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SEMINORM INEQUALITIES FOR VECTOR-ORDER TEMPERED FRACTIONAL INTEGRALS AND APPLICATIONS

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ABSTRACT. The present work investigates vector-order tempered fractional integrals on the half-axis, establishing boundedness and weak boundedness results within a novel framework. Our results extend and refine the well-known norm inequalities for classical scalar fractional integral operators. The main idea of our analysis is the introduction of a generalized locally convex space endowed with a suitably chosen family of vector-valued weighted seminorms defined over compact subsets of the half-axis. We provide seminorm inequalities for both tempered and classical fractional integral operators acting on vector-valued functions multiplied by matrix-valued continuous kernels. These inequalities are expressed in terms of positive matrices whose entries depend on the fractional orders and the tempering parameter. The results are further extended to delayed compositions via weak boundedness estimates. The obtained results are subsequently applied to tempered fractional differential systems, and the resulting framework allows for a more flexible treatment of unbounded domains without imposing restrictive decay conditions at infinity. The results are supported by numerical examples.

1. INTRODUCTION

Fractional calculus has evolved significantly in response to the diverse demands of applied sciences, leading to the development of various fractional integrals and derivatives. While classical fractional operators, defined via power-law kernels, have been extensively studied, they exhibit limitations in modeling dynamics with semi-heavy tails or semi-long-range dependence, such as those observed in anomalous diffusion processes (e.g., Continuous Time Random Walks); see [8, 24]. To address these limitations, tempered fractional operators have been introduced, offering a more realistic framework for capturing transitional behaviors between normal and anomalous diffusion in fields such as physics, hydrology, and finance; see [5, 8, 22, 23]. Owing to their ability to incorporate exponential tempering effects while preserving nonlocal features, these operators have attracted considerable

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attention. Recent advances include the development of new tempered fractional transforms, extensions of tempered fractional calculus, and refined integral inequalities, together with applications to fractional differential equations and related models; see, for example, [17, 18, 32]. Moreover, tempered fractional operators have found valuable applications in the analysis of nonlinear dynamical systems, including the modeling of ion motion in nonlinear Mathieu-type equations [4].

A fundamental aspect of fractional calculus is understanding fractional integrals as they underpin the analysis of fractional differential systems. Although classical fractional integrals have been well-characterized in function spaces such as Lebesgue and Hölder spaces — including their weighted variants (see, e.g., [10, 13, 31, 33])— most existing results pertain to finite intervals. Only a few studies, such as those in [33, 34], have explored boundedness properties on infinite intervals, primarily within normed spaces. Inspired by [7] and aiming to fill these gaps, this study focuses on investigating certain boundedness properties of vector-order tempered fractional integrals on the half-axis through vector-valued seminorm inequalities. To contextualize our boundedness results, we briefly outline the associated locally convex structure. A more complete background is provided in Section 2 and Remark 3.1(ii).

Let Y denote the vector space of all real-valued continuous functions defined on the half-axis $\mathbb{R}_+ = [0, \infty)$. The generalized locally convex space $\mathbb{Y}(W_p^\lambda, \mathscr{X})$ consists of functions in Y^n, equipped with the topology induced by the family of vector-valued weighted seminorms $\{\mathbb{P}_{w_p^\lambda,\xi}\}_{\xi\in\mathscr{X}}$. Here, $\mathscr{X}$ is a collection of compact subsets of $\mathbb{R}_+$, and the weight function w_p^λ is given by a positive continuous function w and two positive parameters λ and p via (2.6). Analogously, the generalized locally convex space $\mathbb{Y}(h_p^\lambda, \mathscr{X})$ is defined in the same way, but with the weight function h_p^λ defined by (3.11), replacing w_p^λ (see Remark 3.1(ii)). $\mathscr{M}_n(\mathbb{R}_+)$ and $\mathscr{M}_n(Y)$ represent the sets of $n\times n$ matrices with entries in $\mathbb{R}_+$ and Y, respectively. A key conclusion derived from our boundedness analyses—relevant to both vector-order classical fractional integrals $\mathbb{I}_{0+}^{\vec{\sigma}}$ and vector-order tempered fractional integrals $\mathbb{I}_{0+}^{\vec{\sigma},\eta}$ (see Definition 2.4)—is as follows

For $\mathscr{K} = (k_{ij})_{1\le i,j\le n} \in \mathscr{M}_n(Y)$, and for every $\lambda > 0$, there exist $\mathbf{A}_{\vec{\sigma},\eta}^{p,\lambda}$, $\tilde{\mathbf{A}}_{\vec{\sigma}}^{p,\lambda} \in \mathscr{M}_n(\mathbb{R}_+)$ (independent of $\mathscr{K}$) such that for every $\xi \in \mathscr{X}$ and $V \in Y^n$, the following vector-valued and weighted seminorm estimates hold (for completeness, see Theorem 3.1 and Theorem 3.2)

$$\mathbb{P}_{h_p^\lambda,\xi}\left(\mathbb{I}_{0+}^{\vec{\sigma}}\,(\mathscr{K}\,V)\right) \le \tilde{\mathbf{A}}_{\vec{\sigma}}^{p,\lambda}\;\mathbb{P}_{h_p^\lambda,\xi}(V); \tag{1.1}$$

$$\mathbb{P}_{w_p^\lambda,\xi}\left(\mathbb{I}_{0+}^{\vec{\sigma},\eta}\,(\mathscr{K}\,V)\right) \le \mathbf{A}_{\vec{\sigma},\eta}^{p,\lambda}\;\mathbb{P}_{w_p^\lambda,\xi}(V). \tag{1.2}$$

Besides generalizing the study of fractional integrals of vector-order (corresponding to $n \ge 2$), inequalities (1.1) and (1.2) both extend and differ from some existing results, even for $n = 1$. Indeed, in the particular case of (1.1) where $\mathscr{K}$ is a constant function, a norm estimate for $I_{0+}^{\sigma}\,(\mathscr{K}\,v)$ (with $v \in Y$) was established in [33]. On the other hand, an extension of this result to non-constant functions $\mathscr{K}$—formulated in appropriately normed spaces entirely embedded in Y—was apparently first addressed and rigorously demonstrated in [7]. Having noted this, we emphasize that (1.1) and (1.2) not only improve but also generalize the previous results to the tempered case ($\eta \neq 0$) within the framework of generalized locally convex structures that equip the entire space Y^n. A key distinction from [7] lies in our approach: the structures we propose inherently ensure that the inequalities (1.1) and (1.2) hold for all vector-valued functions in Y^n, thus eliminating the need for additional boundedness constraints at infinity.

Moreover, by incorporating an appropriately selected mapping $J : \mathscr{X} \longrightarrow \mathscr{X}$, we derive weak bounds for vector-order tempered fractional integral operators with specific delays, as demonstrated in Theorem 3.3, Theorem 3.4, and Corollary 3.2. Notably, the weak boundedness results established here are adequate for practical implementation and align with nonlinear analysis tools applied to the associated fractional differential systems. This adequacy will be illustrated in the last section, which addresses the existence and stability of solutions to such systems. Importantly, our boundedness results enable a more efficient selection of suitable function spaces, thereby avoiding the restrictive assumptions commonly imposed in the literature (see Remark 4.2).

The structure of the paper is as follows. In the next section, we first recall the definitions and properties of fractional calculus along with the necessary auxiliary results. We then provide a brief overview of the generalized locally convex structure and the related notation used later. In Section 3, we establish boundedness and weak boundedness results for vector-order tempered fractional integral operators. Section 4 is devoted to applications of these results to tempered fractional differential systems, where the existence, uniqueness, and stability properties are investigated and illustrated through a numerical example. Finally, Section 5 summarizes the main findings and outlines directions for future research.

2. Preliminaries

In this section, we start by introducing some basic definitions and properties from tempered fractional calculus. For more details, we refer to [12, 19, 21, 22].

Definition 2.1. [19] Let $\eta \geq 0$ be a fixed parameter. The tempered fractional integral of a function $v \in L^1_{loc}(\mathbb{R}_+, \mathbb{R})$ of order $\sigma > 0$ with parameter η is defined by

$$\mathscr{I}^{\sigma,\eta}_{0^+} v\,(t) := e^{-\eta t}\, I^{\sigma}_{0^+}\left(e^{\eta t} v(t)\right) := \frac{1}{\Gamma\,(\sigma)} \int_0^t e^{-\eta(t-\tau)} v\,(\tau)\,(t-\tau)^{\sigma-1} d\tau, \qquad t > 0,$$

where $I^{\sigma}_{0^+}$ denotes the classical Riemann-Liouville fractional integral defined by [14]

$$I^{\sigma}_{0^+} v(t) := \frac{1}{\Gamma\,(\sigma)} \int_0^t (t-\tau)^{\sigma-1} v\,(\tau)\, d\tau.$$

Definition 2.2. [19] Let $\eta \geq 0$ be a fixed parameter and , $\sigma \in (0,1)$. The Riemann-Liouville tempered fractional derivative of a function $v \in \mathscr{C}(\mathbb{R}_+, \mathbb{R})$ of order σ with parameter η is defined by

$$\begin{aligned} {}^{RL}\mathscr{D}^{\sigma,\eta}_{0^+} v\,(t) :=& e^{-\eta t}\, {}^{RL}D^{\sigma}_{0^+}\left(e^{\eta t} v(t)\right) \\ :=& \frac{e^{-\eta t}}{\Gamma\,(1-\sigma)} \frac{d}{dt} \int_0^t e^{\eta \tau} v(\tau)(t-\tau)^{-\sigma} d\tau, \end{aligned}$$

where ${}^{RL}D^{\sigma}_{0+}$ denotes the classical Riemann-Liouville fractional derivative defined by [14]

$$ {}^{RL}D^{\sigma}_{0+}u\,(t) := \frac{d}{dt} I^{1-\sigma}_{0+}u\,(t) := \frac{1}{\Gamma\,(1-\sigma)}\frac{d}{dt}\int_0^t (t-\tau)^{-\sigma}u\,(\tau)\,d\tau. $$

Definition 2.3. [21, Definition 3, for $n=1$, $\psi(t)=t$] Let $\eta \geq 0$ be a fixed parameter and $\sigma \in (0,1)$. The Caputo tempered fractional derivative of a function $v \in \mathscr{C}(\mathbb{R}_+,\mathbb{R})$ of order σ with parameter η is defined by

$$ {}^{C}D^{\sigma,\eta}_{0+}v\,(t) :={}^{RL} D^{\sigma,\eta}_{0+}\left[v(t)-e^{-\eta t}v(0)\right]. \tag{2.1} $$

Remark 2.1. (i) Note that for $\eta=0$, Definition 2.1 and Definition 2.2 reduce to the definition of classical Riemann-Liouville fractional integral and classical Riemann-Liouville fractional derivative, respectively.

(ii) If $v \in \mathscr{C}^1(\mathbb{R}_+,\mathbb{R})$, then using [21, Equation (6)] with $n=1$ and $\psi(t)=t$, we recover the following expression which agrees with the definition of the Caputo tempered fractional derivative given in [19],

$$ {}^{C}\mathscr{D}^{\sigma,\eta}_{0+}v\,(t) = \frac{e^{-\eta t}}{\Gamma\,(n-\sigma)}\int_0^t \frac{d^n}{d\tau^n}\left(e^{\eta\tau}v(\tau)\right)(t-\tau)^{n-\sigma-1}d\tau. \tag{2.2} $$

In other words, (2.1) is an extension of (2.2) to $\mathscr{C}(\mathbb{R}_+,\mathbb{R})$. For the new definitions of tempered ψ-fractional operators of order $\sigma \in (n-1,n)$ for $\mathscr{C}^{n-1}$-functions, see [21].

Lemma 2.1. *Let $0<\eta$ and $0<\sigma<1$. Then*

(i) ${}^{C}D^{\sigma,\eta}_{0+}(e^{-\eta\, t})=0$*; see* [21, Lemma 2: $n=1$, $\psi(t)=t$].

(ii) $\left({}^{C}D^{\sigma,\eta}_{0+}I^{\sigma,\eta}_{0+}v\right)(t)=v(t)$, *for* $v\in\mathscr{C}(\mathbb{R}_+,\mathbb{R})$*; see* [21, Lemma 2: $n=1$, $\psi(t)=t$].

(iii) If ${}^{C}D^{\sigma,\eta}_{0+}v \in \mathscr{C}(\mathbb{R}_+,\mathbb{R})$, *then* $\left(I^{\sigma,\eta}_{0+}\,{}^{C}D^{\sigma,\eta}_{0+}v\right)(t)=v(t)-e^{-\eta t}v(0)$*; see* [28, Lemma 2.7].

We now present extensions of classical fractional integrals, tempered fractional integrals, and Caputo-type fractional derivatives to vector-valued functions.

Definition 2.4. Let $V=(v_1,v_2,...,v_n)^T \in L^1_{loc}(\mathbb{R}_+,\mathbb{R}^n)$, $\eta>0$, and $\vec{\sigma}:=(\sigma_1,\sigma_2,...,\sigma_n)^T \in \mathbb{R}^n$ with $\sigma_i>0$ for $i=1,...,n$.

The vector-order classical fractional integral of V is defined by

$$ \mathbb{I}^{\vec{\sigma}}_{0+}V\,(t) := \left(I^{\sigma_1}_{0+}v_1\,(t)\,,...,I^{\sigma_n}_{0+}v_n\,(t)\right)^T $$

The vector-order tempered fractional integral of V is defined by

$$ \mathbb{I}^{\vec{\sigma},\eta}_{0+}V\,(t) := \left(\mathscr{I}^{\sigma_1,\eta}_{0+}v_1\,(t)\,,...,\mathscr{I}^{\sigma_n,\eta}_{0+}v_n\,(t)\right)^T. $$

Definition 2.5. Let $V=(v_1,..,v_n)^T \in \mathscr{C}(\mathbb{R}_+,\mathbb{R}^n)$, $\vec{\sigma}=(\sigma_1,..,\sigma_n)^T\in\mathbb{R}^n$ and $\sigma_i\in(0,1)$ $(i=1,...,n)$. The tempered fractional derivative of Caputo-type of V is defined by

$$ {}^{C}\mathbb{D}^{\vec{\sigma},\eta}_{0+}V\,(t) := \left({}^{C}\mathscr{D}^{\sigma_1,\eta}_{0+}v_1\,(t)\,,...,{}^{C}\,\mathscr{D}^{\sigma_n,\eta}_{0+}v_n\,(t)\right)^T. $$

From now on, we denote by $\mathscr{I}_{0^+}^{\sigma,\eta}$ and $\mathbb{I}_{0^+}^{\vec{\sigma},\eta}$ the operators corresponding to the case $\eta \neq 0$; otherwise, we use $I_{0^+}^{\sigma}$ and $\mathbb{I}_{0^+}^{\vec{\sigma}}$. In the remainder of this paper, we consider the operators $\mathscr{I}_{0^+}^{\sigma,\eta}$ and $\mathbb{I}_{0^+}^{\vec{\sigma},\eta}$ with $0 < \sigma < 1$ and $0 < \sigma_i < 1$ for $i = 1, ..., n$.

We denote by $\mathscr{M}_n(\mathbb{R}_+)$, the set of all square matrices of order n with positive entries. I and $\mathscr{O}$ denote the identity matrix in $\mathscr{M}_n(\mathbb{R}_+)$ and the zero matrix respectively. The element $(0, 0, ..., 0)^T \in \mathbb{R}^n$ is denoted by $\mathbf{0}$. For $x = (x_1, x_2, ..., x_n)^T, y = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n$, $x \leq y$ means that $x_i \leq y_i$ for $i = 1, ..., n$.

Definition 2.6. [35] Let $A \in \mathscr{M}_n(\mathbb{R}_+)$. A is said to be convergent to zero, if $A^k \longrightarrow \mathscr{O}$, as $k \longrightarrow \infty$.

Definition 2.7. [35]Let $A \in \mathscr{M}_n(\mathbb{R}_+)$ with eigenvalues λ_i, $1 \leq i \leq n$, that is $\lambda_i \in \mathbb{R}$ such that $det(A - \lambda_i I) = \mathscr{O}$. Then the spectral radius of A is defined by

$$r(A) := \max_{1 \leq i \leq n} |\lambda_i|$$

Lemma 2.2. [35, Theorem 3.15] *If $A \in \mathscr{M}_n(\mathbb{R}_+)$ with $r(A) < 1$, then $I - A$ is non singular, and*

$$(I - A)^{-1} = I + A + A^2 + ... + A^k + ...,$$

the series on the right is converging. Conversely, if the series on the right converges, then $r(A) < 1$.

Lemma 2.3. [25] *Let $A = (a_{ij}) \in \mathscr{M}_n(\mathbb{R}_+)$ and $r_i(A) = \sum_{j=1}^{n} a_{ij}$ the i-th row sum of A. Then*

$$\min_{1 \leq i \leq n} r_i(A) \leq r(A) \leq \max_{1 \leq i \leq n} r_i(A).$$

In what follows, we give a brief description of generalized locally convex spaces (GLCS for short). For further details on locally convex spaces (lcs, for short) and GLCS, we refer to [6, 15].

Let E be a linear space over $\mathbb{R}$ or $\mathbb{C}$. We first present the notion of generalized semi-norms.

Definition 2.8. A generalized semi-norm on E is a map $P : E \longrightarrow \mathbb{R}_+^n$ obeying for every $u, v \in E$ and $\mu \in \mathbb{R}$

1. $P(0) = \mathbf{0}$,
2. $P(\mu u) = |\mu| P(u)$,
3. $P(u + v) \leq P(u) + P(v)$.

The P-ball of radius $\epsilon > \mathbf{0}$ centered at $u \in E$ is the set

$$B(u, P, \epsilon) := \{v \in E : P(u - v) < \epsilon\}.$$

Definition 2.9. A family $\mathscr{P} = \{P_\varsigma : \varsigma \in \mathscr{N}\}$ of generalized semi-norms on E is said to be separating if for every $u \neq 0$, there exists $P_\varsigma \in \mathscr{P}$ such that $P_\varsigma(u) \neq \mathbf{0}$.

Definition 2.10. Let $\mathscr{P} = \{P_\varsigma : \varsigma \in \mathscr{N}\}$ be a family of generalized semi-norms on E. The topology generated by the family $\mathscr{P}$ and denoted by $\mathscr{T}(\mathscr{P})$, is the topology whose sub-basis $\mathscr{B}(\mathscr{T})$ is the family of all P_ς-balls, namely

$$\mathscr{B}(\mathscr{T}) = \{B(u, P_\varsigma, \epsilon) : u \in E,\, \epsilon > \mathbf{0},\, \varsigma \in \mathscr{N}\}.$$

The topology $\mathscr{T}(\mathscr{P})$ is locally convex and the pair $(E, \mathscr{T}(\mathscr{P}))$ is called a GLCS, and is Hausdorff if $\mathscr{P}$ is separating.

The notions of convergent sequences, Cauchy sequences, and completeness in GLCS, are similar to those in lcs.

If $\mathscr{P} = \{P_\varsigma : \varsigma \in \mathscr{N}\}$ is a family of separating and generalized semi-norms generating the locally convex topology of E, then the family of generalized pseudo-metrics $\mathscr{D} = \{d_\varsigma : \varsigma \in \mathscr{N}\}$ defined by $d_\varsigma(u, v) = P_\varsigma(u - v)$ generates the same topology of the generalized gauge space $(E, \mathscr{D})$. Therefore, [29, Theorem 2.1] in GLCS setting reduces to the following fixed point theorem.

Theorem 2.1. *Let $(E, \mathscr{P})$ be a Hausdorff complete GLCS, with $\mathscr{P} = \{P_\varsigma : \varsigma \in \mathscr{N}\}$ and let $T : E \longrightarrow E$ be a map such that there exists a function $J : \mathscr{N} \longrightarrow \mathscr{N}$ and $A \in \mathscr{M}_n(\mathbb{R}_+)^{\mathscr{N}}$, $A = \{A_\varsigma\}_{\varsigma \in \mathscr{N}}$ satisfying*

$$P_\varsigma(T(u) - T(v)) \leq A_\varsigma P_{J(\varsigma)}(u - v), \qquad \forall u, v \in E,\ \forall \varsigma \in \mathscr{N} \tag{2.3}$$

$$\sum_{m=1}^{\infty} A_\varsigma A_{J(\varsigma)} \dots A_{J^{m-1}(\varsigma)} P_{J^m(\varsigma)}(u) < \infty, \qquad \forall u \in E,\ \forall \varsigma \in \mathscr{N} \tag{2.4}$$

Then T has a unique fixed point in E, which can be obtained by successive approximations starting from any element of E.

Remark 2.2. Note that if $j = I_{\mathscr{N}}$ (the identity function in $\mathscr{N}$), then Theorem 2.1 reduces to [6, Theorem 2.3].

Throughout the remainder of this paper, we use the following notations.
Y denotes the vector space of all real continuous functions defined on $\mathbb{R}_+$.
$Y(w_p^\lambda, \mathscr{X})$ denotes the complete Hausdorff locally convex space made up of functions in Y with $\{P_{w_p^\lambda,\xi}\}_{\xi \in \mathscr{X}}$ the family of weighted semi-norms generating its topology, defined by

$$P_{w_p^\lambda,\xi}(v) = \max_{t \in \xi} w_p^\lambda(t) \left| v(t) \right|, \quad \xi \in \mathscr{X}; \tag{2.5}$$

where $\mathscr{X}$ denotes the set of all compact subsets ξ of $\mathbb{R}_+$ of the form $[0, x]$, $x \in \mathbb{R}_+$, $p > 1$, $\lambda > 0$ and w_p^λ is the weight function defined via a positive continuous function w by

$$w_p^\lambda(t) = e^{\dfrac{(1-p)\lambda}{p} \displaystyle\int_0^t [w(\tau)]^{\frac{p}{p-1}}\, d\tau}. \tag{2.6}$$

$\mathbb{Y}(W_p^\lambda, \mathscr{X})$ denotes the GLCS made up of elements in $Y^n = \underbrace{Y \times \dots \times Y}_{n\,\text{times}}$ whose topology is generated by the following family of generalized weighted semi-norms $\{\mathbb{P}_{w_p^\lambda,\xi}\}_{\xi \in \mathscr{X}}$ defined for $V = (v_1, \dots, v_n)^T \in Y^n$ by

$$\mathbb{P}_{w_p^\lambda,\xi}(V) = \left(P_{w_p^\lambda,\xi}(v_1), \dots, P_{w_p^\lambda,\xi}(v_n)\right)^T, \tag{2.7}$$

where ξ, $\mathscr{X}$, p, λ and , w_p^λ are as described above.
$J : \mathscr{X} \longrightarrow \mathscr{X}$ is a map that satisfies

$$\xi \subset J(\xi), \ \ \forall \xi \in \mathscr{X}. \tag{2.8}$$

For $U = (u_1, ..., u_n)^T, V = (v_1, ..., v_n)^T \in Y^n$, we put

$$U \circ V := (u_1 \circ v_1, ..., u_n \circ v_n)^T.$$

Generally, all the operators related to the elements of Y^n or $\mathbb{R}^n$ operate component-wise.
$\mathscr{M}_n(Y)$ denotes the set of all square matrices of order n whose entries are continuous functions on the half axis.
For $\mathscr{K} = (k_{ij}) \in \mathscr{M}_n(Y)$ and $V = (v_1, ..., v_n)^T \in Y^n$, we define the vector valued function $\mathscr{K}V$ in Y^n as follows

$$\mathscr{K}V := \left(\sum_{j=1}^n k_{1j} v_j, ..., \sum_{j=1}^n k_{nj} v_j \right)^T.$$

3. Boundedness results

The following boundedness result concerns the purely tempered case, namely $\mathbb{I}_{0^+}^{\vec{\sigma},\eta}$ (i.e., $\eta \neq 0$). The case $\eta = 0$ will be discussed in Remark 3.1(ii) and stated in Theorem 3.2.

Theorem 3.1. *Let $\mathscr{K} = (k_{ij})_{1\leq i,j\leq n} \in \mathscr{M}_n(Y)$. Then, for every $\lambda > 0$, there exists $\mathbf{A}_{\vec{\sigma},\eta}^{p,\lambda} \in \mathscr{M}_n(\mathbb{R}_+)$ (independent of $\mathscr{K}$) such that for every $\xi \in \mathscr{X}$, and $V \in Y^n$, the vector-valued and weighted semi-norm inequality (1.2) holds, where $w := \max\limits_{1\leq i,j\leq n} |k_{ij}|$, and*

$$\mathbf{A}_{\vec{\sigma},\eta}^{p,\lambda} = \begin{pmatrix} {}^1A_{\vec{\sigma},\eta}^{p,\lambda} & \cdots\cdots & {}^1A_{\vec{\sigma},\eta}^{p,\lambda} \\ \vdots & & \\ {}^iA_{\vec{\sigma},\eta}^{p,\lambda} & \cdots\cdots & {}^iA_{\vec{\sigma},\eta}^{p,\lambda} \\ \vdots & & \\ {}^nA_{\vec{\sigma},\eta}^{p,\lambda} & \cdots\cdots & {}^nA_{\vec{\sigma},\eta}^{p,\lambda} \end{pmatrix},$$

with

$${}^iA_{\vec{\sigma},\eta}^{p,\lambda} = \frac{(\eta p)^{(1-\sigma_i)-\frac{1}{p}} \Gamma^{\frac{1}{p}} (p(\sigma_i - 1) + 1)}{\Gamma(\sigma_i) \lambda^{1-\frac{1}{p}}}, \quad 1 \leq i \leq n, \tag{3.1}$$

and

$$1 < p < \min_{1\leq i\leq n} \{\frac{1}{\sigma_i}, \frac{1}{1-\sigma_i}\}. \tag{3.2}$$

Proof. Before proving (1.2), it should be checked that for every $\sigma \in (0, 1)$, we have

$$\mathscr{I}_{0^+}^{\sigma,\eta} v \ \text{ is continuous on } \xi, \quad \forall v \in Y, \ \forall \xi \in \mathscr{X}. \tag{3.3}$$

This ensures that the left-hand side of (1.2) is well-defined. Indeed, note that, for every $v \in Y$ and every $t \in \xi \subset \mathscr{X}$, we have

$$\begin{aligned} \mathscr{I}_{0^+}^{\sigma,\eta} v(t) &= \int_0^t \frac{e^{-\eta(t-\tau)}(t-\tau)^{\sigma-1}}{\Gamma(\sigma)} v(\tau)\, d\tau \\ &= (\varphi_1 * \varphi_2)(t), \end{aligned}$$

where
$$\varphi_1(t) = e^{-\eta\, t}\, t^{\sigma-1}\chi_{\mathbb{R}_+}(t), \qquad \text{and} \qquad \varphi_2(t) = v(t)\,\chi_\xi(t).$$
We have
$$\int_{\mathbb{R}} |\varphi_1(t)|\, dt \quad \leq \quad \int_0^\infty e^{-\eta\, t}\, t^{\sigma-1}\, dt.$$
By changing the variable $x = \eta\, t$, we obtain the integrability of φ_1
$$\begin{aligned}\int_{\mathbb{R}} |\varphi_1(t)|\, dt \quad &\leq \quad \frac{1}{\eta^\sigma}\int_0^\infty e^{-x}\, x^{\sigma-1}\, dx \\ &= \quad \frac{1}{\eta^\sigma}\,\Gamma(\sigma) < \infty.\end{aligned}$$

The continuity of $\mathscr{I}_{0+}^{\sigma,\eta}\, v$ on ξ follows from the continuity under the integral sign, a standard result in the Lebesgue integration theory; see the proof of [27, Lemma 3.2: $\psi(t) = t$] for details. Consequently, Property (3.3) holds.
In accordance with the notations introduced in the preliminary section, we have
$$\mathbb{I}_{0+}^{\vec{\sigma},\eta}\left(\mathscr{K}V(.)\right)(t) = \left(\mathscr{I}_{0+}^{\sigma_1,\eta}\left(\sum_{j=1}^n k_{1j}(.)v_j(.)\right)(t), \ldots, \mathscr{I}_{0+}^{\sigma_n,\eta}\left(\sum_{j=1}^n k_{nj}(.)v_j(.)\right)(t)\right)^T. \tag{3.4}$$
Let $\xi \in \mathscr{X}$ and $t \in \xi$. Then for every $1 \leq i \leq n$, we have
$$\left|\mathscr{I}_{0+}^{\sigma_i,\eta}\left(\sum_{j=1}^n k_{ij}(.)v_j(.)\right)(t)\right| = \left|\sum_{j=1}^n \int_0^t \frac{e^{-\eta(t-\tau)}(t-\tau)^{\sigma_i-1}}{\Gamma(\sigma_i)} k_{ij}(\tau)\, v_j\,(\tau)\, d\tau\right| \tag{3.5}$$
$$\begin{aligned}&\leq \sum_{j=1}^n \frac{1}{\Gamma(\sigma_i)}\int_0^t \left[e^{-\eta(t-\tau)}(t-\tau)^{\sigma_i-1}\right] |k_{ij}(\tau)\, v_j\,(\tau)|\, d\tau \\ &\leq \sum_{j=1}^n \frac{1}{\Gamma(\sigma_i)}\int_0^t \left[e^{-\eta(t-\tau)}(t-\tau)^{\sigma_i-1}\right] [w(\tau)\, |v_j\,(\tau)|]\, d\tau.\end{aligned}$$
Let q be the conjugate of p given in (3.2). Then, Hölder's inequality yields
$$\left|\mathscr{I}_{0+}^{\sigma_i,\eta}\left(\sum_{j=1}^n k_{ij}(.)v_j(.)\right)(t)\right| \leq$$
$$\frac{1}{\Gamma(\sigma_i)}\sum_{j=1}^n \left\{\int_0^t e^{-\eta p(t-\tau)}(t-\tau)^{p(\sigma_i-1)}d\tau\right\}^{\frac{1}{p}} \left\{\int_0^t [w(\tau)\,|v_j(\tau)|]^q\, d\tau\right\}^{\frac{1}{q}}. \tag{3.6}$$

Since
$$[w(\tau)]^q = \frac{\frac{d}{d\tau}\left[\frac{1}{\lambda}e^{\lambda\int_0^\tau (w(\eta))^q d\eta}\right]}{e^{\lambda\int_0^\tau (w(\eta))^q d\eta}},$$
In view of (2.6), the expression above can be interpreted as
$$[w(\tau)]^q = \frac{d}{d\tau}\left[\frac{1}{\lambda}e^{\lambda\int_0^\tau (w(\eta))^q d\eta}\right]\left[w_p^\lambda(\tau)\right]^q. \tag{3.7}$$

On the other hand, making the change of variable $s=\eta p(t-\tau)$, we get

$$\begin{aligned}\int_0^t e^{-\eta p(t-\tau)}(t-\tau)^{p(\sigma_i-1)}d\tau &= \frac{1}{(\eta p)^{p(\sigma_i-1)+1}}\int_0^{\eta pt} e^{-s}\, s^{p(\sigma_i-1)}ds\\ &\leq \frac{1}{(\eta p)^{p(\sigma_i-1)+1}}\int_0^{\infty} e^{-s}\, s^{p(\sigma_i-1)}ds.\end{aligned}$$

Consequently,

$$\int_0^t e^{-\eta p(t-\tau)}(t-\tau)^{p(\sigma_i-1)}d\tau \leq \frac{\Gamma\left(p(\sigma_i-1)+1\right)}{(\eta p)^{p(\sigma_i-1)+1}}. \tag{3.8}$$

Returning now to (3.6) and using (3.7) together with (3.8), we obtain

$$\begin{aligned}&\left|\mathscr{I}_{0^+}^{\sigma_i,\eta}\left(\sum_{j=1}^{n}k_{ij}(.)v_j(.)\right)(t)\right| \leq \frac{1}{\Gamma(\sigma_i)}\left\{\Gamma\left(p(\sigma_i-1)+1\right)\,(\eta p)^{p(1-\sigma_i)-1}\right\}^{\frac{1}{p}}\times\\ &\sum_{j=1}^{n}\left\{\int_0^t\frac{d}{d\tau}\left[\frac{1}{\lambda}e^{\lambda\int_0^{\tau}(w(\rho))^q\,d\rho}\right]\left[w_p^{\lambda}(\tau)\ \left|v_j\left(\tau\right)\right|\right]^q d\tau\right\}^{\frac{1}{q}}\\ &\leq \frac{1}{\Gamma(\sigma_i)}\left\{\Gamma\left(p(\sigma_i-1)+1\right)\,(\eta p)^{p(1-\sigma_i)-1}\right\}^{\frac{1}{p}}\sum_{j=1}^{n}\left\{\frac{\lambda^{-\frac{1}{q}}}{w_p^{\lambda}(t)}\max_{\rho\in[0,t]}\ w_p^{\lambda}(\rho)\left|v_j\left(\rho\right)\right|\right\}\\ &\leq \frac{1}{\Gamma(\sigma_i)}\left\{\Gamma\left(p(\sigma_i-1)+1\right)\,(\eta p)^{p(1-\sigma_i)-1}\right\}^{\frac{1}{p}}\sum_{j=1}^{n}\left\{\frac{\lambda^{-\frac{1}{q}}}{w_p^{\lambda}(t)}\max_{\rho\in\xi}\ w_p^{\lambda}(\rho)\left|v_j\left(\rho\right)\right|\right\}\\ &= \frac{1}{\Gamma(\sigma_i)}\left\{\Gamma\left(p(\sigma_i-1)+1\right)\,(\eta p)^{p(1-\sigma_i)-1}\right\}^{\frac{1}{p}}\sum_{j=1}^{n}\left\{\frac{\lambda^{-\frac{1}{q}}}{w_p^{\lambda}(t)}P_{w_p^{\lambda},\xi}(v_j)\right\}.\end{aligned}$$

Thus, in view of (3.1), the last inequality can be written for every $t\in\xi$ as

$$\left|\mathscr{I}_{0^+}^{\sigma_i,\eta}\left(\sum_{j=1}^{n}k_{ij}(.)v_j(.)\right)(t)\right| \leq \frac{1}{w_p^{\lambda}(t)}\sum_{j=1}^{n}{}^{i}A_{\sigma,\eta}^{p,\lambda}\,P_{w_p^{\lambda},\xi}(v_j). \tag{3.9}$$

Multiplying the inequality in (3.9) by $w_p^{\lambda}(t)$, and then taking the maximum over ξ, we obtain

$$P_{w_p^{\lambda},\xi}\left(\mathscr{I}_{0^+}^{\sigma_i,\eta}\left(\sum_{j=1}^{n}k_{ij}(.)v_j(.)\right)\right) \leq \sum_{j=1}^{n}{}^{i}A_{\sigma,\eta}^{p,\lambda}P_{w_p^{\lambda},\xi}(v_j). \tag{3.10}$$

Now, taking into account (2.7) and (3.4), (1.2) follows immediately from (3.10). $\square$

Remark 3.1.

(i) It should be noted that the choice of the weight functions involves two auxiliary positive parameters, namely p and λ. Unlike fractional orders σ_i and the tempering parameter η, which are intrinsic to the operator $\mathbb{I}_{0^+}^{\vec{\sigma},\eta}$, the exponent p is a free auxiliary parameter introduced solely for the definition of the weighted seminorms. It is required only to satisfy the condition (3.2), which can always be fulfilled and arises from the integrability requirements associated with Hölder's inequality. Due to the factor $\lambda^{1-\frac{1}{p}}$ in ${}^{i}A_{\vec{\sigma},\eta}^{p,\lambda}$ from (3.1), the free parameter λ can be chosen sufficiently large to satisfy the spectral radius condition (4.9) required for the applications considered in Section 4.

(ii) The presence of the exponential damping factor in $\mathbb{I}_{0+}^{\vec{\sigma},\eta}$ when $\eta \neq 0$ plays a crucial role in achieving the boundedness result (1.2). In the absence of this factor (i.e., when $\eta = 0$), approach used in the proof of Theorem 3.1 can be adapted to derive the boundedness result for vector-order Riemann-Liouville fractional integrals $\mathbb{I}_{0+}^{\vec{\sigma}}$, as stated in Theorem 3.2. Indeed, in (3.5) with $\eta = 0$, multiplying e^{τ} by the term $(t-\tau)$ and then dividing it into the term $k_{ij}(\tau)$ allows the proof of Theorem 3.1 to be extended to this case, provided the weight function w_p^{λ} is adjusted appropriately to

$$h_p^{\lambda}(t) = e^{-t + \dfrac{(1-p)\lambda}{p} \displaystyle\int_0^t [h(\tau)]^{\frac{p}{p-1}} \, d\tau}. \tag{3.11}$$

In this case, $Y(h_p^{\lambda}, \mathscr{X})$ and $\mathbb{Y}(h_p^{\lambda}, \mathscr{X})$ denote the complete Hausdorff lcs and GLCS, respectively, consisting of functions in Y and Y^n, respectively. Their topologies are generated by the families of seminorms $\{P_{h_p^{\lambda},\xi}\}_{\xi\in\mathscr{X}}$ and $\{\mathbb{P}_{h_p^{\lambda},\xi}\}_{\xi\in\mathscr{X}}$, defined as in (2.5) and (2.7), respectively, with the weight function w_p^{λ} replaced by the one in (3.11).

The following theorem states a boundedness result for vector-order Riemann-Liouville fractional integral operators $\mathbb{I}_{0+}^{\vec{\sigma}}$ in spaces $\mathbb{Y}(h_p^{\lambda}, \mathscr{X})$.

Theorem 3.2. *Let $\mathscr{K} = (k_{ij})_{1\le i,j\le n} \in \mathscr{M}_n(Y)$. Then, for every $\lambda > 0$, there exists $\tilde{\mathbf{A}}_{\vec{\sigma}}^{p,\lambda} \in \mathscr{M}_n(\mathbb{R}_+)$ (independent of $\mathscr{K}$) such that for every $\xi \in \mathscr{X}$ and $V \in Y^n$, the vector-valued weighted semi-norm inequality (1.2) holds, where $h := \max\limits_{1\le i,j\le n} |k_{ij}|$, p as in (3.2), and each element in the i^{th} line of $\tilde{\mathbf{A}}_{\vec{\sigma}}^{p,\lambda}$ is given by*

$${}^{i}\tilde{A}_{\vec{\sigma}}^{p,\lambda} = \frac{p^{(1-\sigma_i)-\frac{1}{p}}\, \Gamma^{\frac{1}{p}}\left(p(\sigma_i-1)+1\right)}{\Gamma(\sigma_i)\, \lambda^{1-\frac{1}{p}}}, \quad 1 \le i \le n. \tag{3.12}$$

Proof.
See Remark 3.1 (ii). □

For $n = 1$, Theorem 3.1 and Theorem 3.2 reduce to the following Corollary.

Corollary 3.1. *Let f be a function of Y. Then, for every $\lambda > 0$, there exist positive constants $C_{\sigma,\eta}^{p,\lambda}$ and $\tilde{C}_{\sigma}^{p,\lambda}$ (independent of f) such that for every $\xi \in \mathscr{X}$ and $v \in Y$, the following inequalities hold*

$$P_{w_p^{\lambda},\xi}\left(\mathscr{I}_{0+}^{\sigma,\eta}(f\, v)\right) \le C_{\sigma,\eta}^{p,\lambda}\, P_{w_p^{\lambda},\xi}(v); \tag{3.13}$$

$$P_{h_p^{\lambda},\xi}\left(I_{0+}^{\sigma}(f\, v)\right) \le \tilde{C}_{\sigma}^{p,\lambda}\, P_{h_p^{\lambda},\xi}(v), \tag{3.14}$$

where $w = h = |f|$, $C_{\sigma,\eta}^{p,\lambda}$, and $\tilde{C}_{\sigma}^{p,\lambda}$ are given by the right hand side of (3.1), and (3.12), respectively, with σ replacing σ_i.

The following theorem provides a weak boundedness result for the operators $\mathbb{I}_{0+}^{\vec{\sigma},\eta}$ that involve specific delays, see Remark 3.2.

Theorem 3.3. *Let $G = (g_1, ..., g_n)^T \in Y^n$ such that for some positive constant M, for every $i = 1, ..., n$ and $t \geq 0$: $0 \leq g_i(t) \leq \max\{M, t\}$, $\mathscr{K} = (k_{ij}) \in \mathscr{M}_n(Y)$ and $\eta \neq 0$. Then, for every $\lambda > 0$, $\xi \in \mathscr{X}$ and $V \in Y^n$, the following vector valued weighted semi-norm inequality holds true*

$$\mathbb{P}_{w_p^\lambda, \xi}\left(\mathbb{I}_{0^+}^{\vec{\sigma},\eta}\left(\mathscr{K}\left(V \circ G\right)\right)\right) \leq \mathbf{A}_{\vec{\sigma},\eta}^{p,\lambda}\, \mathbb{P}_{w_p^\lambda, J(\xi)}\left(V\right), \tag{3.15}$$

where w, $\mathbf{A}_{\vec{\sigma},\eta}^{p,\lambda}$ are given as in Theorem 3.1, and J satisfies (2.8).

Proof. We have

$$\mathbb{I}_{0^+}^{\vec{\sigma},\eta}\left(\mathscr{K}\left(V \circ G\right)(.)\right)(t) = \left(\mathscr{I}_{0^+}^{\sigma_1,\eta}\left(\sum_{j=1}^{n} k_{1j}(.)(v_j \circ g_j)(.)\right)(t), ..., \mathscr{I}_{0^+}^{\sigma_n,\eta}\left(\sum_{j=1}^{n} k_{nj}(.)(v_j \circ g_j)(.)\right)(t)\right)^T. \tag{3.16}$$

By following the steps used to prove (3.6) in Theorem 3.1, we obtain, for every $i = 1, ..., n$ and $t \in \xi \in \mathscr{X}$, the following inequality

$$\left|\mathscr{I}_{0^+}^{\sigma_i,\eta}\left(\sum_{j=1}^{n} k_{ij}(.)(v_j \circ g_j)(.)\right)(t)\right| \leq \frac{1}{\Gamma(\sigma_i)}\sum_{j=1}^{n}\left\{\int_0^t e^{-\eta p(t-\tau)}(t-\tau)^{p(\sigma_i-1)}d\tau\right\}^{\frac{1}{p}}\left\{\int_0^t \left[w(\tau)\left|v_j\left(g_j(\tau)\right)\right|\right]^q d\tau\right\}^{\frac{1}{q}}. \tag{3.17}$$

Since we have for every $j = 1, ..., n$

$$0 \leq g_j(\tau) \leq \max\{\tau, M\} := \tau_M,$$

the following inequality holds for all t such that $0 < \tau \leq t$,

$$w_p^\lambda(\tau)\,\left|v_j\left(g_j(\tau)\right)\right| \leq \max_{\rho \in [0, t_M]} w_p^\lambda(\tau)\,\left|v_j\left(\tau\right)\right|, \tag{3.18}$$

where $t_M := \max\{t,\, M\}$.
Combining (3.7) with (3.18) and taking into account (3.8), we deduce from (3.17) that the following inequality holds for every $t \in \xi$

$$\left|\mathscr{I}_{0^+}^{\sigma_i,\eta}\left(\sum_{j=1}^{n} k_{ij}(.)(v_j \circ g_j)(.)\right)(t)\right| \leq \frac{1}{w_p^\lambda(t)}\sum_{j=1}^{n} {}^{i}A_{\sigma,\eta}^{p,\lambda} \max_{\rho \in [0, t_M]} w_p^\lambda(\rho)\,\left|v_j\left(\rho\right)\right|. \tag{3.19}$$

Let us define the map J by

$$J(\xi) = \left[0,\, \max\{\sup \xi, M\}\right]. \tag{3.20}$$

Note that clearly, the map J given in (3.20) satisfies (2.8). Moreover, for every $t \in \xi$, we have

$$[0,\, t_M] \subset J(\xi). \tag{3.21}$$

Now multiplying the inequality in (3.19) by $w_p^\lambda(t)$, then taking the maximum over ξ, we obtain in view of (3.21) the following inequality

$$P_{w_p^\lambda, \xi}\left(\mathscr{I}_{0^+}^{\sigma_i,\eta}\left(\sum_{j=1}^{n} k_{ij}(.)(v_j \circ g_j)(.)\right)\right) \leq \sum_{j=1}^{n} {}^{i}A_{\sigma,\eta}^{p,\lambda}\, P_{w_p^\lambda, J(\xi)}(v_j). \tag{3.22}$$

Finally, taking into account (2.7) and (3.16), (3.15) yields immediately from (3.22). □

Remark 3.2. In view of (2.8), the right-hand side of (3.15) involves the seminorm over the larger compact set $J(\xi)$, which yields a weak boundedness estimate. This weakening reflects the intrinsic delay effect induced by the map G and remains sufficient for the applications of delayed fractional differential equations developed in Section 4.

Note that the weak boundedness result of Theorem 3.3 concerns the purely tempered case, namely $\eta \neq 0$. The following theorem states the result for case $\eta = 0$, which can be obtained, as explained again in Remark 3.1(ii).

Theorem 3.4. *Let $\mathscr{K} = (k_{ij}) \in \mathscr{M}_n(Y)$ and G as given in Theorem 3.3. Then, for every $\lambda > 0$, $\xi \in \mathscr{X}$ and $V \in Y^n$ the following vector valued weighted semi-norm inequality holds*

$$\mathbb{P}_{h_p^\lambda,\xi}\left(\mathbb{I}_{0+}^{\vec{\sigma}}\left(\mathscr{K}\left(V \circ G\right)\right)\right) \leq \tilde{\mathbf{A}}_{\vec{\sigma}}^{p,\lambda}\ \mathbb{P}_{h_p^\lambda,J(\xi)}\left(V\right), \tag{3.23}$$

where h, $\tilde{\mathbf{A}}_{\vec{\sigma}}^{p,\lambda}$ are given as in Theorem 3.2, and J satisfies (2.8).

Letting $n = 1$, Theorem 3.3 and Theorem 3.4 reduce to the following Corollary.

Corollary 3.2. *Let f and g be functions of Y such that $0 \leq g(t) \leq t$. Then, for every $\lambda > 0$, there exist positive constants $C_{\sigma,\eta}^{p,\lambda}$ and $\tilde{C}_{\sigma}^{p,\lambda}$ (independent of f and g) such that for every $\xi \in \mathscr{X}$ and $v \in Y$, the following inequalities hold*

$$P_{w_p^\lambda,\xi}\left(\mathscr{I}_{0+}^{\sigma,\eta}\left(f\left(v \circ g\right)\right)\right) \leq C_{\sigma,\eta}^{p,\lambda}\ P_{w_p^\lambda,J(\xi)}(v), \tag{3.24}$$

$$P_{h_p^\lambda,\xi}\left(I_{0+}^{\sigma}\left(f\left(v \circ g\right)\right)\right) \leq \tilde{C}_{\sigma}^{p,\lambda}\ P_{h_p^\lambda,J(\xi)}(v), \tag{3.25}$$

where $w = h = |f|$, J satisfying (2.8) , $C_{\sigma,\eta}^{p,\lambda}$ and $\tilde{C}_{\sigma}^{p,\lambda}$ are given by the right hand side of (3.1) and (3.12), respectively, with σ replacing σ_i.

4. Related existence and stability results

We consider the following nonlinear multi-order Caputo tempered fractional functional differential equations

$$\begin{cases} {}^{C}D^{\sigma_1,\eta}v_1(t) = \psi_1\left(t, v_1(g_1(t)), ..., v_n(g_n(t))\right), & t > 0, \\ {}^{C}D^{\sigma_2,\eta}v_2(t) = \psi_2\left(t, v_1(g_1(t)), ..., v_n(g_n(t))\right), & t > 0, \\ \vdots \\ {}^{C}D^{\sigma_n,\eta}v_n(t) = \psi_n\left(t, v_1(g_1(t)), ..., v_n(g_n(t))\right), & t > 0, \end{cases} \tag{4.1}$$

with the following initial condition

$$(v_1(0), ..., v_n(0))^T = (v_1^0, ..., v_n^0)^T, \tag{4.2}$$

where ${}^{C}D^{\sigma_i,\eta}$ denotes the tempered fractional derivative operator of Caputo type of order $\sigma_i \in (0,1)$. For $i = 1, ..., n$, $\psi_i : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a nonlinear continuous function, $g_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a continuous function such that $0 \leq g_i(t) \leq \max\{M, t\}$, where M is a constant, and v_i^0 is a given constant.

In accordance with Definition 2.5 and the notation introduced in the Preliminary section, (4.1)-(4.2) can be written in a vector form as follows

$$\begin{cases} {}^{C}\mathbb{D}_{0^+}^{\vec{\sigma},\eta} V(t) = \Psi\left(t, (V \circ G)(t)\right), \quad t > 0 \\ V(0) = V^0, \end{cases} \tag{4.3}$$

where $V = (v_1, ..., v_n)^T$ is a vector valued function with $V(t) = (v_1(t), ..., v_n(t))^T$, $\vec{\sigma} = (\sigma_1, ..., \sigma_n)^T$, $V^0 = (V_1^0, ..., V_n^0)^T$, $\Psi : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a vector valued function defined for $x \in \mathbb{R}^n$ by $\Psi(t, x) = (\psi_1(t, x), ..., \psi_n(t, x))^T$, and $G : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ defined by $G(t) = (g_1(t), ..., g_n(t))^T$.
In the sense of Definition 2.5, the extension to the vector case of [11, Lemma 4.1] yields to the following lemma, see also [1].

Lemma 4.1. *Let $\chi = (\chi_1, ..., \chi_n)^T \in Y^n$. Then the solution of the following tempered fractional vector order differential equation*

$$\begin{cases} {}^{C}\mathbb{D}_{0^+}^{\vec{\sigma},\eta} V(t) = \chi(t), \quad t > 0, \\ V(0) = V^0, \end{cases}$$

is given by the following symbolic vector form of integral equation

$$V(t) = e^{-\eta t} V^0 + \int_0^t \frac{e^{-\eta(t-\tau)}(t-\tau)^{\vec{\sigma}-1}}{\Gamma(\vec{\sigma})} \chi(\tau) d\tau, \tag{4.4}$$

where multiplication and integration are understood componentwise.

The following definition clarifies what we mean by a solution to the problem (4.3).

Definition 4.1. A function $V \in Y^n$ is said to be a solution to (4.3) if it satisfies the integral equation (4.4) with $\chi(t) = \Psi\left(t, (V \circ G)(t)\right)$.

Suppose that for all $i = 1, ..., n$, there exist continuous functions k_{ij} with $j = 1, ..., n$ such that for all $(\delta_1, ..., \delta_n)^T$, $(\theta_1, ..., \theta_n)^T \in \mathbb{R}^n$ and for all $t \in \mathbb{R}_+$, we have

$$\left|\psi_i\left(t, \delta_1, ..., \delta_n\right) - \psi_i\left(t, \theta_1, ..., \theta_n\right)\right| \leq \sum_{j=1}^{n} |k_{ij}(t)| \, |\delta_j - \theta_j| \,. \tag{4.5}$$

Note that hypothesis (4.5) reads in vector form as follows

($\mathscr{H}_1$) $\exists \mathscr{K} = (k_{ij}) \in \mathscr{M}_n(Y)$ such that for all $\Delta = (\delta_1, ..., \delta_n)^T, \Theta = (\theta_1, ..., \theta_n)^T \in \mathbb{R}^n$, the following vector form inequality holds for every $t > 0$

$$\left|\Psi(t, \Delta) - \Psi(t, \Theta)\right| \leq |\mathscr{K}(t)| \, |\Delta - \Theta| \,,$$

where $|\,.\,|$ is defined component wisely in $\mathbb{R}^n$.
We are now ready to prove the existence-uniqueness result for (4.3).

Theorem 4.1. *Under hypothesis ($\mathscr{H}_1$), the problem (4.3) admits a unique global solution in Y^n in both cases: $\eta \neq 0$ and $\eta = 0$.*

Proof. Let us start with the purely tempered case, namely $\eta \neq 0$.
We consider the GLCS $\mathbb{Y}(w_p^\lambda, \mathscr{X})$ where w, and p are given as in Theorem 3.3, and λ is a positive auxiliary parameter to be designated later. In view of Definition 4.1, the solutions of (4.3) are the fixed points of the operator $\mathscr{R}$ defined on $(w_p^\lambda, \mathscr{X})$ by

$$\mathscr{R}V(t) = e^{-\eta t}\, V^0 + \int_0^t \frac{e^{-\eta(t-\tau)}(t-\tau)^{\vec{\sigma}-1}}{\Gamma(\vec{\sigma})} \Psi\left(\tau, (V \circ G)(\tau)\right) d\tau. \tag{4.6}$$

Since Ψ and G are componentwise continuous functions, it follows from (3.3) that the right hand side of (4.6) is componentwise continuous for every componentwise continuous function V. Hence, the operator $\mathscr{R}$ is well defined. To establish the desired result, it suffices to verify that $\mathscr{R}$ satisfies (2.3) and (2.4) with $\mathscr{N} = \mathscr{X}$ and $p_\nu = \mathbb{P}_{w_p^\lambda, \xi}$, and then apply Theorem 2.1.
Let $V = (v_1, ..., v_n)^T, Z = (z_1, ..., z_n)^T \in Y^n$. Then using $(\mathscr{H}_1)$, we have for every $t \in \mathbb{R}_+$

$$\begin{aligned}
|\mathscr{R}V(t) - \mathscr{R}Z(t)| &\leq \int_0^t \frac{e^{-\eta(t-\tau)}(t-\tau)^{\vec{\sigma}-1}}{\Gamma(\vec{\sigma})} |\mathscr{K}(\tau)| \left|(V \circ G)(\tau) - (Z \circ G)(\tau)\right| d\tau \\
&= \int_0^t \frac{e^{-\eta(t-\tau)}(t-\tau)^{\vec{\sigma}-1}}{\Gamma(\vec{\sigma})} |\mathscr{K}(\tau)| \left|((V - Z) \circ G)(\tau)\right| d\tau \\
&= \mathbb{I}_{0^+}^{\vec{\sigma},\eta} \left(|\mathscr{K}| \left(|V - Z| \circ G\right)(.)\right)(t).
\end{aligned}$$

Then, in view of Theorem 3.3, we have for every $\xi \in \mathscr{X}$

$$\mathbb{P}_{w_p^\lambda, \xi}\left(\mathscr{R}V - \mathscr{R}Z\right) \leq \mathbf{A}_{\vec{\sigma},\eta}^{p,\lambda}\, \mathbb{P}_{w_p^\lambda, J(\xi)}\left(V - Z\right).$$

In other words, (2.3) yields with J the map defined by (3.20) which implies that

$$J^m(\xi) = J(\xi), \quad \forall \xi \in \mathscr{X},\ \forall m \geq 2, \tag{4.7}$$

and $A_k = \mathbf{A}_{\vec{\sigma},\eta}^{p,\lambda}$, which is independent of k, and consequently (2.4) becomes

$$\sum_{m=0}^{\infty} \left(\mathbf{A}_{\vec{\sigma},\eta}^{p,\lambda}\right)^{m+1} \mathbb{P}_{w_p^\lambda, J^m(\xi)}(V). \tag{4.8}$$

Let us choose λ sufficiently large such that

$$n \max_{1 \leq i \leq n} {}^{i}A_{\vec{\sigma},\eta}^{p,\lambda} < 1, \tag{4.9}$$

which implies by means of Lemma 2.3 that

$$r\left(\mathbf{A}_{\vec{\sigma},\eta}^{p,\lambda}\right) < 1. \tag{4.10}$$

On the other hand, in view of (4.7), for every $V \in Y^n$ and $\xi \in \mathscr{X}$, we have

$$\text{The set } \left\{\mathbb{P}_{w_p^\lambda, J^m(\xi)}(V) : m = 0, 1, 2, \dots\right\} \text{ is finite }. \tag{4.11}$$

Now, combining (4.10) and (4.11), we deduce from Lemma 2.2 that the series in (4.8) is convergent, and therefore (2.4) holds true.In summary, $\mathscr{R}$ admits a unique fixed point in $\mathbb{Y}(w_p^\lambda, \mathscr{X})$, which is the unique solution of (4.3) when $\eta \neq 0$.
In the case $\eta = 0$, we consider the operator $\mathscr{R}$ defined on $\mathbb{Y}(h_p^\lambda, \mathscr{X})$ by (4.6), where h and p are given as in Theorem 3.4. By proceeding as above, but using Theorem 3.4 instead of Theorem 3.3, we obtain the unique global solution in Y^n of (4.3) for the case $\eta = 0$. □

The existence-uniqueness result for the following associated problem to (4.3) on finite intervals, will be discussed in Remark 4.1 and stated in Corollary 4.1.

$$\begin{cases} \mathbb{I}_{0^+}^{\vec{\sigma},\eta} V(t) = \Phi\left(t, (V \circ L)(t)\right), \;\; 0 < t \leq H < \infty \\ V(0) = V^0, \end{cases} \tag{4.12}$$

where $\Phi : [0,\, H] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $L : [0,\, H] \longrightarrow \mathbb{R}^n$ are component-wise continuous vector valued functions such that $L(t) \leq (t, ..., t)^T$, for all $0 < t \leq H$.

Remark 4.1. Although the approach used in the proof of Theorem 4.1 is formulated for systems defined at infinite intervals, it can be easily adapted to the problem (4.12). Indeed, it suffices to extend Φ to $\mathbb{R}_+$, say by a function $\tilde{\Phi}$, and then apply Theorem 4.1 to the corresponding problem (4.3). This yields a global solution $V \in Y^n$. The restriction of V to $[0,\, H]$ then provides the desired solution of (4.12).

Corollary 4.1. *Suppose that the following condition holds*

($\mathscr{H}_2$) *There exists a square matrix $\mathscr{S}$ of order n whose entries are positive continuous functions defined on $[0, H]$ such that for all Δ, $\Theta \in \mathbb{R}^n$, the following vector form inequality holds, for every $t \in [0, H]$*

$$|\Phi\left(t, \Delta\right) - \Phi\left(t, \Theta\right)| \leq |\mathscr{S}(t)| \; |\Delta - \Theta| \,.$$

Then (4.12) *admits a unique continuous solution on* $[0, H]$ *in both cases:* $\eta \neq 0$ *and* $\eta = 0$.

Remark 4.2. Concerning the existence-uniqueness results for fractional differential equations, many hypotheses imposed in the literature are too restrictive compared to ($\mathscr{H}_1$) and ($\mathscr{H}_2$). Therefore, Theorem 4.1 and Corollary 4.1 provide significant improvements and generalizations of many results, such as those in [1, 2, 9, 16]. Furthermore, we believe that our approach can be adapted and applied to obtain improved results for many other problems, such as those in [3, 20, 26], which require further investigations.

In what follows, we introduce and discuss the uniform stability of solutions of (4.3) in some generalized sense.

Definition 4.2. (Weak uniform stability). Let w, h, p, λ, and J be as given in Theorem 4.1. We say that the solution of (4.3) is weakly uniformly stable (with respect to $\{\mathbb{P}_{w_p^\lambda, \xi}\}_{\xi \in \mathscr{X}}$) if for every $\boldsymbol{\epsilon} = (\epsilon_1, ..., \epsilon_n) > \mathbf{0}$, there exists $\boldsymbol{\delta} = (\delta_1, ..., \delta_n) > \mathbf{0}$ such that for every two solutions V and $\tilde{V}$ of (4.3) with the initial condition $V(0) = V^0 = (v_1^0, ..., v_n^0)^T$ and $\tilde{V}(0) = \tilde{V}^0 = (\tilde{v}_1^0, ..., \tilde{v}_n^0)^T$ respectively, the following condition holds for every $\xi \in \mathscr{X}$

$$\left| V^0 - \tilde{V}^0 \right| < \boldsymbol{\delta} \;\text{ implies }\; \mathbb{P}_{w_p^\lambda, J(\xi)}\,(V - \tilde{V}) < \boldsymbol{\epsilon}, \;\text{ when } \eta \neq 0, \tag{4.13}$$

$$\left| V^0 - \tilde{V}^0 \right| < \boldsymbol{\delta} \;\text{ implies }\; \mathbb{P}_{h_p^\lambda, J(\xi)}\,(V - \tilde{V}) < \boldsymbol{\epsilon}, \;\text{ when } \eta = 0. \tag{4.14}$$

Remark 4.3. This notion generalizes the classical uniform stability. When $J(\xi) = \xi$ (i.e., no delay-induced expansion), Definition 4.2 reduces to the classical uniform stability criterion expressed in terms of the weighted seminorm family $\{\mathbb{P}_{w_p^\lambda, \xi}\}_{\xi \in \mathscr{X}}$. The appearance of $J(\xi)$ reflects the effect of the delay map G (see Theorem 3.3) and is essential to establish the estimates used in the proof of Theorem 4.2.

Theorem 4.2. *Let* $(\mathscr{H}_1)$ *be satisfied. Then, the solution of* (4.3) *is uniformly stable.*

Proof. For $\eta \neq 0$, let V and $\tilde{V}$ be tow solutions of (4.3) with the initial condition $V(0) = V^0 = (v_1^0, ..., v_n^0)^T$ and $\tilde{V}(0) = \tilde{V}^0 = (\tilde{v}_1^0, ..., \tilde{v}_n^0)^T$, respectively. Suppose that w, p, λ, and J are as given in Theorem 4.1.
Let $\xi \in \mathscr{X}$, and let $t \in J(\xi)$. In view of Definition 4.1 together with $(\mathscr{H}_1)$, we have the following inequalities, for every $i = 1, ..., n$

$$|v_i(t) - \tilde{v}_i(t)| \leq$$

$$e^{-\eta t}\left|v_i^0 - \tilde{v}_i^0\right| + \sum_{j=1}^{n} \frac{1}{\Gamma(\sigma_i)} \int_0^t e^{-\eta(t-\tau)}(t-\tau)^{\sigma_i - 1}|k_{ij}(\tau)| \left|v_j\left(g_j(\tau)\right) - \tilde{v}_j\left(g_j(\tau)\right)\right| d\tau$$

$$\leq e^{-\eta t}\left|v_i^0 - \tilde{v}_i^0\right| + \sum_{j=1}^{n} \frac{1}{\Gamma(\sigma_i)} \int_0^t e^{-\eta(t-\tau)}(t-\tau)^{\sigma_i - 1} w(\tau) \left|v_j\left(g_j(\tau)\right) - \tilde{v}_j\left(g_j(\tau)\right)\right| d\tau.$$

Reputing an argument similar to that used in the proof of Theorem 3.3, we derive the following inequality

$$|v_i(t) - \tilde{v}_i(t)| \leq e^{-\eta t}\left|v_i^0 - \tilde{v}_i^0\right| + \frac{1}{w_p^\lambda(t)} \sum_{j=1}^{n} {}^{i}A_{\vec{\sigma},\eta}^{p,\lambda} \max_{\rho \in J^2(\xi)} w_p^\lambda(\rho) \left|v_j(\rho) - \tilde{v}_j(\rho)\right|;$$

where ${}^{i}A_{\vec{\sigma},\eta}^{p,\lambda}$ is given by (3.1).
Multiplying the above inequality by $w_p^\lambda(t)$, and taking into account that $w_p^\lambda \leq 1$, we obtain

$$\begin{aligned} w_p^\lambda(t)\left|v_i(t) - \tilde{v}_i(t)\right| &\leq w_p^\lambda(t)\, e^{-\eta t}\left|v_i^0 - \tilde{v}_i^0\right| + \sum_{j=1}^{n} {}^{i}A_{\vec{\sigma},\eta}^{p,\lambda} \max_{\rho \in J^2(\xi)} w_p^\lambda(\rho) \left|v_j(\rho) - \tilde{v}_j(\rho)\right| \\ &\leq \left|v_i^0 - \tilde{v}_i^0\right| + \sum_{j=1}^{n} {}^{i}A_{\vec{\sigma},\eta}^{p,\lambda} \max_{\rho \in J^2(\xi)} w_p^\lambda(\rho) \left|v_j(\rho) - \tilde{v}_j(\rho)\right| \\ &= \left|v_i^0 - \tilde{v}_i^0\right| + {}^{i}A_{\vec{\sigma},\eta}^{p,\lambda} \sum_{j=1}^{n} P_{w_p^\lambda, J^2(\xi)}(v_j - \tilde{v}_j). \end{aligned}$$

Taking the maximum over $J(\xi)$ in the above inequality, and noting that $J^2(\xi) = J(\xi)$, we get for every $i = 1, ..., n$

$$P_{w_p^\lambda, J(\xi)}(v_i - \tilde{v}_i) \leq \left|v_i^0 - \tilde{v}_i^0\right| + {}^{i}A_{\vec{\sigma},\eta}^{p,\lambda} \sum_{j=1}^{n} P_{w_p^\lambda, J(\xi)}(v_j - \tilde{v}_j). \tag{4.15}$$

Summing (4.15) over i, and using the fact that λ is chosen sufficiently large so that

$$n \max_{1 \leq i \leq n} {}^{i}A_{\vec{\sigma},\eta}^{p,\lambda} < 1, \tag{4.16}$$

we derive the following inequality

$$\sum_{j=1}^{n} P_{w_p^\lambda, J(\xi)}(v_j - \tilde{v}_j) \leq L_{\vec{\sigma},\eta}^{p,\lambda} \sum_{j=1}^{n} \left|v_j^0 - \tilde{v}_j^0\right|, \tag{4.17}$$

where

$$L_{\vec{\sigma},\eta}^{p,\lambda} := \frac{1}{1 - n \max\limits_{1 \leq i \leq n} {}^{i}A_{\vec{\sigma},\eta}^{p,\lambda}}.$$

Combining (4.15) with (4.17), we deduce that for every $i = 1, ..., n$, we have

$$\begin{aligned} P_{w_p^\lambda, J(\xi)}(v_i - \tilde{v}_i) &\leq \left|v_i^0 - \tilde{v}_i^0\right| + C_{\vec{\sigma},\eta}^{p,\lambda}\, {}^i A_{\vec{\sigma},\eta}^{p,\lambda} \sum_{j=1}^{n} \left|v_j^0 - \tilde{v}_j^0\right| \\ &= \left(1 + {}^i A_{\vec{\sigma},\eta}^{p,\lambda}\, L_{\vec{\sigma},\eta}^{p,\lambda}\right) \left|v_i^0 - \tilde{v}_i^0\right| + L_{\vec{\sigma},\eta}^{p,\lambda}\, {}^i A_{\vec{\sigma},\eta}^{p,\lambda} \sum_{j=1, j\neq i}^{n} \left|v_j^0 - \tilde{v}_j^0\right|. \end{aligned}$$

Consequently, for every $\boldsymbol{\epsilon} = (\epsilon_1, ..., \epsilon_n) > \mathbf{0}$, there exists $\boldsymbol{\delta} = (\delta_1, ..., \delta_n) > \mathbf{0}$ given by

$$\delta_i = \min\left\{ \frac{\epsilon_i}{n\left(1 + {}^i A_{\vec{\sigma},\eta}^{p,\lambda}\, L_{\vec{\sigma},\eta}^{p,\lambda}\right)}, \frac{\epsilon_j}{n\, {}^i A_{\vec{\sigma},\eta}^{p,\lambda}\, L_{\vec{\sigma},\eta}^{p,\lambda}}, \, j = 1, .., n, \, j \neq i \right\},$$

such that (4.13) holds. This proves the uniform stability of the solution of (4.3) when $\eta \neq 0$. For the case $\eta = 0$, (4.14) can be obtained in a similar way; taking into account Remark 3.1(ii) and using the function h instead of w, that is

$$\delta_i = \min\left\{ \frac{\epsilon_i}{n\left(1 + {}^i \tilde{A}_{\vec{\sigma},\eta}^{p,\lambda}\, \tilde{L}_{\vec{\sigma},\eta}^{p,\lambda}\right)}, \frac{\epsilon_j}{n\, {}^i \tilde{A}_{\vec{\sigma},\eta}^{p,\lambda}\, \tilde{L}_{\vec{\sigma},\eta}^{p,\lambda}}, \, j = 1, .., n, \, j \neq i \right\},$$

with

$$\tilde{L}_{\vec{\sigma},\eta}^{p,\lambda} := \frac{1}{1 - n \max\limits_{1 \leq i \leq n} {}^i \tilde{A}_{\vec{\sigma},\eta}^{p,\lambda}}.$$

□

Example 4.1. Consider the following system

$$\begin{cases} {}^C D^{0.5,\eta} v_1(t) = \sum\limits_{i=1}^{3} (1 + t^2) \tanh(v_i(t)), & t > 0 \\ {}^C D^{0.5,\eta} v_2(t) = \sum\limits_{i=1}^{3} e^t \tanh(v_i(t)), & t > 0 \\ {}^C D^{0.5,\eta} v_3(t) = \sum\limits_{i=1}^{3} (t + e^t) \tanh(v_i(t)), & t > 0 \\ (v_1(0), v_2(0), v_3(0)) = (0, 0, 0), \end{cases} \tag{4.18}$$

where $\eta \geq 0$. The system (4.18) is identified to (4.1)-(4.2) with $n = 3$, $\sigma_i = 0.5\,(i = 1, 2, 3)$, and

$$\psi_1(t, x_1, x_2, x_3) = (1 + t^2) \sum_{i=1}^{3} \tanh(x_i), \;\; \psi_2(t, x_1, x_2, x_3) = e^t \sum_{i=1}^{3} \tanh(x_i),$$

$$\psi_3(t, x_1, x_2, x_3) = (t + e^t) \sum_{i=1}^{3} \tanh(x_i), \;\; g_i(t) = t, \;\; v_i^0 = 0 \;\; (i = 1, 2, 3).$$

It is clear that g_i satisfies the condition $0 \le g_i(t) \le \max\{M, t\}$ with $M = 0$. Furthermore, for every $t > 0$ and $x_i, y_i \in \mathbb{R}$ with $i = 1, 2, 3$, we have

$$|\psi_1(t, x_1, x_2, x_3) - \psi_1(t, y_1, y_2, y_3)| = (1+t^2)\left|\sum_{i=1}^{3}\Big(\tanh(x_i) - \tanh(y_i)\Big)\right|$$
$$\le (1+t^2)\sum_{i=1}^{3}\Big|\tanh(x_i) - \tanh(y_i)\Big| \le (1+t^2)\sum_{i=1}^{3}|x_i - y_i|.$$

Similarly, we obtain the following inequalities

$$|\psi_2(t, x_1, x_2, x_3) - \psi_2(t, y_1, y_2, y_3)| \le e^t \sum_{i=1}^{3}|x_i - y_i|,$$
$$|\psi_3(t, x_1, x_2, x_3) - \psi_3(t, y_1, y_2, y_3)| \le (t+e^t) \sum_{i=1}^{3}|x_i - y_i|.$$

That is, $(\mathscr{H}_1)$ is satisfied with

$$k_{1j}(t) = 1+t^2, \quad k_{2j}(t) = e^t \ \text{ and } \ k_{3j}(t) = t + e^t \text{ with } j = 1, 2, 3.$$

Then, in view of Theorem 4.1, (4.18) admits a unique global solution in Y^n. Additionally, the solution of (4.18) is uniformly stable according to Theorem 4.2.
Consider $p = 1.5$, then for $\eta = 1.5$ and $\eta = 0$, respectively, the corresponding constants and the $(\boldsymbol{\epsilon}, \boldsymbol{\delta})$ pairs for $\lambda = 50$, 80, 100, and 300 are summarized in Table 1 and Table 2, respectively.

TABLE 1. Numerical results for (4.18) at $p = \eta = 1.5$ with varying λ

Parameter	λ			
	50	80	100	300
${}^{i}A^{p,\lambda}_{\vec{\sigma},\eta}$	0.3157	0.2699	0.2506	0.1738
$3\,{}^{i}A^{\vec{\sigma},\eta}_{p,\lambda}$	0.9471	0.8097	0.7518	0.5214
$L^{p,\lambda}_{\vec{\sigma},\eta}$	18.9034	5.2549	4.0290	2.0894
Different values of $\boldsymbol{\epsilon} = (\epsilon, \epsilon, \epsilon)$ and corresponding values of $\boldsymbol{\delta} = (\delta, \delta, \delta)$				
ϵ	δ	δ	δ	δ
1.000	0.0478	0.1378	0.1658	0.2446
0.100	0.00478	0.01378	0.01658	0.02446
0.050	0.00239	0.00689	0.00829	0.01223
0.010	0.000478	0.001378	0.001658	0.002446
0.005	0.000239	0.000689	0.000829	0.001223
0.001	0.0000478	0.0001378	0.0001658	0.0002446

TABLE 2. Numerical results for (4.18) at $p = 1.5$, $\eta = 0$ with varying λ

Parameter	λ			
	50	80	100	300
${}^{i}\tilde{A}^{p,\lambda}_{\vec{\sigma}}$	0.3378	0.2888	0.2682	0.1859
$3\,{}^{i}\tilde{A}^{\vec{\sigma}}_{p,\lambda}$	1.0134	0.8664	0.8046	0.5577
$\tilde{L}^{p,\lambda}_{\vec{\sigma}}$	—	7.4850	5.1177	2.2609
Corresponding $\boldsymbol{\delta} = (\delta, \delta, \delta)$-values for different $\boldsymbol{\epsilon} = (\epsilon, \epsilon, \epsilon)$-values				
ϵ	δ	δ	δ	δ
1.000	—	0.1054	0.1405	0.2347
0.100	—	0.01054	0.01405	0.02347
0.050	—	0.00527	0.00703	0.01174
0.010	—	0.001054	0.001405	0.002347
0.005	—	0.000527	0.000703	0.001174
0.001	—	0.0001054	0.0001405	0.0002347

In Table 2, "—" denotes an undefined value. A comparison of the tempered case (Table 1) with the non-tempered case (Table 2) shows that, for the same values of p and $\vec{\sigma}$, the key relation (4.16)—which guarantees the existence and stability of solutions—is satisfied for a smaller value of λ in the tempered case.
Furthermore, Table 1 and Table 2 indicate that a larger value of λ permits larger initial perturbations δ while still maintaining the same bound ϵ on the solution perturbation. This confirms that the enhancement of uniform stability, as stated in Theorem 4.2, is fundamentally governed by the free parameter λ.

5. Conclusions

In this work, we addressed tempered fractional operators, which multiply the power-law kernel by an exponential damping factor, thereby providing a generalization of classical fractional operators. By introducing generalized locally convex spaces equipped with carefully chosen families of vector-valued weighted semi-norms, we established new bounds for tempered fractional integrals of vector-order on the half-axis. As a consequence, one of the well-known boundedness results for classical scalar fractional integral operators in normed spaces established in [33] was improved and generalized. It should be noted that, unlike many earlier findings in this area, our bounds do not require additional boundedness restrictions at infinity.

As an application, a vector approach for the coupled systems of nonlinear tempered differential equations of Caputo-type (4.1)-(4.2), was adopted. Based on our boundedness findings, and via Perov's type fixed point theorem, global existence-uniqueness results were reached under a Lipschitz condition on nonlinearity with merely continuous Lipschitz functions. This approach led to improved results compared to those often imposed in the literature, such as those in [1,2,9,16]. The uniform stability of the solution was also obtained under the same conditions.

The present framework suggests several directions for future research. Possible extensions include the treatment of variable-order tempered fractional operators and other definitions of tempered derivatives. The flexibility of the weighted seminorm approach and the auxiliary parameters p and λ also points to potential applications in the stability and control of delayed tempered fractional systems.

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