2$. The first inequality in (1.1) is due to Adamović and Mitrinović [25, p. 238], and the second inequality was discovered by German philosopher Cusa (1401 − 1464) by using a certain geometrical construction. A rigorous proof of the second inequality was given by Huygens [17] in 1664 while considering the estimation of π. In literature the first inequality in (1.1) is known as Adamović-Mitrinović inequality, and the second inequality is known as Cusa-Huygens inequality [28, 29].

Key words and phrases. Cusa-Huygens inequality, elementary functions, l'Hôpital's rule, inequalities.

2010 *Mathematics Subject Classification.* Primary: 26D05, 26D07. Secondary: 26D99, 33B10.

Received: 03/11/2025 *Accepted*: 04/04/2026.

Cited this article as: Y.J. Bagul, B.A. Bhayo, On a new refinement of Cusa-Huygens inequality, Turkish Journal of Inequalities, 10(1) (2026), 57-66.

The hyperbolic version of (1.1) is

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{2+\cosh x}{3}\,, \tag{1.2}$$

for $x \neq 0$. The first inequality in (1.2) was obtained by Lazarević [25, p. 270], and second inequality appeared in [29].

In highly cited paper [28], Klén et al. refined the inequalities in (1.1) by showing the monotonicty of the function $p \mapsto (\cos px)^{1/p}$ as follows:

$$\cos^2 \frac{x}{2} < \frac{\sin x}{x} < \cos^3 \frac{x}{2} < \frac{2+\cos x}{3}\,, \tag{1.3}$$

for $x \in (-\sqrt{27/5}, \sqrt{27/5})$.

The following improvement of (1.3) appeared in [20, 29],

$$\cos^{4/3} \frac{x}{2} < \frac{\sin x}{x} < \cos^m \frac{x}{2}, \quad 0 < x < \frac{\pi}{2}\,,$$

where $m = 2(\ln \pi - \ln 2)/\ln \approx 1.303$. A very strong improvement of (1.3) was given by Yang [32], as follows:

$$\left(\cos \frac{x}{\sqrt{5}}\right)^p < \frac{\sin x}{x} < \left(\cos \frac{x}{\sqrt{5}}\right)^{5/3}, \quad 0 < x < \frac{\pi}{2},$$

where $p = \ln(2/\pi)/\cos(\sqrt{5}\pi/10) \approx 1.6714$.

The inequalities involving trigonometric and hyperbolic functions have been considered by many researchers for various purposes. We refer the reader to [2–6, 10–12, 14, 22–24, 26, 29–31, 33–35] for the extensions, refinements, generalizations, and applications of inequalities in (1.1) and (1.2).

In 2021, Bagul et al. [3] studied that for what functions $\phi(x)$ and $\psi(x)$, the following inequalities

$$\frac{2+\cos x}{3} - \phi(x) < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \psi(x)\,,$$

hold true for $x \in (0, \pi/2)$? As an answer they established the following inequalities:

$$\frac{2+\cos x}{3} - \left(\frac{2}{3} - \frac{2}{\pi}\right)\phi_1(x) < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \left(\frac{2}{3} - \frac{2}{\pi}\right)\phi_2(x) \tag{1.4}$$

and

$$\frac{2+\cos x}{3} - \left(\frac{2}{3} - \frac{2}{\pi}\right)\psi_1(x) < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \left(\frac{2}{3} - \frac{2}{\pi}\right)\psi_2(x), \tag{1.5}$$

hold for $x \in (0, \pi/2)$, where

$$\phi_1(x) = \frac{x - \sin x}{\pi/2 - 1}, \;\; \phi_2(x) = \left(\frac{x - \sin x}{\pi/2 - 1}\right)^2$$

and

$$\psi_1(x) = \sin x - x\cos x, \;\; \psi_2(x) = (\sin x - x\cos x)^2.$$

The following inequality

$$\phi_1(x) = \frac{x - \sin x}{\pi/2 - 1} < (\sin x - x\cos x)^2 = \psi_2(x) \tag{1.6}$$

can be written as

$$\frac{2+(\pi-2)\cos x}{\pi}=\frac{\cos x+\pi/(\pi-2)-1}{\pi/(\pi-2)}<\frac{\sin x}{x},\quad -\frac{\pi}{2}<x<\frac{\pi}{2}$$

which holds true by [9, Theorem 1].

Now it is easy to observe that inequality (1.6) implies that the first inequality in (1.4) improves the first inequality in (1.5), and the second inequality in (1.5) is improving the second inequality in (1.4). So from (1.4) and (1.5) we conclude the following refined inequalities

$$\frac{2+\cos x}{3}-\left(\frac{2}{3}-\frac{2}{\pi}\right)\phi_1(x)<\frac{\sin x}{x}<\frac{2+\cos x}{3}-\left(\frac{2}{3}-\frac{2}{\pi}\right)\psi_2(x). \tag{1.7}$$

In 2021, Zhu [33] drew two conclusions about the improvement of inequality (1.1):

$$\frac{\sin x}{x}<\frac{2+\cos x}{3}-\frac{1}{180}x^4\left(\frac{\sin x}{x}\right)^{2/7}, \tag{1.8}$$

$$\frac{\sin x}{x}<\frac{2+\cos x}{3}-\left(\frac{2}{3}-\frac{2}{\pi}\right)(\sin x-x\cos x)^{c_1}, \tag{1.9}$$

for $0<x<\pi/2$, with $c_1=(\pi^2-12)/(3\pi^2-\pi^3)\approx 1.5244$. Inequality (1.8) improves (1.9) for $x\in(0,\,1.0399)$. Clearly, inequality (1.9) is better than the second inequality in (1.7).

In our discussion we will also consider the following simple refined version of the Cusa-Huygens inequality

$$\frac{2+\cos x}{3}-\left(\frac{x}{\pi}\right)^3<\frac{\sin x}{x}<\frac{2+\cos x}{3}-\left(\frac{x}{\pi}\right)^4,\ 0<x<\frac{\pi}{2} \tag{1.10}$$

which was established by Bhayo and Sándor [8] in 2016. The second inequality in (1.10) improves (1.8) and (1.9) for $x\in(1.22,\,\pi/2)$ and $x\in(0,\,0.4422)$, respectively. Moreover, the first inequality in (1.10) is weaker than the first inequality in (1.7).

Motivated by the work of Zhu [33], we give a new refinement of (1.1) by finding the functions $j_1(x)$ and $j_2(x)$ such that for all $x\in(-\pi/2,\,\pi/2)$ the following inequalities

$$\left(\frac{2+\cos x}{3}\right)j_1(x)<\frac{\sin x}{x}<\left(\frac{2+\cos x}{3}\right)j_2(x)$$

hold true.

Now we are in the position to state our first main result, which reads as follows.

Theorem 1.1. *For $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, the following inequalities*

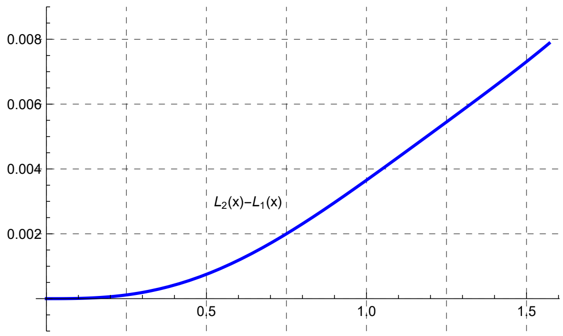
$$\left(\frac{2+\cos x}{3}\right)e^{-\alpha x^4}\le\frac{\sin x}{x}\le\left(\frac{2+\cos x}{3}\right)e^{-\beta x^4}, \tag{1.11}$$

hold true, where $\alpha=16\ln(\pi/3)/\pi^4\approx 0.0076$ and $\beta=1/180\approx 0.0056$ are the best possible constants.

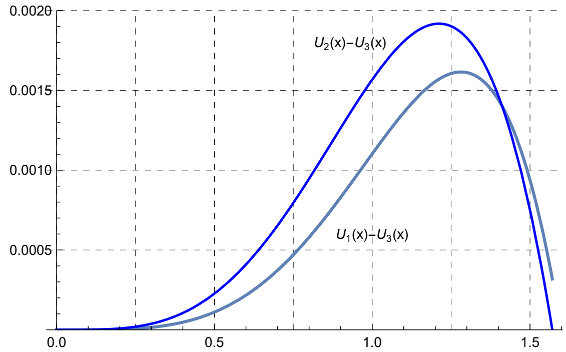
Moreover, for a real number $a<\pi$ and $0<x<a<\pi$, we have

$$\left(\frac{2+\cos x}{3}\right)e^{-\gamma_1 x^4}<\frac{\sin x}{x}<\left(\frac{2+\cos x}{3}\right)e^{-\beta x^4} \tag{1.12}$$

where $\gamma_1=\frac{\ln[\alpha(2+\cos a)/(3\sin a)]}{a^4}$.



(a)



(b)

FIGURE 1. Graphs of the lower and upper bounds of $\frac{\sin x}{x}$ for $x \in (0, \pi/2)$.

Remark 1. In (1.11), for values $x \in [-\pi/2, \pi/2]$ the difference between the function and the lower bound is less than 0.0016 and the difference between the function and the upper bound is less than 0.0079.

We denote the lower bounds in (1.7) and (1.11) by $L_1(x)$ and $L_2(x)$, respectively. One can prove that $L_1(x) < L_2(x)$, for $x \in (0, \pi/2)$, but we are not able to prove that, so we show in Figure 1(a) that $0 < L_2(x) - L_1(x)$ for $x \in (0, \pi/2)$.

Similarly, we denote the upper bounds in (1.8) , (1.9) and (1.11) by $U_1(x)$, $U_2(x)$ and $U_3(x)$, respectively. It is easy to see from then from Figure (1)(b), that bound $U_3(x)$ is better than $U_1(x)$ and $U_2(x)$.

In our next theorem we present the hyperbolic variant of the first result.

Theorem 1.2. *For $0 < x < c < \infty$, the following inequalities hold*

$$\left(\frac{2+\cosh x}{3}\right) e^{-\beta x^4} < \frac{\sinh x}{x} < \left(\frac{2+\cosh x}{3}\right) e^{-\gamma_2 x^4}, \tag{1.13}$$

where $\beta = 1/180 \approx 0.0056$ and $\gamma_2 = \frac{\ln[\alpha(2+\cosh c)/(3\sinh c)]}{c^4}$ are the best possible constants.

2. LEMMAS AND PROOFS OF THEOREMS

In this section we recall few lemmas from the literature which will be used as a main tool in the proofs of theorems. The following lemma was originally proved in [27], but can also be found in [16].

Lemma 2.1. *Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$ be any two real series converging on the interval $(-R, R)$, where $R > 0$ and $b_k > 0$ for all k. Then the function $A(x)/B(x)$ is increasing (decreasing) on $(0, R)$ if the sequence $\{a_k/b_k\}$ is increasing (decreasing).*

For $|x| < \pi$, the following power series expansions can be found in [15, 1.411] and [19, 1.3, 1.4].

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \tag{2.1}$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \tag{2.2}$$

$$\frac{x}{\sin x} = 1 + \sum_{k=1}^{\infty} \frac{2(2^{2k-1}-1)}{(2k)!} |B_{2k}| x^{2k};\ |x| < \pi, \tag{2.3}$$

and

$$x \cot x = 1 - \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} |B_{2k}| x^{2k},\ |x| < \pi. \tag{2.4}$$

where B_{2n} are the even-indexed Bernoulli numbers, see [18, p. 231].

Lemma 2.2. [1]*(l'Hôpital Monotone Rule) Suppose a and b are any two real numbers such that $a < b$. Let $f(x)$ and $g(x)$ be two real-valued functions which are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$, for all $x \in (a, b)$. Let,*

$$A(x) = \frac{f(x) - f(a)}{g(x) - g(a)}, \quad B(x) = \frac{f(x) - f(b)}{g(x) - g(b)}.$$

Then, the functions $A(x)$ and $B(x)$ are increasing (decreasing) on (a, b) if $f'(x)/g'(x)$ is increasing (decreasing) on (a, b). The strictness of the monotonicity of $A(x)$ and $B(x)$ depends on the strictness of the monotonicity of $f'(x)/g'(x)$.

Lemma 2.3. *For $k = 3, 4, 5, \cdots$, it holds that*

$$\frac{1}{2^{2k}+2} \leq |B_{2k}|.$$

Proof. The lower bound for $|B_{2k}|$ is obtained in [13] as follows:

$$\frac{2 \cdot (2k)!}{\pi^{2k}(2^{2k}-1)} < |B_{2k}|,\ k = 1, 2, 3, \cdots.$$

We get the required inequality if

$$2 \cdot (2k)!(2^{2k}+2) > \pi^{2k}(2^{2k}-1),$$

which is true as $2 \cdot (2k)! > \pi^{2k}$ for $k = 3, 4, 5, \cdots$. □

Lemma 2.4. *The following function*

$$f(x) = \frac{\ln[x(2+\cos x)/(3\sin x)]}{x^4}$$

is strictly increasing from $(0, \pi)$ onto (β, ∞).

Proof. Write

$$f(x) = \frac{\ln[x(2+\cos x)/(3\sin x)]}{x^4} = \frac{f_1(x)}{f_2(x)},$$

where

$$f_1(x) = \ln\left[\frac{x(2+\cos x)}{3\sin x}\right] \quad \text{and} \quad f_2(x) = x^4\,.$$

Clearly $f_1(0+) = 0 = f_2(0)$. Differentiation with respect to x and simple calculations yield

$$\frac{f_1'(x)}{f_2'(x)} = \frac{2\sin x + \sin x\cos x - x\sin^2 x - 2x\cos x - x\cos^2 x}{4x^4 \sin x(2+\cos x)} = \frac{1}{4}h_1(x)h_2(x),$$

where $h_1(x) = 1/(2+\cos x)$ and

$$\begin{aligned} h_2(x) &= \frac{2\sin x + \sin x\cos x - x - 2x\cos x}{x^4 \sin x} \\ &= \frac{1}{x^4}\left(2 + \cos x - \frac{x}{\sin x} - 2x\cot x\right). \end{aligned}$$

Clearly, h_1 is positive and increasing in $(0,\,\pi)$. Proving the monotonicity of h_2, we utilize (2.1), (2.3), and (2.4), and get

$$\begin{aligned} h_2(x) &= \frac{1}{x^4}\left[2 + \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} - 1 - \sum_{k=1}^{\infty}\frac{2(2^{2k-1}-1)}{(2k)!}|B_{2k}|x^{2k} - 2 + \sum_{k=1}^{\infty}\frac{2^{2k+1}}{(2k)!}|B_{2k}|x^{2k}\right] \\ &= \frac{1}{x^4}\sum_{k=1}^{\infty}\left[2^{2k+1}|B_{2k}| + (-1)^k - 2(2^{2k-1}-1)|B_{2k}|\right]\frac{1}{(2k)!}x^{2k} \\ &= \sum_{k=1}^{\infty}\left[(2^{2k}+2)|B_{2k}| + (-1)^k\right]\frac{1}{(2k)!}x^{2k-4} \\ &= \sum_{k=2}^{\infty}P_k x^{2k-4}, \quad \text{where } P_k = \frac{\left[(2^{2k}+2)|B_{2k}| + (-1)^k\right]}{(2k)!}. \end{aligned}$$

Now, $P_k > 0$ for $k = 2, 4, 6, \cdots$ and $P_k > 0$ for $k = 3, 5, 7, \cdots$ by Lemma 2.3. Hence $h_2(x)$ is also positive and increasing in $(0, \pi)$ by Lemma 2.1. Therefore, $f_1'(x)/f_2'(x)$ is strictly increasing and consequently, $f(x) = f_1(x)/f_2(x)$ is strictly increasing in $(0, \pi)$ by Lemma 2.2. Applying the l'Hôpital rule, we get $\lim_{x\to 0} f(x) = 1/180 = \beta$, and $\lim_{x\to\pi} f(x) = \infty$. This completes the proof. ∎

Lemma 2.5. *The following function*

$$g(x) = \frac{\ln[(x(2+\cosh x))/(3\sinh x)]}{x^4}$$

is strictly decreasing from $(0,\,\infty)$ *onto* $(\beta,\,0)$.

Proof. Write

$$g(x) = \frac{\ln[(x(2+\cosh x))/(3\sinh x)]}{x^4} = \frac{g_1(x)}{g_2(x)},$$

where

$$g_1(x) = \ln\left[\frac{x(2+\cosh x)}{3\sinh x}\right] \quad \text{and} \quad g_2(x) = x^4\,.$$

Clearly, $g_1(0) = g_2(0) = 0$. Differentiating g_1 and g_2 with respect to x, we obtain

$$\begin{aligned}\frac{g_1'(x)}{g_2'(x)} &= \frac{2\sinh x + \sinh x \cosh x + x\sinh^2 x - 2x\cosh x - x\cosh^2 x}{4x^4 \sinh x(2+\cosh x)} \\ &= \frac{1}{4} \cdot \frac{2\sinh x + \sinh x \cosh x - x - 2x\cosh x}{2x^4 \sinh x + x^4 \sinh x \cosh x} \\ &= \frac{1}{4} \frac{2\sinh x + \frac{1}{2}\sinh 2x - x - 2x\cosh x}{2x^4\sinh x + \frac{1}{2}x^4 \sinh 2x}.\end{aligned}$$

Using (2.2), we get

$$\begin{aligned}\frac{g_1'(x)}{g_2'(x)} &= \frac{\sum_{k=0}^{\infty} \frac{2\cdot x^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{2^{2k}x^{2k+1}}{(2k+1)!} - x - \sum_{k=0}^{\infty} \frac{2\cdot x^{2k+1}}{(2k)!}}{\sum_{k=0}^{\infty} \frac{2\cdot x^{2k+5}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{2^{2k}x^{2k+5}}{(2k+1)!}} \\ &= \frac{\sum_{k=1}^{\infty} \frac{[2^{2k}+2-2(2k+1)]}{(2k+1)!} x^{2k+1}}{\sum_{k=0}^{\infty} \frac{(2^{2k}+2)}{(2k+1)!} x^{2k+5}} \\ &= \left[\sum_{k=2}^{\infty} \frac{(2^{2k}-4k)}{(2k+1)!} x^{2k+1}\right] \Big/ \left[\sum_{k=0}^{\infty} \frac{(2^{2k}+2)}{(2k+1)!} x^{2k+5}\right] \\ &= \left[\sum_{k=0}^{\infty} \frac{(2^{2k+4}-4k-8)}{(2k+5)!} x^{2k+5}\right] \Big/ \left[\sum_{k=0}^{\infty} \frac{(2^{2k}+2)}{(2k+1)!} x^{2k+5}\right] \\ &= \frac{\sum_{k=0}^{\infty} a_k x^{2k+5}}{\sum_{k=0}^{\infty} b_k x^{2k+5}},\end{aligned}$$

where

$$\frac{a_k}{b_k} = \frac{(2^{2k+2}-k-2)}{(k+1)(k+2)(2k+3)(2k+5)(2^{2k}+2)} = A_k.$$

We prove that A_k is decreasing for $k = 0, 1, 2, \cdots$. In particular, $A_0 = 1/45$, $A_1 = 13/1260$, $A_2 = 5/1134$ implying that $A_0 > A_1 > A_2$. We write

$$A_k = B_k \cdot C_k,$$

where

$$B_k = \frac{1}{(k+2)(2k+3)(2k+5)}, \quad C_k = \frac{(2^{2k+2}-k-2)}{(k+1)(2^{2k}+2)}.$$

Clearly, B_k is positive decreasing for $k = 0, 1, 2, \cdots$. For the monotonicity of C_k, we need to prove that

$$C_k > C_{k+1} = \frac{(2^{2k+4}-k-3)}{(k+2)(2^{2k+2}+2)}.$$

This is equivalent to

$$(k+2)(2^{2k+2}+2)(2^{2k+2}-k-2) > (k+1)(2^{2k}+2)(2^{2k+4}-k-3)$$

$\Longleftrightarrow$

$$(k+2)(16\cdot 2^{4k}-4k\cdot 2^{2k}-2k-4) > (k+1)(16\cdot 2^{4k}-k\cdot 2^{2k}+29\cdot 2^{2k}-2k-6)$$

$\Longleftrightarrow$

$$16\cdot 2^{2k}\cdot 2^{2k} > (3k^2+36k+29)2^{2k}+2$$

which is true for $k=2,3,4,\cdots$. Thus C_k is also positive and decreasing for $k=2,3,4,\cdots$. Consequently, A_k is decreasing. By Lemma 2.2, $g_1'(x)/g_2'(x)$ is decreasing, this implies that $g(x)$ is strictly decreasing for $x>0$. Applying L'Hôpital rule, we get $\lim_{x\to 0} g(x) = 1/180$ and $\lim_{x\to\infty} g(x)=0$. This completes the proof of lemma.

□

Proof of Theorem 1.1 and 1.2. By Lemma 2.4 the function $f(x)$ is strictly increasing in $(0,\,\pi)$ and we have

$$\beta = \frac{1}{180} = \lim_{x\to 0} f(x) < f(x) < \lim_{x\to\pi/2} f(x) = 16\ln(\pi/3)/\pi^4 = \alpha,$$

and

$$\beta = \frac{1}{180} = \lim_{x\to 0} f(x) < f(x) < \lim_{x\to a} f(x) = f(a).$$

This implies the proof of Theorem 1.1. Similarly, Lemma 2.5 gives

$$\beta = \frac{1}{180} = \lim_{x\to 0} g(x) > g(x) > \lim_{x\to c} g(x) = g(c),$$

and this implies the proof of Theorem 1.2. □

Remark 2.1. Theorems 1.1 and 1.2 can also be proved and the corresponding best possible constants can be determined with the help of techniques given by Malešević et. al. in a recently published paper [21].

The proof of the following corollary follows immediately from (1.11) and (1.13).

Corollary 2.1. *The following inequalities*

$$3e^{\beta x^4} \leq \frac{x}{\tan x} + 2\frac{x}{\sin x} \leq 3e^{\alpha x^4},\ \ x\in[-\pi/2,\,\pi/2],$$

and

$$\frac{x}{\tanh x} + 2\frac{x}{\sinh x} \leq 3e^{\beta x^4},\ \ x>0$$

hold true, where $\alpha = 16\ln(\pi/3)/\pi^4 \approx 0.0076$ *and* $\beta = 1/180 \approx 0.0056$.

We finish the paper by posing the following conjecture, which gives a new refinement of Cusa-Huygens inequality.

Conjecture. For $x\in(-\pi,\,\pi)$, the following inequalities

$$\frac{3(1-\xi_1 x^4)}{2+\cosh x} < \frac{\sin x}{x} < \frac{3(1-\xi_2 x^4)}{2+\cosh x}$$

hold true, where $\xi_1 = 1/\pi^4 \approx 0.01027$ and $\xi_2 = \beta = 1/180$ are the best possible constants.

Acknowledgements. The authors are indebted to the anonymous referee for their valuable comments, which have significantly improved the clarity of the paper.

References

[1] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, *Monotonicity rules in calculus,* Amer. Math. Monthly., **133**(9) (2006), 805–816. https://doi.org/10.1080/00029890.2006.11920367
[2] Y.J. Bagul, *Remark on the paper of Zheng Jie Sun and Ling Zhu*, J. Math. Ineqal., **13**(3) (2019), 801–803. http://dx.doi.org/10.7153/jmi-2019-13-55
[3] Y.J. Bagul, B. Banjac, C. Chesneau, M. Kostić, B. Malešević, *New refinements of Cusa-Huygens inequality,* Results Math., **76** (2021), 107. https://doi.org/10.1007/s00025-021-01392-8
[4] Y.J. Bagul, C. Chesneau, *Refined forms of Oppenheim and Cusa-Huygens type inequalities*, Acta Comment. Univ. Tartu. Math., **24**(2) (2020), 183–194. https://doi.org/10.12697/ACUTM.2020.24.12
[5] Y.J. Bagul, C. Chesneau, M. Kostić, *On the Cusa-Huygens inequality,* RACSAM., **115** (2021), 29. https://doi.org/10.1007/s13398-020-00978-1
[6] Y.J. Bagul, C. Chesneau, M. Kostić, *The Cusa-Huygens inequality revisited,* Novi Sad J. Math., **50**(2) (2020), 149–159. https://doi.org/10.30755/NSJOM.10667
[7] Y.J. Bagul, C. Chesneau, *Some sharp circular and hyperbolic bounds of exp*$(-x^2)$ *with applications,* Appl. Anal. Discrete Math., **14**(1) (2020), 239–254. https://doi.org/10.2298/AADM190123010B
[8] B.A. Bhayo, J. Sándor, *On Jordan's, Redheffer's and Wilker's inequality*, Math. Ineq. Appl., **19**(3) (2016), 823–839, dx.doi.org/10.7153/mia-19-60.
[9] B.A. Bhayo, R. Klén, J. Sándor, *New trigonometric and hyperbolic inequalities*, Miskolc Math. Notes, **18**(1) (2017), 125–137, https://doi.org/10.18514/MMN.2017.1560
[10] B. Chaouchi, V. E. Fedorov, M. Kostić, *Monotonicity of certain classes of functions related with Cusa-Huygens inequality,* Chelyab. Fiz.-Mat. Zh., **6**(3) (2021), 331–337. https://doi.org/10.47475/2500-0101-2021-16307
[11] C.-P. Chen, W.-S. Cheung, *Sharp Cusa and Becker-Stark inequalities*, J. Inequal. Appl., **2011**, Article no. 136, 2011. https://doi.org/10.1186/1029-242X-2011-136
[12] C.-P. Chen, J. Sándor, *Inequality chains for Wilker, Huygens and Lazarević type inequalities,* J. Math. Inequal., **8**(1) (2014), 55–67. http://dx.doi.org/10.7153/jmi-08-02
[13] C. D'Aniello, *On some inequalities for the Bernoulli numbers,* Rend. Circ. Mat. Palermo (2), **43**(3) (1994), 329–332. https://doi.org/10.1007/BF02844246
[14] R.M. Dhaigude, C. Chesneau, Y.J. Bagul, *About trigonometric-polynomial bounds of sinc function,* Math. Sci. Appl. E-Notes, **8**(1) (2020), 100–104. https://doi.org/10.36753/mathenot.585735
[15] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products,* Elsevier, edn. 2007.
[16] V. Heikkala, M.K. Vamanamurthy, M. Vuorinen, *Generalized elliptic integrals,* Comput. Methods Funct. Theory, **9**(1) (2009), 75–109. https://doi.org/10.1007/BF03321716
[17] C. Huygens, *Oeuvres Completes*, Société Hollandaise des Sciences, Haga, 1888–1940.
[18] K. Ireland, M. Rosen, *A classical introduction to modern number theory*, 2nd ed., ser. Graduate Texts in Mathematics. Springer-Verlag, New York, **84** (1990). http://dx.doi.org/10.1007/978-1-4757-2103-4.doi:10.1007/978-1-4757-2103-4
[19] A. Jeffrey, H.-H. Dai, *Handbook of mathematical formulas and integrals*, 4th ed. Elsevier Academic Press, 2008.
[20] Y.-P. Lv, G.-D. Wang, Y.-M. Chu, *A note on Jordan type inequalities for hyperbolic functions*, Appl. Math. Lett., **25**(3) (2012), 505–508. https://doi.org/10.1016/j.aml.2011.09.046
[21] B. Malešević, M. Mićović, B. Mihailović, T. Lutovac, *A new approach to selecting constants for some analytic inequalities,* J. Math. Inequal., **19**(1) (2025), 223–245. http://dx.doi.org/10.7153/jmi-2025-19-15
[22] B. Malešević, M. Nenezić, L. Zhu, B. Banjac, M. Petrović, *Some new estimates of precision of Cusa-Huygens and Huygens approximations,* Appl. Anal. Discrete Math., **15**(1) (2021), 243–259. http://dx.doi.org/10.2298/AADM190904055M

[23] B. Malešević, T. Lutovac, M. Rašajski, C. Mortici, *Extensions of the natural approach to refinements and generalizations of some trigonometric inequalities*, Adv. Difference Equ., **2018**, 2018:90. https://doi.org/10.1186/s13662-018-1545-7

[24] B. Malešević, M. Makragić, *A Method for Proving Some Inequalities on Mixed Trigonometric Polynomial Functions*, J. Math. Inequal., **10**(3) (2016), 849–876. http://dx.doi.org/10.7153/jmi-10-69

[25] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[26] C. Mortici, *The natural approach of Wilker-Cusa-Huygens Inequalities*, Math. Inequal. Appl., **14**(3) (2011), 535–541. https://dx.doi.org/10.7153/mia-14-46

[27] Q. I. Rahman, *On the monotonicity of certain functionals in the theory of analytic functions*, Canad. Math. Bull., **10** (1967), 723–729.

[28] R. Klén, M. Visuri, M. Vuorinen, *On Jordan type inequalities for hyperbolic functions*, J. Inequal. Appl., **2010** (2010), Art. ID 362548, 14 pages. https://doi.org/10.1155/2010/362548

[29] E. Neuman, J. Sándor, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities*, Math. Inequal. Appl., **13**(4) (2010), 715–723. https://dx.doi.org/10.7153/mia-13-50

[30] J. Sándor, *Sharp Cusa-Huygens and related inequalities*, Notes on Number Theory and Discrete Mathematics, **19**(1) (2013), 50–54.

[31] J. Sándor, R. Oláh-Gal, *On Cusa-Huygens type trigonometric and hyperbolic inequalities*, Acta. Univ. Sapientiae Mathematica **4**(2) (2012), 145–153.

[32] Z.-H. Yang, *Refinements of a two-sided inequality for trigonometric functions*, J. Math. Ineq., **7**(4) (2013), 601–615. http://dx.doi.org/10.7153/jmi-07-57

[33] L. Zhu, *New inequalities of Cusa-Huygens type,* Mathematics, **9**(17) (2021), 2101. https://doi.org/10.3390/math9172101

[34] L. Zhu, *New bounds for the sine function and tangent function,* Mathematics, **9**(19) (2021), 2373. https://doi.org/10.3390/math9192373

[35] L. Zhu, *New Cusa-Huygens type inequalities,* AIMS Mathematics, **5**(5) (2020), 5320–5331. https://doi.org/10.3934/math.2020341

[1]Department of Mathematics,
K. K. M. College, Manwath,
Parbhani (M. S.) - 431505, India
Email address: yjbagul@gmail.com

[2]School of Energy Systems,
Lappeenranta-Lahti University of Technology,
15210 Lahti, Finland
Email address: barkat.bhayo@lut.fi