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ESTIMATION OF COEFFICIENTS FOR A CLASS OF BI-UNIVALENT FUNCTIONS WITH BOUNDED BOUNDARY ROTATION THROUGH THE APPLICATION OF THE FABER POLYNOMIAL

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Abstract. In this article, we introduce a new subclass, referred to as $\mathcal{N}_{\Sigma}^{\alpha,\beta}[\vartheta, m, \eta]$, of bi-univalent function with bounded boundary rotation. Through the application of Faber polynomials, we investigate the coefficient estimate $|c_n|$ for the functions in this subclass. Additionally, we derive the initial estimates for the coefficients $|c_2|$ and $|c_3|$, and we calculate the Fekete-Szegö functional $|c_3 - \aleph c_2^2|$ for this subclass. The results presented in this paper would improve some recent works of several earlier authors.

1. Introduction

Let $\mathcal{A}$ illustrate the family of functions γ that are analytic within the open unit disk $\mathbb{E} := \{\nu \in \mathbb{C} : |\nu| < 1\}$. Consequently, let $\mathcal{S}$ indicate the subclass of functions within $\mathcal{A}$ that are univalent in $\mathbb{E}$, normalized such that $\gamma(0) = 0$ and $\gamma'(0) = 1$, and can be expressed as:

$$\gamma(\nu) = \nu + \sum_{n=2}^{\infty} c_n \nu^n. \tag{1.1}$$

Two significant and thoroughly researched subclasses of the analytic and univalent function class $\mathcal{S}$ are the class $\mathcal{S}^*(\eta)$ of starlike functions of order η in $\mathbb{E}$ and the class $\mathcal{C}(\eta)$ of convex functions of order η in $\mathbb{E}$. According to the definition, we have

$$\mathcal{S}^*(\eta) := \left\{\gamma \in \mathcal{S} : \Re\left(\frac{\nu\gamma'(\nu)}{\gamma(\nu)}\right) > \eta, \quad 0 \le \eta < 1\right\}$$

and

$$\mathcal{C}(\eta) := \left\{\gamma \in \mathcal{S} : \Re\left(1 + \frac{\nu\gamma''(\nu)}{\gamma'(\nu)}\right) > \eta, \quad 0 \le \eta < 1\right\}.$$

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It is a well-known [12] fact that every function $\gamma \in \mathcal{S}$ has an inverse $\xi = \gamma^{-1}$, which satisfies the equations $\xi(\gamma(\nu)) = \nu$ and $\gamma(\xi(u)) = u$ (where $\nu \in \mathbb{E}$ and $|u| \le \rho : \rho \ge 1/4$) and is defined by

$$\gamma^{-1}(u) = \xi(u) = u - c_2u^2 + (2c_2^2 - c_3)u^3 - (5c_2^3 - 5c_2c_3 + c_4)u^4 + \cdots . \tag{1.2}$$

Let γ be a function in $\mathcal{S}$, and its inverse ξ also belonging to $\mathcal{S}$; then, γ is referred to as bi-univalent in $\mathbb{E}$ and is denoted by Σ. Subclasses of Σ include bi-starlike (bi-convex) of order η $(0 \le \eta < 1)$, which have been established by Brannan and Taha [6]. These classes provide non-sharp coefficient estimates $|c_2|$ and $|c_3|$. For further information, refer to [32]; additionally, consider some of the recent studies [2,3,7–9,11,19,22–25,27–31,34]. Nevertheless, the n^{th} Taylor–Maclaurin coefficients $|c_n|$ $(n \in \{3, 4, \dots\})$ remain an unresolved issue. Lewin [20] investigated the class Σ of bi-univalent functions and derived the bound for the second coefficient. Additionally, Lewin [20] established that $|c_2| \le 1.51$. Furthermore, Brannan and Clunie conjectured [5] that $|c_2| \le \sqrt{2}$.

The Faber polynomials, as introduced by Faber [13], hold significant importance in numerous fields of mathematical sciences, particularly in the realm of geometric function theory. In the existing literature, only a limited number of studies have established the general coefficient bounds $|c_n|$ for analytic bi-univalent functions as defined by (1.1) through the use of Faber polynomial expansions (refer to, for instance, [15,17,18]). With the Faber polynomial expansion applied to functions $\gamma \in \mathcal{A}$ of the structure (1.1), the coefficients of the inverse mapping $\xi = \gamma^{-1}$ may be detailed as per [1]:

$$\begin{aligned}\xi(u) = \gamma^{-1}(u) &= u + \sum_{n=2}^{\infty} C_n u^n \\ &= u + \sum_{n=2}^{\infty} \frac{1}{n}\mathcal{F}_{n-1}^{-n}(c_2, c_3, \dots, c_n)u^n,\end{aligned}$$

where

$$\begin{aligned}\mathcal{F}_{n-1}^{-n} = &\frac{(-n)!}{(1-2n)!(n-1)!}c_2^{n-1} + \frac{(-n)!}{(2(1-n))!(-3+n)!}c_2^{n-3}c_3 \\ &+ \frac{(-n)!}{(3-2n)!(n-4)!}c_2^{n-4}c_4 \\ &+ \frac{(-n)!}{(2(2-n))!(n-5)!}c_2^{n-5}\left[c_5 + (2-n)c_3^2\right] \\ &+ \frac{(-n)!}{(5-2n)!(n-6)!}c_2^{n-6}\left[c_6 + (5-2n)c_2c_4\right] + \sum_{r=1}^{\infty} c^{n-r}\mathcal{Z}_r,\end{aligned}$$

and $\mathcal{Z}_r$ for $j \in [7, n]$ in a homogeneous polynomial in the variable $c_2, c_3, \dots, c_n$. The first three terms of $\mathcal{F}_{n-1}^{-n}$ are,

$$\begin{aligned}\mathcal{F}_1^{-2} &= -2c_2, \\ \mathcal{F}_2^{-3} &= 3(2c_2^2 - c_3)\end{aligned}$$

and

$$\mathcal{F}_3^{-4} = -4(5c_2^3 - 5c_2c_3 + c_4).$$

In most cases, the expansion for $\mathcal{F}_n^a$ is provided by

$$\mathcal{F}_n^a = ac_n + \frac{a(a-1)}{2}\mathcal{V}_n^2 + \frac{a!}{(a-3)!3!}\mathcal{V}_n^3 + \ldots + \frac{a!}{(a-n)!n!}\mathcal{V}_n^n$$

where $\mathcal{V}_n^a = B_n^a(c_1, c_2, \ldots, c_n)$ and

$$B_n^q(c_1, c_2, \ldots, c_n) = \sum_{n=1}^{\infty} \frac{(k)!(c_1)^{b_1}(c_2)^{b_2}\ldots(c_n)^{b_n}}{b_1!b_2!\ldots b_n!}.$$

where $c_1 = 1$ and the sum is taken over all non-negative integers $b_1!, b_2!, \ldots, b_n!$ such that

$$b_1 + b_2 + \ldots + b_n = k$$
$$b_1 + 2b_2 + \ldots + nb_n = n.$$

In 1975, Padmanabhan and Parvatham [26] established the class $\mathcal{P}_m(\eta)$. A function $\hbar$ is normalized such that $\hbar(0) = 0$ and $\hbar'(0) > 1$. It is classified under the class $\mathcal{P}_m(\eta)$ if it fulfills the condition

$$\int_0^{2\pi} \left| \frac{\Re(\hbar(\nu)) - \eta}{1-\eta} \right| ds \leq m\pi,$$

where $\nu = e^{is} \in \partial\mathbb{E}$. The function $\hbar$ belongs to the class $\mathcal{P}_m(\eta)$ if and only if there exists a non-decreasing function χ defined on the interval $[0, 2\pi]$ with

$$\int_0^{2\pi} d\chi(s) = 2 \quad \textit{and} \quad \int_0^{2\pi} |d\chi(s)| \leq m$$

satisfying

$$\hbar(\nu) = \int_0^{2\pi} \frac{1 + (1-2\eta)e^{-is}}{1 - \eta e^{-is}} d\chi(s).$$

Lemma 1.1. [4] *Should $\hbar(\nu) = 1 + \hbar_1\nu + \hbar_2\nu^2 + \hbar_3\nu^3 + \cdots$ be classified under $\mathcal{P}_m(\eta)$, then for any $n \geq 1$,*

$$|\hbar_n| \leq m(1-\eta).$$

2. Main Results

Definition 2.1. A function $\gamma \in \Sigma$, as defined by (1.1), is regarded as being in the class $\mathcal{N}_{\Sigma}^{\alpha,\beta}[\vartheta, m, \eta]$ for $\alpha \geq 1$, $\beta \geq 0$, $\vartheta \geq 0$, $2 \leq m \leq 4$ and $0 \leq \eta < 1$, if the conditions listed below hold true for every $\nu, \varsigma \in \mathbb{E}$:

$$(1-\alpha)\left(\frac{\gamma(\nu)}{\nu}\right)^{\beta} + \alpha\gamma'(\nu)\left(\frac{\gamma(\nu)}{\nu}\right)^{\beta-1} + \mu\vartheta\nu\gamma''(\nu) \in \mathcal{P}_m(\eta)$$

and

$$(1-\alpha)\left(\frac{\xi(\varsigma)}{\varsigma}\right)^{\beta} + \alpha\xi'(\varsigma)\left(\frac{\xi(\varsigma)}{\varsigma}\right)^{\beta-1} + \mu\vartheta\varsigma\xi''(\varsigma) \in \mathcal{P}_m(\eta),$$

where the function $\xi(\varsigma)$ is defined as $\gamma^{-1}(\nu)$ and $\vartheta = \dfrac{2\alpha+\beta}{2\alpha+1}$.

Remark 2.1. In the following particular scenarios of Definition 2.1; we illustrate how the class of analytic bi-univalent functions $\mathcal{N}_{\Sigma}^{\alpha,\beta}[\vartheta, m, \eta]$ with suitable selections of α, β, ϑ and m leads to both new and previously recognized classes of analytic bi-univalent functions that have been explored in the literature.
(i) When $\vartheta = 0$ and $\beta = 1$, we arrive at the bi-univalent function with bounded boundary rotation class $\mathcal{N}_{\Sigma}^{\alpha,1}[0, m, \eta] := \mathcal{G}_{\Sigma}^{\alpha}(m, \eta)$ proposed by Sharma et al. [27].

(ii) When $\vartheta = 0$, $\alpha = 1$ and $\beta = 1$, we arrive at the bi-univalent function with bounded boundary rotation class $\mathcal{N}_{\Sigma}^{1,1}[0, m, \eta] := \mathcal{H}_{\Sigma}(m, \eta)$ proposed by Li et al. [21].

(iii) When $\vartheta = 0$, $m = 2$ and $\beta = 1$, we arrive at the bi-univalent function class $\mathcal{N}_{\Sigma}^{\alpha,1}[0, 2, \eta] := \mathcal{G}_{\Sigma}^{\alpha}(\eta)$ proposed by Frasin and Aouf [14].

(iv) When $\vartheta = 0$, $\alpha = 1$, $m = 2$ and $\beta = 1$, we arrive at the bi-univalent function class $\mathcal{N}_{\Sigma}^{1,1}[0, 2, \eta] := \mathcal{H}_{\Sigma}(\eta)$ proposed by Srivastava et al. [32].

(v) When $\vartheta = 0$, $\alpha = 1$ and $\beta = 0$, we arrive at the bi-univalent function with bounded boundary rotation class $\mathcal{N}_{\Sigma}^{1,0}[0, m, \eta] := \mathcal{S}_{\Sigma}^{*}(m, \eta)$ proposed by Li et al. [21].

(vi) When $\vartheta = 0$, $\alpha = 1$, $m = 2$ and $\beta = 0$, we derive the well-known class $\mathcal{N}_{\Sigma}^{1,0}[0, 2, \eta] := \mathcal{S}_{\Sigma}^{*}(\eta)$ of bi-starlike functions of order η.

(vii) When $\vartheta = 0$ and $m = 2$, we arrive at the bi-univalent function class $\mathcal{N}_{\Sigma}^{\alpha,\beta}[0, 2, \eta] := \mathcal{B}_{\Sigma}^{\beta}(\alpha, \eta)$ proposed by Çağlar et al. [10].

Theorem 2.1. *Given $\alpha \geq 1$, $\beta \geq 0$, $\vartheta \geq 0$, $2 \leq m \leq 4$ and $0 \leq \eta < 1$, let us define the function $\gamma \in \mathcal{N}_{\Sigma}^{\alpha,\beta}[\vartheta, m, \eta]$ as specified in (1.1). If $c_k = 0$ (where $2 \leq k \leq n-1$), then*

$$|c_n| \leq \frac{m(1-\eta)}{\beta + (n-1)\alpha + n(n-1)\vartheta\mu}, \quad n \geq 4.$$

Proof. In accordance with Definition 2.1, a function $\gamma \in \mathcal{N}_{\Sigma}^{\alpha,\beta}[\vartheta, m, \eta]$ is defined; therefore, two functions $\phi(\nu)$ and $\psi(\varsigma)$ exist within the family $\mathcal{P}_m(\eta)$. Hence, we have

$$(1-\alpha)\left(\frac{\gamma(\nu)}{\nu}\right)^{\beta} + \alpha\gamma'(\nu)\left(\frac{\gamma(\nu)}{\nu}\right)^{\beta-1} + \mu\vartheta\nu\gamma''(\nu) = \phi(\nu) \tag{2.1}$$

and

$$(1-\alpha)\left(\frac{\xi(\varsigma)}{\varsigma}\right)^{\beta} + \alpha\xi'(\varsigma)\left(\frac{\xi(\varsigma)}{\varsigma}\right)^{\beta-1} + \mu\vartheta\varsigma\xi''(\varsigma) = \psi(\varsigma), \tag{2.2}$$

where

$$\phi(\nu) = 1 + \phi_1\nu + \phi_2\nu^2 + \phi_3\nu^3 + \cdots \tag{2.3}$$

and

$$\psi(\varsigma) = 1 + \psi_1\varsigma + \psi_2\varsigma^2 + \psi_3\varsigma^3 + \cdots. \tag{2.4}$$

For functions γ represented by the equation (1.1), we can express

$$(1-\alpha)\left(\frac{\gamma(\nu)}{\nu}\right)^{\beta} + \alpha\gamma'(\nu)\left(\frac{\gamma(\nu)}{\nu}\right)^{\beta-1} + \mu\vartheta\nu\gamma''(\nu) = 1 + \sum_{n=1}^{\infty}\Gamma_{n-1}(c_2, c_3, \ldots, c_n)\nu^{n-1}, \tag{2.5}$$

where

$$\begin{aligned}\Gamma_{n-1}(c_2, c_3, \ldots, c_n) =&[\beta + (n-1)\alpha + n(n-1)\vartheta\mu] \times [(\beta-1)!]\\ &\times \left[\sum_{n=2}^{\infty} \frac{c_2^{j_1} c_3^{j_2} \ldots c_n^{j_{n-1}}}{j_1! j_2! \ldots j_n! [\beta - (j_1 + j_2 + \cdots + j_{n-1})]!}\right].\end{aligned}$$

For the inverse function $\xi = \gamma^{-1}$, taking into account (1.2), we derive

$$(1-\alpha)\left(\frac{\xi(\varsigma)}{\varsigma}\right)^{\beta} + \alpha\xi'(\varsigma)\left(\frac{\xi(\varsigma)}{\varsigma}\right)^{\beta-1} + \mu\vartheta\varsigma\xi''(\varsigma) = 1 + \sum_{n=1}^{\infty} \Gamma_{n-1}(C_2, C_3, \ldots, C_n)\varsigma^{n-1}, \quad (2.6)$$

with

$$C_n = \frac{1}{n}\mathcal{F}_{n-1}^{-n}(c_2, c_3, \ldots, c_n).$$

Alternatively,

$$(1-\alpha)\left(\frac{\gamma(\nu)}{\nu}\right)^{\beta} + \alpha\gamma'(\nu)\left(\frac{\gamma(\nu)}{\nu}\right)^{\beta-1} + \mu\vartheta\nu\gamma''(\nu) = 1 + \sum_{n=1}^{\infty} \mathcal{F}_n^1(\phi_1, \phi_2, \ldots, \phi_n)\nu^n \quad (2.7)$$

and

$$(1-\alpha)\left(\frac{\xi(\varsigma)}{\varsigma}\right)^{\beta} + \alpha\xi'(\varsigma)\left(\frac{\xi(\varsigma)}{\varsigma}\right)^{\beta-1} + \mu\vartheta\varsigma\xi''(\varsigma) = 1 + \sum_{n=1}^{\infty} \mathcal{F}_n^1(\psi_1, \psi_2, \ldots, \psi_n)\varsigma^n. \quad (2.8)$$

Evaluating the matching coefficients of (2.5) and (2.7) for any $n \geq 2$, we obtain

$$\Gamma_{n-1}(c_2, c_3, \ldots, c_n) = \mathcal{F}_n^1(\phi_1, \phi_2, \ldots, \phi_n). \quad (2.9)$$

Evaluating the matching coefficients of (2.6) and (2.8) for any $n \geq 2$, we obtain

$$\Gamma_{n-1}(C_2, C_3, \ldots, C_n) = \mathcal{F}_n^1(\psi_1, \psi_2, \ldots, \psi_n). \quad (2.10)$$

It is important to note that when $c_k = 0$ (for $2 \leq k \leq n-1$), we derive $C_n = -c_n$, and

$$[\beta + (n-1)\alpha + n(n-1)\vartheta\mu]c_n = \phi_{n-1} \quad (2.11)$$

and

$$-[\beta + (n-1)\alpha + n(n-1)\vartheta\mu]c_n = \psi_{n-1}. \quad (2.12)$$

By utilizing the absolute values of Equations (2.11) and (2.12), and referencing lemma 1.1, we conclude

$$|c_n| = \frac{|\phi_{n-1}|}{\beta + (n-1)\alpha + n(n-1)\vartheta\mu} = \frac{|\psi_{n-1}|}{\beta + (n-1)\alpha + n(n-1)\vartheta\mu} \leq \frac{m(1-\eta)}{\beta + (n-1)\alpha + n(n-1)\vartheta\mu}.$$

Thus, the proof of Theorem 2.1 is complete. □

With the choice of $\vartheta = 0$ and $m = 2$ in Theorem 2.1, we can deduce the following corollary.

Corollary 2.1. *Given $\alpha \geq 1$, $\beta \geq 0$, and $0 \leq \eta < 1$, let us define the function $\gamma \in \mathcal{B}_{\Sigma}^{\beta}(\alpha, \eta)$ as specified in (1.1). If $c_k = 0$ (where $2 \leq k \leq n-1$), then*

$$|c_n| \leq \frac{2(1-\eta)}{\beta + (n-1)\alpha}, \quad n \geq 4.$$

With the choice of $\vartheta = 0$ and $\beta = 1$ in Theorem 2.2, we can deduce the following corollary.

Corollary 2.2. *Given $\alpha \geq 1$, $2 \leq m \leq 4$ and $0 \leq \eta < 1$, let us define the function $\gamma \in \mathcal{G}_{\Sigma}^{\alpha}(m,\eta)$ as specified in (1.1). If $c_k = 0$ (where $2 \leq k \leq n-1$), then*

$$|c_n| \leq \frac{m(1-\eta)}{1+(n-1)\alpha}, \quad n \geq 4.$$

With the choice of $\vartheta = 0$, $\alpha = 1$ and $\beta = 1$ in Theorem 2.2, we can deduce the following corollary.

Corollary 2.3. *Given $2 \leq m \leq 4$ and $0 \leq \eta < 1$, let us define the function $\gamma \in \mathcal{H}_{\Sigma}(m,\eta)$ as specified in (1.1). If $c_k = 0$ (where $2 \leq k \leq n-1$), then*

$$|c_n| \leq \frac{m(1-\eta)}{n}, \quad n \geq 4.$$

With the choice of $\vartheta = 0$, $\alpha = 1$ and $\beta = 0$ in Theorem 2.2, we can deduce the following corollary.

Corollary 2.4. *Given $2 \leq m \leq 4$ and $0 \leq \eta < 1$, let us define the function $\gamma \in \mathcal{S}_{\Sigma}^{*}(m,\eta)$ as specified in (1.1). If $c_k = 0$ (where $2 \leq k \leq n-1$), then*

$$|c_n| \leq \frac{m(1-\eta)}{n-1}, \quad n \geq 4.$$

Remark 2.2. (i) Corollary 2.2, Corollary 2.3 and Corollary 2.4, supports the coefficient bounds proposed by Huo Tang [33].

(ii) With the choice of $m = 2$ in Corollary 2.2, Corollary 2.3 and Corollary 2.4, supports the coefficient bounds proposed by Hamidi and Jahangiri [16].

Theorem 2.2. *Given $\alpha \geq 1$, $\beta \geq 0$, $\vartheta \geq 0$, $2 \leq m \leq 4$ and $0 \leq \eta < 1$, let us define the function $\gamma \in \mathcal{N}_{\Sigma}^{\alpha,\beta}[\vartheta,m,\eta]$ as specified in (1.1), then*

$$|c_2| \leq \sqrt{\frac{2m(1-\eta)}{(\beta+2\alpha+6\vartheta\mu)(\beta+1)}}, \tag{2.13}$$

$$|c_3| \leq \frac{2m(1-\eta)}{(\beta+2\alpha+6\vartheta\mu)(\beta+1)} \tag{2.14}$$

and

$$|c_3 - \aleph c_2^2| \leq \begin{cases} \dfrac{2m(1-\eta)(1-\aleph)}{(\beta+2\alpha+6\vartheta\mu)(\beta+1)} & \text{if } \aleph < \dfrac{1-\beta}{2}, \\ \dfrac{m(1-\eta)}{\beta+2\alpha+6\vartheta\mu} & \text{if } \dfrac{1-\beta}{2} \leq \aleph \leq \dfrac{\beta+3}{2}, \\ \dfrac{2m(1-\eta)(\aleph-1)}{(\beta+2\alpha+6\vartheta\mu)(\beta+1)} & \text{if } \aleph > \dfrac{\beta+3}{2}. \end{cases} \tag{2.15}$$

Proof. By inserting $n = 2$ into Equations (2.9) and (2.10), we achieve

$$(\beta+2\alpha+6\vartheta\mu)\left[\frac{\beta-1}{2}c_2^2 + \left(1+\frac{6\mu}{2\alpha+1}\right)c_3\right] = \phi_2 \tag{2.16}$$

and

$$(\beta+2\alpha+6\vartheta\mu)\left[\frac{\beta+3}{2}c_2^2 - \left(1+\frac{6\mu}{2\alpha+1}\right)c_3\right] = \psi_2. \tag{2.17}$$

Summing equations (2.16) and (2.17) yields

$$c_2^2 = \frac{\phi_2 + \psi_2}{(\beta + 2\alpha + 6\vartheta\mu)(\beta + 1)}. \tag{2.18}$$

By utilizing the absolute values of Equation (2.18), and referencing lemma 1.1, we conclude

$$|c_2|^2 \leq \frac{2m(1-\eta)}{(\beta + 2\alpha + 6\vartheta\mu)(\beta + 1)}. \tag{2.19}$$

Equation (2.19) defines the bound of $|c_2|$ as shown in equation (2.13). By deducting equation (2.16) from equation (2.17), we obtain

$$c_3 = c_2^2 + \frac{\phi_2 - \psi_2}{2(\beta + 2\alpha + 6\vartheta\mu)}. \tag{2.20}$$

When the value of c_2^2 from (2.18) is substituted into (2.20), it results in

$$c_3 = \frac{(\beta+3)\phi_2 + (1-\beta)\psi_2}{2(\beta + 2\alpha + 6\vartheta\mu)(\beta + 1)}. \tag{2.21}$$

By utilizing the absolute values of Equation (2.21), and referencing lemma 1.1, we conclude

$$|c_3| \leq \frac{2m(1-\eta)}{(\beta + 2\alpha + 6\vartheta\mu)(\beta + 1)}. \tag{2.22}$$

Equation (2.22) defines the bound of $|c_3|$ as shown in equation (2.14). For any real number $\aleph$, by using equations (2.18) and (2.21), one obtains the following

$$c_3 - \aleph c_2^2 = \frac{(\beta + 3 - 2\aleph)\phi_2 + (1 - \beta - 2\aleph)\psi_2}{2(\beta + 2\alpha + 6\vartheta\mu)(\beta + 1)}. \tag{2.23}$$

By utilizing the absolute values of Equation (2.23), and referencing lemma 1.1, we conclude

$$|c_3 - \aleph c_2^2| \leq \frac{m(1-\eta)[|\beta + 3 - 2\aleph| + |1 - \beta - 2\aleph|]}{2(\beta + 2\alpha + 6\vartheta\mu)(\beta + 1)}. \tag{2.24}$$

Equation (2.24) defines the bound of $|c_3 - \aleph c_2^2|$ as shown in equation (2.15). This signifies the end of the proof for Theorem 2.2. □

With the choice of $\vartheta = 0$ and $m = 2$ in Theorem 2.2, we can deduce the following corollary.

Corollary 2.5. *Given $\alpha \geq 1$, $\beta \geq 0$, and $0 \leq \eta < 1$, let us define the function $\gamma \in \mathcal{B}_\Sigma^\beta(\alpha, \eta)$ as specified in (1.1), then*

$$|c_2| \leq \sqrt{\frac{4(1-\eta)}{(\beta + 2\alpha)(\beta + 1)}},$$

$$|c_3| \leq \frac{4(1-\eta)}{(\beta + 2\alpha)(\beta + 1)}$$

and

$$|c_3 - \aleph c_2^2| \leq \begin{cases} \dfrac{4(1-\eta)(1-\aleph)}{(\beta + 2\alpha)(\beta + 1)} & \text{if } \aleph < \dfrac{1-\beta}{2}, \\ \dfrac{2(1-\eta)}{\beta + 2\alpha} & \text{if } \dfrac{1-\beta}{2} \leq \aleph \leq \dfrac{\beta+3}{2}, \\ \dfrac{4(1-\eta)(\aleph - 1)}{(\beta + 2\alpha)(\beta + 1)} & \text{if } \aleph > \dfrac{\beta+3}{2}. \end{cases}$$

With the choice of $\vartheta = 0$ and $\beta = 1$ in Theorem 2.2, we can deduce the following corollary.

Corollary 2.6. *Given $\alpha \geq 1$, $2 \leq m \leq 4$ and $0 \leq \eta < 1$, let us define the function $\gamma \in \mathcal{G}_{\Sigma}^{\alpha}(m, \eta)$ as specified in (1.1), then*

$$|c_2| \leq \sqrt{\frac{m(1-\eta)}{1+2\alpha}},$$

$$|c_3| \leq \frac{m(1-\eta)}{1+2\alpha}$$

and

$$|c_3 - \aleph c_2^2| \leq \begin{cases} \dfrac{m(1-\eta)(1-\aleph)}{1+2\alpha} & \text{if } \aleph < 0, \\ \dfrac{m(1-\eta)}{1+2\alpha} & \text{if } 0 \leq \aleph \leq 2, \\ \dfrac{m(1-\eta)(\aleph-1)}{1+2\alpha} & \text{if } \aleph > 2. \end{cases}$$

With the choice of $\vartheta = 0$, $\alpha = 1$ and $\beta = 1$ in Theorem 2.2, we can deduce the following corollary.

Corollary 2.7. *Given $2 \leq m \leq 4$ and $0 \leq \eta < 1$, let us define the function $\gamma \in \mathcal{H}_{\Sigma}(m, \eta)$ as specified in (1.1), then*

$$|c_2| \leq \sqrt{\frac{m(1-\eta)}{3}},$$

$$|c_3| \leq \frac{m(1-\eta)}{3}$$

and

$$|c_3 - \aleph c_2^2| \leq \begin{cases} \dfrac{m(1-\eta)(1-\aleph)}{3} & \text{if } \aleph < 0, \\ \dfrac{m(1-\eta)}{3} & \text{if } 0 \leq \aleph \leq 2, \\ \dfrac{m(1-\eta)(\aleph-1)}{3} & \text{if } \aleph > 2. \end{cases}$$

With the choice of $\vartheta = 0$, $\alpha = 1$ and $\beta = 0$ in Theorem 2.2, we can deduce the following corollary.

Corollary 2.8. *Given $2 \leq m \leq 4$ and $0 \leq \eta < 1$, let us define the function $\gamma \in \mathcal{S}_{\Sigma}^{*}(m, \eta)$ as specified in (1.1), then*

$$|c_2| \leq \sqrt{\frac{m(1-\eta)}{3}},$$

$$|c_3| \leq \frac{m(1-\eta)}{3}$$

and

$$|c_3 - \aleph c_2^2| \leq \begin{cases} \dfrac{m(1-\eta)(1-\aleph)}{3} & \text{if } \aleph < \dfrac{1}{2}, \\ \dfrac{m(1-\eta)}{3} & \text{if } \dfrac{1}{2} \leq \aleph \leq \dfrac{3}{2}, \\ \dfrac{m(1-\eta)(\aleph-1)}{3} & \text{if } \aleph > \dfrac{3}{2}. \end{cases}$$

Remark 2.3. (i) Corollary 2.7 and Corollary 2.8, confirms the bound of $|c_2|$ and enhances the bound of $|c_3|$ established by Y. Li [21]. Corollary 2.7 and Corollary 2.8, substantiates the findings presented by Sharma [27].

(ii) Corollary 2.6, supports the coefficient bounds proposed by Sharma [27].

3. Concluding remarks and observations

In this article, using the Faber polynomial expansions, we found general coefficient bounds for newly defined subclasses of bi-univalent functions with bounded boundary rotation. We also derived interesting and familiar Fekete–Szegö inequality for these new classes. In addition, we gave interesting remarks and corollaries for certain choices of parameters. Similar coefficient problems may also be analyzed for other function classes existing in the literature. The study also suggests that by employing $q-$calculus for values of $0 < q < 1$, along with orthogonal polynomials and different kinds of operators.

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