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A GENERALIZATION OF THE LOWER BOUND FOR THE SPECTRAL RADIUS OF NONNEGATIVE OR COMPLEX MATRICES

MARYAM SHAMS SOLARY[1]

Abstract. In this paper, we try to find a lower bound for the spectral radius of nonnegative or complex matrices by using Laguerre's theorem on the polar derivative of the characteristic polynomial. This method is a generalization of the work is done in the paper of Lin et al (2023) and its results have been compared with the results related to the bound found by Horne (1997) for matrices with different dimensions. The numerical and simulation results can be used and competed in different problems and applications.

1. Introduction and Preliminaries

Finding the lower bound for the spectral radius (the largest eigenvalue in modulus) of nonnegative or complex matrix A

$$\rho(A) = \max\{|\lambda| : A - \lambda I \text{ is singular}\},$$

is an important and practical requirement in various problems. Because it is possible to determine the stability and instability of matrices and whether they are convergent or divergent without spending a lot of time and memory, refer to the following references [4,6,7] for more study on this matter. Here from [5], we suppose $f(z)$ is a complex polynomial of degree n and for $1 \le k \le n$ the kth polar derivative $f_k(z)$ (a generalization of the ordinary derivative) of $f(z)$ with respect to $\xi_1, \cdots, \xi_k \in \mathbb{C}$ is defined by

$$f_k(z) = (n - k + 1)f_{k-1}(z) + (\xi_k - z)f'_{k-1}(z).$$

Then, for $k = 1$, the polar derivative of $f(z)$ with respect to $\xi_1 \in \mathbb{C}$ is defined by

$$f_1(z) = nf(z) + (\xi_1 - z)f'(z).$$

Now, two important and practical tools to find this lower bound are Lucas's theorem [[1], p. 18] and Laguerre's theorem [[1], p. 20], which the first one used in Horne's paper [3] and the second one in Lin et al's paper[4].

Key words and phrases. Spectral radius, Polar derivative, Lower bound.

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Theorem 1.1. *(Lucas) Let $f(z)$ be a nonconstant complex polynomial. Then the critical points (i.e., the zeros of $f'(z)$) lie in the convex hull of the zeros of $f(z)$.*

Theorem 1.2. *(Laguerre) Suppose $f(z)$ is a complex polynomial of degree n and has all its zeros in a disk D. If $\xi_1 \notin D$ then $f_1(z)$ has all its zeros in the disk D.*
By repeatedly using this tool, we can generalize the location of zeros of the kth polar derivative as follows: If $\xi_1, \cdots, \xi_k \notin D$, $1 \le k \le n$, then all zeros of each polar derivative $f_1(z), \cdots, f_k(z)$ lie in the disk D.

Now, let the following notation on the $n \times n$ complex matrix A.

The conjugate transpose of A is denoted by $A^\star$.
The Frobenius norm of A is denoted by $\|A\|$, trA for the trace of A and $\|A\| = \sqrt{tr(A^\star A)}$.
The ith compound matrix of A, where $1 \le i \le n$ is shown by $C_i(A)$, see [2,11].
Let $p(z) = \sum\limits_{i=0}^{n} a_i z^i$ be the characteristic polynomial of A.
Then $a_i = (-1)^{n-i} tr\ C_{n-i}(A), i = 0, 1, \cdots, n$.

After these preliminary definitions that we have stated so far, it is time to find the lower limit for the spectral radius with the help of the following two theorems:

Theorem 1.3. *Let A be an $n \times n$ complex matrix with $m = rank\ A \ge 2$. Then*

$$\rho(A) \ge \max \left\{ \left| \frac{trA}{m} \pm \sqrt{\frac{1}{m(m-1)} \left(trA^2 - \frac{1}{m} tr^2 A \right)} \right| \right\}$$

Theorem 1.4. *Let A be an $n \times n$ complex matrix with $m = rank\ A \ge 2$. Then*

$$\rho(A) \ge \max \left\{ \left| \frac{-b_1 \pm \sqrt{b_1^2 - b_0 b_2}}{b_2} \right| \right\},$$

where

$$b_0 = \sum_{i=0}^{n-2} (-1)^{n-i} (n-i)(n-i-1) \|A\|^i tr\ C_{n-i}(A),$$
$$b_1 = \sum_{i=0}^{n-2} (-1)^{n-i-1} (i+1)(n-i-1) \|A\|^i tr\ C_{n-i-1}(A),$$
$$b_2 = \sum_{i=0}^{n-2} (-1)^{n-i-2} (i+1)(i+2) \|A\|^i tr\ C_{n-i-2}(A).$$

For the proof of the following theorems see [1,3,4].

2. Main Theorem

Theorem 2.1. *Let A be an $n \times n$ complex matrix with $m = \mathrm{rank}\, A \ge 2$ and $j = 1, 2, \cdots, n-1$. Then*

$$\rho(A) \ge \max_{1 \le k \le j} \left\{ \left| z_k^{(n-j)} \right| \right\}, \tag{2.1}$$

and

$$\rho(A) \geq \frac{\sum\limits_{k=1}^{j} \left| z_k^{(n-j)} \right|}{j}, \tag{2.2}$$

where $z_k^{(n-j)}$ *for* $1 \leq k \leq j$ *are the roots of the following polynomial*

$$p_{n-j}(z) = \frac{(n-j)!}{j!} \left(b_0 + \binom{j}{1} b_1 z + \binom{j}{2} b_2 z^2 + \cdots + b_j z^j \right), \tag{2.3}$$

$$b_0 = \sum_{i=0}^{n-j} (n-i)(n-i-1)\cdots(n-i-j+1)\|A\|^i a_i,$$

$$b_k = \sum_{i=0}^{n-j} (i+1)\cdots(i+k)(n-i-k)\cdots(n-i-j+1)\|A\|^i a_{i+k}, \quad 1 \leq k \leq j-1,$$

$$b_j = \sum_{i=0}^{n-j} (i+1)(i+2)\cdots(i+j)\|A\|^i a_{i+j}.$$

Proof. We can prove the following theorem by mathematical induction.
Let $p(z) = \sum\limits_{i=0}^{n} a_i z^i := \sum\limits_{i=0}^{n} \binom{n}{i} c_i z^i$ be the characteristic polynomial of A. Since

$$a_i = (-1)^{n-i} \operatorname{tr} C_{n-i}(A), \; i = 0, 1, \cdots, n, \tag{2.4}$$

then

$$c_i = \frac{a_i}{\binom{n}{i}} = (-1)^{n-i} \frac{\operatorname{tr} C_{n-i}(A)}{\binom{n}{i}}, \; i = 0, 1, \cdots, n.$$

Since rank $A \geq 2$, we have $\|A\| > \rho(A)$. Let $\xi_1 = \xi_2 = \cdots = \xi_k = \|A\|$.
For every $\epsilon \in (0, \|A\| - \rho(A))$, we let D_ϵ be a disk centered at the origin with radius $\rho(A) + \epsilon$. Note that, $\xi_1, \cdots, \xi_k$ are all outside the disk D_ϵ, then from Theorem 1.2 all the zeros $z_i^{(k)}$ (there are at most $n-k$ zeros) of $p_k(z)$ lie in the disk D_ϵ and we have

$$\max_{1 \leq i \leq n-k} \left| z_i^{(k)} \right| < \rho(A) + \epsilon.$$

When $\epsilon \to 0$, then

$$\max_{1 \leq i \leq n-k} \left| z_i^{(k)} \right| \leq \rho(A). \tag{2.5}$$

Now, in the next step, we can check that this theorem is true for
$k = n-1, n-2, n-3, \cdots, 1$: From Eq.(13.9) and Eq.(13.11) in [5], we have

$$p(z) = \sum_{i=0}^{n} \binom{n}{i} c_i z^i,$$

$$p_k(z) = \sum_{i=0}^{n-k} \binom{n-k}{i} c_i^{(k)} z^i,$$

If $k = n-1$, then

$$p_{(n-1)}(z) = \sum_{i=0}^{1} \binom{n-1}{i} c_i^{(n-1)} z^i,$$

where

$$c_0^{(n-1)} = n!\sum_{i=0}^{n-1}\binom{n-1}{i}\|A\|^i c_i = n!\sum_{i=0}^{n-1}\binom{n-1}{i}\|A\|^i \frac{a_i}{\binom{n}{i}}$$
$$= (n-1)!\sum_{i=0}^{n-1}(n-i)\|A\|^i a_i,$$
$$c_1^{(n-1)} = n!\sum_{i=0}^{n-1}\binom{n-1}{i}\|A\|^i c_{i+1} = n!\sum_{i=0}^{n-1}\binom{n-1}{i}\|A\|^i \frac{a_{i+1}}{\binom{n}{i+1}}$$
$$= (n-1)!\sum_{i=0}^{n-1}(i+1)\|A\|^i a_i.$$

That is,

$$p_{n-1}(z) = (n-1)!\left(\sum_{i=0}^{n-1}(n-i)\|A\|^i a_i + \sum_{i=0}^{n-1}(i+1)\|A\|^i a_{i+1} z\right) = (n-1)!(b_0 + b_1 z).$$

Next, we show that the degree of $p_{n-1}(z)$ is exactly 1, namely, $b_1 \neq 0$. For this work, compute the first derivative of the characteristic polynomial $p(z)$, then

$$p'(z) = \sum_{i=0}^{n-1}(i+1)a_{i+1}z^i.$$

Using Lucas' theorem, we find that the root of $p(z)$ in modulus is no larger than $\rho(A)$, that is,

$$b_1 = p'(\|A\|) \neq 0.$$

$p_{n-1}(z)$ has only one root

$$z_1^{(n-1)} = \frac{-b_0}{b_1},$$

that this root applies to relations (2.1) and (2.2).

If $k = n-2$ then

$$p_{n-2}(z) = \sum_{i=0}^{2}\binom{n-2}{i}c_i^{(n-2)}z^i = c_0^{(n-2)} + 2c_1^{(n-2)}z + c_2^{(n-2)}z^2$$

where $c_0^{(n-2)} = \frac{(n-2)!}{2}b_0$, $c_1^{(n-2)} = \frac{(n-2)!}{2}b_1$, $c_2^{(n-2)} = \frac{(n-2)!}{2}b_2$ according to coefficient in Theorem 1.4.

If $k = n-3$ then

$$p_{n-3}(z) = \sum_{i=0}^{3}\binom{n-3}{i}c_i^{(n-3)}z^i = c_0^{(n-3)} + 3c_1^{(n-3)}z + 3c_2^{(n-3)}z^2 + c_3^{(n-3)}z^3,$$

where

$$c_0^{(n-3)} = \frac{n!}{6}\sum_{i=0}^{n-3}\binom{n-3}{i}\|A\|^i c_i = \frac{n!}{6}\sum_{i=0}^{n-3}\binom{n-3}{i}\|A\|^i \frac{a_i}{\binom{n}{i}}$$

$$= \frac{(n-3)!}{6}\sum_{i=0}^{n-3}(n-i)(n-i-1)(n-i-2)\|A\|^i a_i,$$

$$c_1^{(n-3)} = \frac{n!}{6}\sum_{i=0}^{n-3}\binom{n-3}{i}\|A\|^i c_{i+1} = \frac{n!}{6}\sum_{i=0}^{n-3}\binom{n-3}{i}\|A\|^i \frac{a_{i+1}}{\binom{n}{i+1}}$$

$$= \frac{(n-3)!}{6}\sum_{i=0}^{n-3}(i+1)(n-i-1)(n-i-2)\|A\|^i a_{i+1},$$

$$c_2^{(n-3)} = \frac{n!}{6}\sum_{i=0}^{n-3}\binom{n-3}{i}\|A\|^i c_{i+2} = \frac{n!}{6}\sum_{i=0}^{n-3}\binom{n-3}{i}\|A\|^i \frac{a_{i+2}}{\binom{n}{i+2}}$$

$$= \frac{(n-3)!}{6}\sum_{i=0}^{n-3}(i+1)(i+2)(n-i-2)\|A\|^i a_{i+2},$$

$$c_3^{(n-3)} = \frac{n!}{6}\sum_{i=0}^{n-3}\binom{n-3}{i}\|A\|^i c_{i+3} = \frac{n!}{6}\sum_{i=0}^{n-3}\binom{n-3}{i}\|A\|^i \frac{a_{i+3}}{\binom{n}{i+3}}$$

$$= \frac{(n-3)!}{6}\sum_{i=0}^{n-3}(i+1)(i+2)(i+3)\|A\|^i a_{i+3},$$

where

$$c_0^{(n-3)} = \frac{(n-3)!}{3!}b_0,\ c_1^{(n-3)} = \frac{(n-3)!}{3!}b_1,\ c_2^{(n-3)} = \frac{(n-3)!}{3!}b_2,\ c_3^{(n-3)} = \frac{(n-3)!}{3!}b_3.$$

Next, we show that the degree of $p_{n-3}(z)$ is exactly 3, namely, $b_3 \neq 0$. For this work, compute the third derivative of the characteristic polynomial $p(z)$, then

$$p^{(3)}(z) = \sum_{i=0}^{n-3}(i+1)(i+2)(i+3)a_{i+3}z^i.$$

Using Lucas' theorem, we find that the roots of $p(z)$ in modulus are no larger than $\rho(A)$, they are strictly less than $\|A\|$, namely from (2.5), we have

$$b_3 = p^{(3)}(\|A\|) \neq 0.$$

The zeros of $p_{n-3}(z)$ are found by using Shengjin formulas are given in [8–10], Denote the coefficients $p_{n-3}(z)$ of Equation (2.3) with

$$\Delta_1 = b_1^2 - 3b_0b_2,\ \Delta_2 = b_1b_2 - 9b_0b_3,\ \Delta_3 = b_2^2 - 3b_1b_3,\ \Delta = \Delta_2^2 - 4\Delta_1\Delta_3.$$

Then we have

(1) If $\Delta_1 = \Delta_2 = 0$, $p_{n-3}(z)$ has only one real triple root;
(2) If $\Delta > 0$, $p_{n-3}(z)$ has one real root and a pair of conjugate imaginary roots;
(3) If $\Delta = 0$, $p_{n-3}(z)$ has three real roots: one simple and the other double;
(4) If $\Delta < 0$, $p_{n-3}(z)$ has three different real roots.

Then $z_1^{(n-3)}, z_2^{(n-3)}$, and $z_3^{(n-3)}$, are the zeros of $p_{n-3}(z)$ that apply to relations (2.1) and

(2.2). If we continue inductively this process for $j = 4, \cdots, n-1$, the roots of this polynomial can be find by the 'roots' command in Matlab, then for jth derivative of the characteristic polynomial $p(z)$, we derive

$$p^{(j)}(z) = \sum_{i=0}^{n-j} (i+1)\cdots(i+j) a_{i+j} z^i,$$

that by using Lucas' theorem, we get

$$b_j = p^{(j)}(\|A\|) \neq 0,$$

applies to relation (2.5). Then the proof of the theorem is complete. □

3. Numerical Examples

In this section, we compute the lower bound for some nonnegative or complex matrices by Theorem 1.3 and Theorem 2.1 when $j = 1$, $j = 2$, and $j = 3$.
In this way, we use some numerical simulations by Matlab software.
We can conduct Theorem 2.1 for $j > 3$, but the complexity of our computations will be increase.
As the matrix size decreases, the quality of the bounds increases. The simulations are presented in Figures 1, 2, and 3 are the performance of our main Theorem and Horne by the average over 100000 random trials of the ratio of the lower bound to the actual spectral radius and the percentage.
We see from simulations that main Theorem with Re. (2.2) when $j = 1, 2$ often gives a better estimate than that of Horne and Re. (2.1).
Also, the results for the matrices being considered are entrywise nonnegative matrices are better than complex matrices by using the percentage that shows the ratio of numerical bounds to the spectral radius.
Namely, if the percentage is equal to 1, the main theorem works very well and the ratio of the our bound it finds to the spectral radius is 100% correct.

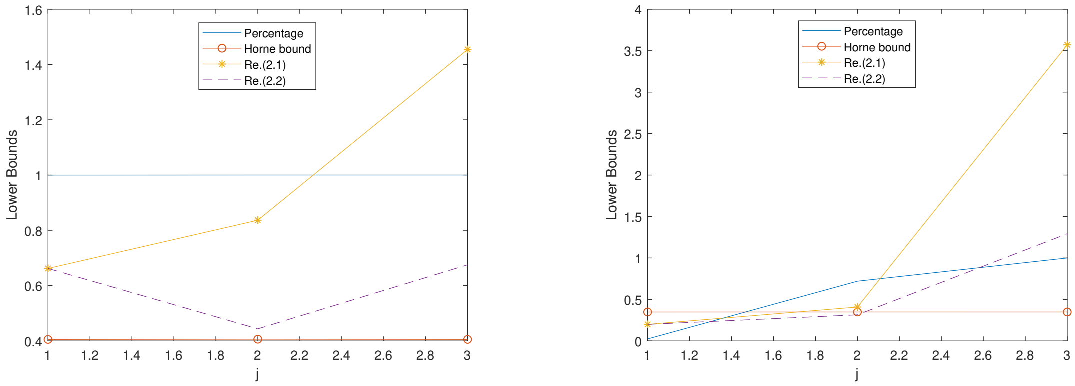


FIGURE 1. Comparison of the performance of our new bounds and that of Horne for entrywise nonnegative matrices (left) and general complex matrices (right) when $n = 5$

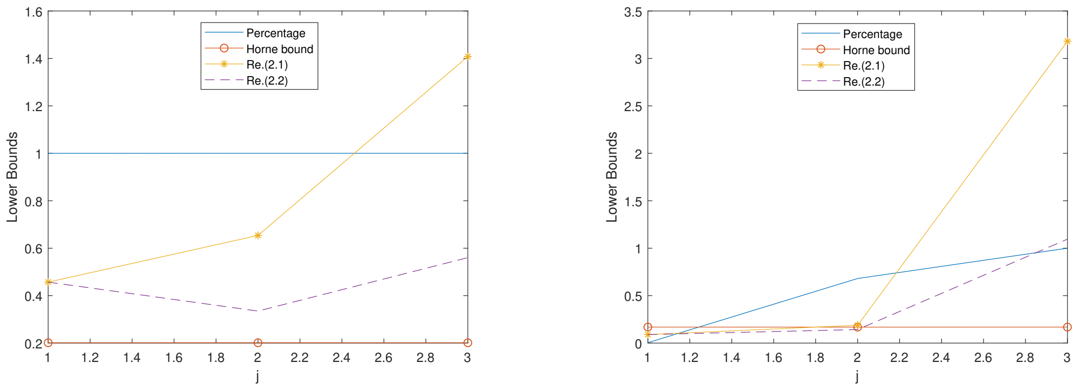
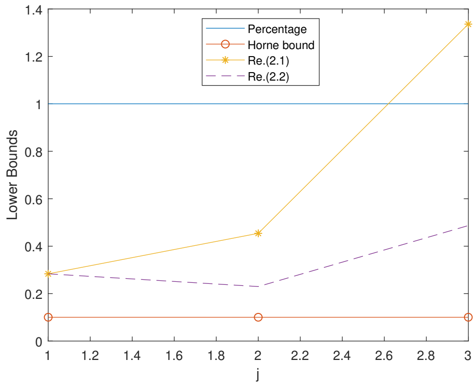


FIGURE 2. Comparison of the performance of our new bounds and that of Horne for entrywise nonnegative matrices (left) and general complex matrices (right) when $n = 10$



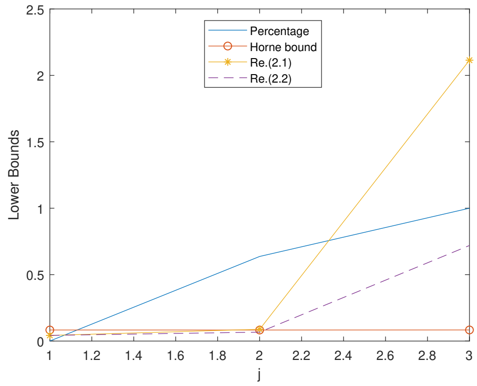


FIGURE 3. Comparison of the performance of our new bounds and that of Horne for entrywise nonnegative matrices (left) and general complex matrices (right) when $n = 20$

APPENDIX: MATLAB CODES

The following MATLAB codes were used to generate the numerical results presented in Section 3.

Code 1: Complex Matrices.

```
clear all
c=0;
n=20; % matrix size, change it to 6, 8, 12

s1=0;
s2=0;
s3=0;
jj=[1:1:3];
ii=1;
for j=1:3
x= cell(1, j); y= cell(1, j); r= cell(1, j+1);
st=10000; % simulation times
for k=1:st
    for i=1:j
        x{i}=j-i+1:n-i+1;
    end

    xx=n-j:-1:0;
    for i=1:j
        y{i}=n-j+i:-1:i;
    end

    A=randn(n,n)+1i*randn(n,n); % generate random complex matrices

    % new bound
    t=norm(A,'fro');
```

```
    nn=t.^(xx);
    p=poly(A);
    for i=1:j+1
        r{i}=p(i:n-j+i);
    end
    rho=max(abs(eig(A)));
    x1=x{1};
    for i=2:j
        x1=x1.*x{i};
    end
    b0=sum(((x1).*nn).*r{j+1});
    if j>1
        x1=x{2};y1=y{1};
        for k=1:j-1
            for i=3:j
                x1=x1.*x{i};
            end
            for i=2:j-k
                y1=y1.*y{i};
            end
            xy1=(y1.*x1);
            b(k)=sum((xy1.*nn).*r{k+1});
        end
    end
    y1=y{1};
    for i=2:j
        y1=y1.*y{i};
    end
    b(j)=sum(((y1).*nn).*r{1});
    q(j+1)=b0;
    for i=0:j-1
        h=factorial(j)/(factorial(i+1)*factorial(j-i-1));
        q(j-i)=h*b(i+1);
    end
    z=roots(q);
    newbound=max(abs(z));

    d1=newbound/rho;
    s1=s1+d1;

    newbound1=sum(abs(z))/j;
    d3= newbound1/rho;
    s3=s3+d3;

    % Horne's bound
    m=rank(A);
    delta=4*(trace(A^2)-((trace(A))^2)/m)/(m*(m-1));
    bb=-2*trace(A)/m;
    z3=(-bb + sqrt(delta))/2;
    z4=(-bb - sqrt(delta))/2;
```

```
    hornebound=max(abs(z3),abs(z4));

    d2=hornebound/rho;
    s2=s2+d2;

    if newbound>hornebound
        c=c+1;
    end
end
s1=s1/st;
s2=s2/st;
c=c/st;
s3=s3/st;
cc(ii)=c;
ss1(ii)=s1;
ss2(ii)=s2;
ss3(ii)=s3;
ii=ii+1;
end

plot(jj,cc,'-',jj,ss2,'-o',jj,ss1,'-*',jj,ss3,'--')
xlabel('j'),ylabel('Lower Bounds')
legend('Percentage','Horne bound','Re.(2.1)','Re.(2.2)')
```

Listing 1. MATLAB code for complex matrices

Code 2: Nonnegative Matrices.

```
clear all
c=0;
n=20; % matrix size, change it to 6, 8, 12

s1=0;
s2=0;
s3=0;
jj=[1:1:3];
ii=1;
for j=1:3
x= cell(1, j); y= cell(1, j); r= cell(1, j+1);
st=10000; % simulation times
for k=1:st
    for i=1:j
        x{i}=j-i+1:n-i+1;
    end

    xx=n-j:-1:0;
    for i=1:j
        y{i}=n-j+i:-1:i;
    end

    A=rand(n,n); % generate random nonnegative matrices

```

```
    % new bound
    t=norm(A,'fro');
    nn=t.^(xx);
    p=poly(A);
    for i=1:j+1
        r{i}=p(i:n-j+i);
    end
    rho=max(abs(eig(A)));
    x1=x{1};
    for i=2:j
        x1=x1.*x{i};
    end
    b0=sum(((x1).*nn).*r{j+1});
    if j>1
        x1=x{2};y1=y{1};
        for k=1:j-1
            for i=3:j
                x1=x1.*x{i};
            end
            for i=2:j-k
                y1=y1.*y{i};
            end
            xy1=(y1.*x1);
            b(k)=sum((xy1.*nn).*r{k+1});
        end
    end
    y1=y{1};
    for i=2:j
        y1=y1.*y{i};
    end
    b(j)=sum(((y1).*nn).*r{1});
    q(j+1)=b0;
    for i=0:j-1
        h=factorial(j)/(factorial(i+1)*factorial(j-i-1));
        q(j-i)=h*b(i+1);
    end
    z=roots(q);
    if j==3
        z=solve3Polynomial(q(1), q(2), q(3), q(4));
    end
    newbound=max(abs(z));

    d1=newbound/rho;
    s1=s1+d1;

    newbound1=sum(abs(z))/j;
    d3= newbound1/rho;
    s3=s3+d3;

    % Horne's bound
```

```
    m=rank(A);
    delta=4*(trace(A^2)-((trace(A))^2)/m)/(m*(m-1));
    bb=-2*trace(A)/m;
    z3=(-bb + sqrt(delta))/2;
    z4=(-bb - sqrt(delta))/2;
    hornebound=max(abs(z3),abs(z4));

    d2=hornebound/rho;
    s2=s2+d2;

    if newbound>hornebound
        c=c+1;
    end
end
s1=s1/st;
s2=s2/st;
c=c/st;
s3=s3/st;
cc(ii)=c;
ss1(ii)=s1;
ss2(ii)=s2;
ss3(ii)=s3;
ii=ii+1;
end

plot(jj,cc,'-',jj,ss2,'-o',jj,ss1,'-*',jj,ss3,'--')
xlabel('j'),ylabel('Lower Bounds')
legend('Percentage','Horne bound','Re.(2.1)','Re.(2.2)')
```

Listing 2. MATLAB code for nonnegative matrices

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[1]Department of Mathematics,
Payame Noor University, P. O. Box 19395-3697,
Tehran, IRAN
Email address: shamssolary@pnu.ac.ir
Email address: shamssolary@gmail.com