Turkish J. Ineq., 10 (1) (2026), Pages 1 –11 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

ON k-GENERALIZED PADOVAN NUMBERS CLOSE TO A POWER OF 2

HUNAR SHERZAD TAHER[1] AND SAROJ KUMAR DASH[1]

Abstract. Let $(\mathcal{P}_r^{(k)})_{r\geq -(k-3)}$ be k-generalized Padovan sequence for a fixed integer $k \geq 3$. Its first k terms are $0, 0, \ldots, 0, 1, 1$. In this paper, we determine all k-generalized Padovan numbers that are close to a power of two. We apply Matveev's theorem to derive lower bounds for nonzero linear forms in logarithms and subsequently employ the LLL algorithm to reduce them.

1. Introduction

The Padovan numbers $(\mathcal{P}_r)_{r\geq 0}$ are defined by the recurrence relation

$$\mathcal{P}_r = \mathcal{P}_{r-2} + \mathcal{P}_{r-3} \quad \text{for all } r \geq 3,$$

with initial conditions $\mathcal{P}_0 = 0$ and $\mathcal{P}_1 = \mathcal{P}_2 = 1$. The Padovan sequence corresponds to sequence $A000931$ in the Online Encyclopedia of Integer Sequences (OEIS).

In 2022, Bravo and Herrera [6] introduced a generalization of the Padovan sequence, denoted by $\left(\mathcal{P}_r^{(k)}\right)_{r\geq -(k-3)}$, for a fixed integer $k \geq 3$. This sequence is defined by the linear recurrence relation

$$\mathcal{P}_r^{(k)} = \mathcal{P}_{r-2}^{(k)} + \mathcal{P}_{r-3}^{(k)} + \cdots + \mathcal{P}_{r-k}^{(k)} \quad \text{for all } r \geq 3, \tag{1.1}$$

with initial conditions

$$\mathcal{P}_{-(k-3)}^{(k)} = \mathcal{P}_{-(k-4)}^{(k)} = \cdots = \mathcal{P}_0^{(k)} = 0 \quad \text{and} \quad \mathcal{P}_1^{(k)} = \mathcal{P}_2^{(k)} = 1.$$

Observe that when $k = 3$ in (1.1), the generalized sequence coincides with the classical Padovan sequence, satisfying $\mathcal{P}_r^{(3)} = \mathcal{P}_r$ for all $r \geq 0$. Subsequently, some researchers proposed further generalizations of the Padovan numbers, as discussed in [13, 19].

Key words and phrases. k-generalized Padovan numbers, Linear forms in logarithms, Baker's method, LLL reduction

2010 *Mathematics Subject Classification.* Primary: 11J86, 11B39. Secondary: 11D75, 11J25.

Received: 17/07/2025 *Accepted*: 26/01/2026.

Cite this article as: H.S. Taher, K. Dash, On k-generalized Padovan numbers close to a power of 2, Turkish Journal of Inequalities, 10(1) (2026), 1–11.

In 2014, Chern and Cui [8] introduced the concept that an integer r is said to be close to a positive integer s if it satisfies the following inequality

$$|r - s| < \sqrt{s}.$$

Then, they demonstrated that all Fibonacci numbers are close to a power of two. Specifically, they established the complete set of solutions to the inequality

$$|F_n - 2^m| < \sqrt{2^m},$$

in positive integers n and m. Bravo et al. [5] extended the previous result by considering the k-generalized Fibonacci numbers. Açikel et al. [1] solved the Diophantine inequality

$$\left|L_n^{(k)} - 2^m\right| < \sqrt{2^m}.$$

The authors of [10, 18] demonstrated that the sums of two and three Fibonacci numbers are close to powers of two, respectively.

Recently, Bachabi et al. [2, 3] discovered all members of the k-generalized Pell and Pell-Lucas numbers close to a power of two, while Pomeo and Bravo [14] investigated all Pell numbers close to a Fibonacci number. Gaha and Mezroui [9] studied the Diophantine inequality

$$|F_n + F_m - 3^a| < 3^{a/2},$$

in positive integers n, m, and a with $n \geq m$.

In this paper, we study the Diophantine inequality

$$\left|\mathcal{P}_r^{(k)} - 2^s\right| < \sqrt{2^s}, \tag{1.2}$$

for positive integers r, s, and $k \geq 3$. Our main result is as follows.

Theorem 1.1. *All solutions* $\left(r, s, \mathcal{P}_r^{(k)}, k\right)$ *of the Diophantine inequality* (1.2) *are given by*

$$\left\{\begin{array}{c}(3,1,1,k\geq 3),(5,1,3,k\geq 4),(5,2,3,k\geq 4),(6,2,5,k\geq 5),(7,3,8,k\geq 6),\\(8,4,13,k\geq 7),(10,5,34,k\geq 9),(5,1,2,3),(6,1,3,3),(6,2,3,3),(7,2,4,3),\\(8,2,5,3),(9,3,7,3),(10,3,9,3),(12,4,16,3),(14,5,28,3),(15,5,37,3),\\(17,6,65,3),(22,8,265,3),(6,2,4,4),(7,3,6,4),(8,3,9,4),(9,4,13,4),\\(10,4,19,4),(11,5,28,4),(13,6,60,4),(15,7,129,4),(7,3,7,5),(9,4,17,5),\\(12,6,61,5),(17,9,520,5),(9,4,19,6),(10,5,30,6),(10,5,32,7),(13,7,129,7),\\(16,9,518,7),(19,11,2083,7),(10,5,33,8),(13,7,136,8),(20,12,4089,11),\\(20,12,4130,12),(20,12,4153,13).\end{array}\right\}.$$

Remark 1.1. In Theorem 1.1, we consider only the case $r \geq 3$ with $r \neq 4$, since $\mathcal{P}_1^{(k)} = \mathcal{P}_2^{(k)} = \mathcal{P}_3^{(k)} = 1$ and $\mathcal{P}_4^{(k)} = \mathcal{P}_5^{(k)} = 2$.

2. Preliminaries

2.1. **Properties of the k-generalized Padovan sequence.** The characteristic polynomial of the k-generalized Padovan sequence is given by

$$\Psi_k(x) = x^k - x^{k-2} - x^{k-3} - \cdots - x - 1.$$

As established in [6, Lemma 2.1], the polynomial $\Psi_k(x)$ has k simple zeros. When k is odd, $\Psi_k(x)$ possesses a unique real zero $\xi(k) \in (1, \phi)$, where $\phi = \frac{1+\sqrt{5}}{2}$, along with $k-1$ complex zeros. Otherwise, when k is even, then all zeros of $\Psi_k(x)$ consist of a real zero $\xi(k) \in (1, \phi), -1$, and $k-2$ complex zeros. Notably, all complex zeros lie within the unit circle.

Furthermore, they demonstrated that the following inequalities

$$\phi\left(1 - \phi^{-k/2}\right) < \xi(k) < \phi \quad \text{for all } k \geq 3, \tag{2.1}$$

hold.

For $k \geq 3$, the function $q_k(x)$ is defined as

$$q_k(x) = \frac{x^2 - 1}{(k+1)x^2 - kx - k + 1}.$$

The fundamental properties of k-Padovan sequences are summarized in the following lemma.

Lemma 2.1. *Let $k \geq 3$. With the above notations, we have the following properties.*

(a) $\mathcal{P}_r^{(k)} = \sum_{i=1}^{k} q_k(\xi_i)(\xi_i)^{r-1}$, *for all $r \geq 3-k$, where ξ_i represent the roots of the characteristic polynomial $\Psi_k(x)$.*

(b) $0.44 < q_k(\xi) < 0.73$, *and $|q_k(\xi_i)| < 1$, for $1 \leq i \leq k$.*

(c) $\left|\mathcal{P}_r^{(k)} - q_k(\xi)\xi^{r-1}\right| < \frac{5}{2}$, *for all $r \geq 3-k$.*

(d) $\mathcal{P}_r^{(k)} = q_k(\xi)\xi^{r-1} + e_k(r)$, *where $|e_k(r)| < \frac{5}{2}$.*

(e) $\xi^{r-3} \leq \mathcal{P}_r^{(k)} \leq \xi^{r-1}$, *for $r \geq 1$.*

Proof. The proof of the results is available in [6, Lemmas 3.1, 3.3, 3.4]. □

2.2. **Linear forms in logarithms.** To obtain a nontrivial lower bound on the absolute value of a linear form in logarithms, we apply Matveev's theorem. This result ensures that any nonzero expression of the form $\Lambda := \prod_{i=1}^{t} \gamma_i^{b_i} - 1$, cannot be arbitrarily small, which is critical in bounding the variables in Diophantine inequalities.

Definition 2.1. (Absolute logarithmic height)
Let γ be an algebraic number of degree d with the following minimal polynomial

$$c_0 x^d + c_1 x^{d-1} + \ldots + c_d = c_0 \prod_{i=1}^{d}\left(x - \gamma^{(i)}\right) \in \mathbb{Z}[x],$$

where $\gamma^{(i)}$ represents the conjugates of γ, and the coefficients c_i are relatively prime integers with $c_0 > 0$. The logarithmic height of γ is defined as

$$h(\gamma) = \frac{1}{d}\left(\log c_0 + \sum_{i=1}^{d} \log\left(\max\left\{\left|\gamma^{(i)}\right|, 1\right\}\right)\right).$$

In the special case where $\gamma = \frac{a}{b}$ is a rational number with $\gcd(a, b) = 1$ and $b > 0$, the logarithmic height simplifies to $h(\gamma) = \log(\max\{|a|, b\})$.

The logarithmic height function satisfies the following fundamental properties, which will be employed throughout the subsequent analysis.

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \quad h\left(\eta\gamma^{\pm 1}\right) \leq h(\eta) + h(\gamma) \quad \text{and } h\left(\eta^k\right) = |k| h(\eta).$$

The following result is a modification of Matveev's theorem [11]. Let $\mathbb{L}$ be a real algebraic number field of degree D over $\mathbb{Q}$. Let $\gamma_1, \ldots, \gamma_t \in \mathbb{L}$ be positive real algebraic numbers, and let $b_1, \ldots, b_t$ be nonzero integers. Put

$$B \geq \max\{|b_1|, \ldots, |b_t|\}, \quad A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\} \quad \text{for each } 1 \leq i \leq t,$$

and

$$\Lambda := \prod_{i=1}^{t} \gamma_i^{b_i} - 1.$$

With this notation, the main result of Matveev [11], the following theorem gives the explicit lower bound for linear forms in logarithms, refined by Bugeaud, Mignotte, and Siksek [7].

Theorem 2.1. ([7, Theorem 9.4]) *If $\Lambda \neq 0$, then*

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2 \times \left(\prod_{i=1}^{t} A_i\right)(1 + \log D)(1 + \log B).$$

The above results relies on modern techniques from transcendental number theory. Further information on this theory, especially its connection with Diophantine approximation, can be found in [4, 20].

2.3. **Reduction method.** In our calculations, we apply the LLL algorithm (Lenstra-Lenstra-Lovász lattice basis reduction algorithm) is a fundamental tool in reducing large upper bounds obtained via logarithmic inequalities. By constructing a suitable lattice and computing a reduced basis, one can estimate the minimal distance from a target vector to the lattice, thus deriving a lower bound for the associated linear form. This significantly reduces the search space for possible integer solutions.

Let n be a positive integer. A lattice $\mathcal{L}$ is defined as a discrete $\mathbb{Z}$-module in $\mathbb{R}^n$, spanned by n linearly independent vectors $b_1, \ldots, b_n$, which form a basis for the lattice [12]. Formally, the lattice is the set

$$\mathcal{L} = \left\{\sum_{i=1}^{n} x_i b_i : x_i \in \mathbb{Z}\right\},$$

where the vectors $b_1, \ldots, b_n$ constitute a basis for $\mathbb{R}^n$. The matrix B, whose columns are $b_1, \ldots, b_n$, is called the basis matrix of $\mathcal{L}$. For further details on the determinant of $\mathcal{L}$, the Gram-Schmidt orthogonalization technique, and the definition of a reduced basis, see [16, V.3].

We define the function $\ell(\mathcal{L}, y)$ as follows

$$\ell(\mathcal{L}, y) = \begin{cases} \min_{x \in \mathcal{L}} \|x - y\|, & y \notin \mathcal{L}, \\ \min_{0 \neq x \in \mathcal{L}} \|x\|, & y \in \mathcal{L}, \end{cases}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. To determine a lower bound for $\ell(\mathcal{L}, y)$, we apply the LLL-algorithm. Let $y \in \mathbb{R}^n$, and consider the following cases.

Case 1: $y \notin \mathcal{L}$ (Non-homogeneous linear forms)
Define the vector $\sigma \in \mathbb{R}^n$ by $\sigma = B^{-1}y$, where B is the basis matrix of an LLL-reduced basis for $\mathcal{L}$. For a real number x, let $\{x\}$ denote the distance from x to the nearest integer. Then, we have the following theorem.

Theorem 2.2. ([16, Theorem V.10]) *Let i_0 be the largest index with $\{\sigma_{i_0}\} \neq 0$. Then, for all $x \in \mathcal{L}$,*

$$\|x-y\|^2 \geq c_1^{-1}\{\sigma_{i_0}\}^2 \|b_1\|^2 =: c_4^2,$$

where

$$c_1 := \max\left\{\frac{\|b_1\|^2}{\|b_i^*\|^2} : 1 \leq i \leq n\right\}.$$

Case 2: $y \in \mathcal{L}$ (Homogeneous linear forms)
In this case, we set $\{\sigma_{i_0}\} = 1$. This statement is clarified by the following theorem.

Theorem 2.3. ([16, Theorem V.9]) *Let B be a reduced basis for a lattice $\mathcal{L}$. Then, for all $x \neq 0$ in $\mathcal{L}$,*

$$\|x\|^2 \geq c_1^{-1}\|b_1\|^2 := c_4^2.$$

For application to linear forms consider real numbers $\gamma_0, \gamma_1, \ldots, \gamma_n$ that are linearly independent over $\mathbb{Q}$, along with positive constants c_2, c_3 satisfying

$$\left|\gamma_0 + \sum_{i=1}^{n} x_i\gamma_i\right| \leq c_2 e^{-c_3 H^q},$$

for some positive integer q, where the integers $x_1, \ldots, x_n$ are bounded by $|x_i| \leq X_i$. Let $X_0 = \max_{1\leq i\leq n}\{X_i\}$. We now consider the lattice $\mathcal{L}$ generated by the columns of the matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \lfloor C\gamma_1\rceil & \lfloor C\gamma_2\rceil & \cdots & \lfloor C\gamma_{n-1}\rceil & \lfloor C\gamma_n\rceil \end{pmatrix} \in \mathbb{Z}^{n\times n}$$

where $\lfloor x\rceil$ denotes the nearest integer to x, and C is chosen such that the determinant of the lattice is approximately X_0^n. Using de Weger's LLL-algorithm and Theorems 2.2 and 2.3, we can compute, in polynomial time, a lower bound c_4 on $\ell(\mathcal{L}, y)$, where $y = (0, 0, \ldots, -\lfloor C\gamma_0\rceil)$.

To derive a bound on H, we apply the following result with the above notations (see [16, Lemma VI.1]).

Lemma 2.2. *Let $S := \sum_{i=1}^{n-1} X_i^2$ and $T := \frac{1}{2}\left(1+\sum_{i=1}^{n} X_i\right)$. If $c_4^2 \geq T^2 + S$, then either*

$$H \leq \left[\frac{1}{c_3}\left(\log(Cc_2) - \log\left(\sqrt{c_4^2 - S} - T\right)\right)\right]^{1/q},$$

or $x_1 = \cdots = x_{n-1} = 0$ and $x_n = -\frac{\lfloor C\gamma_0\rceil}{\lfloor C\gamma_n\rceil}$.

In the final part, we present the following auxiliary results.

Lemma 2.3. ([15, Lemma 7]) *For parameters $c \geq 1$ and $S > (4c^2)^c$ satisfying $\frac{L}{(\log L)^c} < S$, we have*

$$L < 2^c S(\log S)^c.$$

Lemma 2.4. *Let a and b be real numbers such that $|e^a - 1| < b < \frac{3}{4}$. Then*

$$|a| < 2b.$$

Lemma 2.5. ([17, Lemma 2.5]) *For all integers $k \geq 3$, the following inequality*

$$h\left(q_k(\xi)\right) < 2.5 \log k,$$

holds.

3. Proof of the main Theorem 1.1

To complete the proof of Theorem 1.1, we consider the following cases.

3.1. The case $3 \leq r \leq k+1$, where $k \geq 3$.

We know from [6, equation (3)] that $\mathcal{P}_r^{(k)} = \mathcal{F}_{r-1}$, where $2 \leq r \leq k+1$ and $\mathcal{F}_r$ is a Fibonacci number. Thus, from (1.2), we obtain

$$|\mathcal{F}_{r-1} - 2^s| < \sqrt{2^s}. \tag{3.1}$$

This result was previously established in [8, Theorem 1.1]. Now, comparing inequality (3.1) with the result in [8, Theorem 1.1], we deduce that the solutions are

$$(r, s, \mathcal{F}_{r-1}, 2^s) \in \left\{ \begin{array}{c} (3,1,1,2), (5,1,3,2), (5,2,3,4), (6,2,5,4) \\ (7,3,8,8), (8,4,13,16), (10,5,34,32) \end{array} \right\}.$$

3.2. The case $r \geq k+2$, where $k \geq 3$.

3.2.1. *Relation between r and s.* Combining inequality (1.2) with Lemma 2.1(e), we have

$$2^{s-1} \leq 2^s - 2^{s/2} < \mathcal{P}_r^{(k)} \leq \xi^{r-1}$$

and

$$\xi^{r-3} \leq \mathcal{P}_r^{(k)} < 2^s + 2^{s/2} < 2^{s+1}.$$

Taking logarithms on both sides of the above inequalities with inequality (2.1), we get

$$(r-3)\frac{\log \xi}{\log 2} - 1 < s < (r-1)\frac{\log \xi}{\log 2} + 1 < (r-1)\frac{\log \phi}{\log 2} + 1 = 0.69r + 0.31.$$

3.2.2. *Bounding r in terms of k.* The following lemma determines an upper bound for r in terms of k.

Lemma 3.1. *If r, s and k are positive integers where $r \geq k+2$ with $k \geq 3$, then the following inequalities*

$$s < r < 2.2 \cdot 10^{14} k^4 (\log k)^3, \tag{3.2}$$

hold.

Proof. Substituting Lemma 2.1(d) into inequality (1.2), we obtain

$$\left|q_k(\xi)\xi^{r-1} - 2^s\right| < 2^{s/2} + |e_k(r)| < 2^{s/2} + \frac{5}{2}.$$

Dividing both sides by $q_k(\xi)\xi^{r-1}$ and applying Lemma 2.1(b), we deduce that

$$\left|1 - \frac{2^s\xi^{-(r-1)}}{q_k(\xi)}\right| < \frac{2^{s/2}}{q_k(\xi)\xi^{r-1}} + \frac{5}{2q_k(\xi)\xi^{r-1}} < \frac{2.27 \cdot 2^{s/2}}{\xi^{r-1}} + \frac{2.27 \cdot 5}{\xi^{r-1}}.$$

Note that $2^{s/2} < \left(2\xi^{r-1}\right)^{1/2}$. Thus, we get

$$\left|1 - \frac{2^s\xi^{-(r-1)}}{q_k(\xi)}\right| < \frac{2.27 \cdot 2^{1/2}\xi^{(r/2-1/2)}}{\xi^{r-1}} + \frac{2.27 \cdot 5}{\xi^{r-1}} < \frac{14.6}{\xi^{(r-1)/2}}. \tag{3.3}$$

Let

$$\Lambda_1 := 2^s\xi^{-(r-1)}\left(q_k(\xi)\right)^{-1} - 1.$$

By Theorem 2.1, we consider

$$t := 3, \quad (\gamma_1, b_1) := (2, s), \quad (\gamma_2, b_2) := (\xi, -(r-1)), \quad (\gamma_3, b_3) := (q_k(\xi), -1).$$

Since $\gamma_1, \gamma_2, \gamma_3$ are elements of $\mathbb{L} := \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3)$ and $D := [\mathbb{L}, \mathbb{Q}] = k$. If $\Lambda_1 = 0$, then $q_k(\xi) = 2^s\xi^{-(r-1)}$. It is clear that 2 and ξ are algebraic integers. Since $\xi^{-(r-1)}$ is also an algebraic integer, and the product of algebraic integers is again an algebraic integer, it follows that $q_k(\xi)$ is an algebraic integer. This leads to a contradiction, since $q_k(\xi)$ is only an algebraic number. Thus, $\Lambda_1 \neq 0$. Furthermore, by Lemma 2.5 and the properties of absolute logarithmic height, we estimate the heights of γ_1, γ_2, and γ_3 as follows

$$h(\gamma_1) \leq \log 2, \quad h(\gamma_2) < \frac{\log\phi}{k}, \quad \text{and} \quad h(\gamma_3) < 2.5\log k \text{ for all } k \geq 3.$$

Thus, $A_1 := k\log 2, A_2 := \log\phi$, and $A_3 := 2.5k\log k$. We have $\max\{1, |-(r-1)|, s\} = r-1 = B$. By Theorem 2.1, we obtain

$$\frac{14.6}{\xi^{(r-1)/2}} > |\Lambda_1| > \exp\{-G(1+\log(r-1))(k\log 2)(\log\phi)(2.5k\log k)\},$$

where $G = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot k^2(1+\log k)$. Using the facts that $(1+\log k) < 2\log k$ for all $k \geq 3$ and $(1+\log(r-1)) < 2\log(r-1)$ for all $r \geq 5$, we conclude that

$$(r-1)\log\xi \leq 9.6 \cdot 10^{11}k^4(\log k)^2\log(r-1).$$

Dividing by $\log\xi$ and using the fact that the dominant root $\xi > 1.32$ for all $k \geq 3$, we have

$$\frac{r-1}{\log(r-1)} < 3.5 \cdot 10^{12}k^4(\log k)^2.$$

Applying Lemma 2.3 with $S := 3.5 \cdot 10^{12}k^4(\log k)^2, L := r-1$, and $c := 1$, we deduce that

$$\begin{aligned} r-1 &< 2\left(3.5 \cdot 10^{12}k^4(\log k)^2\right) \cdot \log\left(3.5 \cdot 10^{12}k^4(\log k)^2\right) \\ &< 2\left(3.5 \cdot 10^{12}k^4(\log k)^2\right) \cdot (28.9 + 4\log k + 2\log(\log k)) \\ &< 2.17 \cdot 10^{14}k^4(\log k)^3, \end{aligned}$$

using the fact that $28.9 + 4\log k + 2\log(\log k) < 31\log k$ for all $k \geq 3$. Hence, $r < 2.2 \cdot 10^{14} k^4 (\log k)^3$.

□

3.2.3. *The case $k > 1200$.* In this case, we determine the upper bound of r for large values of k.

Lemma 3.2. *All solutions $\left(r, s, \mathcal{P}_r^{(k)}, k\right)$ of inequality (1.2) with $k > 1200$ satisfy the following inequalities*

$$k < 1.8 \cdot 10^{16} \quad \textit{and} \quad s < r < 1.3 \cdot 10^{89}.$$

Proof. The proof of this lemma follows the same method as in [17, Lemma 3.2]. Here, we have

$$r < 2.2 \cdot 10^{14} k^4 (\log k)^3 < \phi^{k/4},$$

holds for all $k > 1200$, and

$$q_k(\xi)\xi^{r-1} = \frac{\phi^r}{\phi + 2} + \phi^{r-1}\gamma + \frac{\phi\delta}{\phi + 2} + \gamma\delta, \tag{3.4}$$

where

$$|\delta| < \frac{2\phi^{r-1}}{\phi^{k/4}}, \quad \text{and} \quad |\gamma| < \frac{k}{\phi^{(k-2)/2}}.$$

Now, equation (3.4) can be written as

$$\begin{aligned}
\left|2^s - \frac{\phi^r}{\phi+2}\right| &= \left|\left(2^s - q_k(\xi)\xi^{r-1}\right) + \phi^{r-1}\gamma + \frac{\phi\delta}{\phi+2} + \gamma\delta\right| \\
&\leq \left|2^s - q_k(\xi)\xi^{r-1}\right| + \phi^{r-1}|\gamma| + \frac{\phi|\delta|}{\phi+2} + |\gamma||\delta| \\
&< \frac{5}{2} + 2^{s/2} + \frac{k\phi^r}{\phi^{k/2}} + \frac{2\phi^r}{(\phi+2)\phi^{k/4}} + \frac{2k\phi^r}{\phi^{3k/4}}.
\end{aligned}$$

Dividing both sides by $\frac{\phi^r}{\phi+2}$ and using the inequality $s < 0.69r + 0.31$ together with $2 < \phi^2$, we obtain

$$\begin{aligned}
\left|2^s\phi^{-r}(\phi+2) - 1\right| &< \frac{5(\phi+2)}{2\phi^r} + \frac{2^{s/2}(\phi+2)}{\phi^r} + \frac{k(\phi+2)}{\phi^{k/2}} + \frac{2}{\phi^{k/4}} + \frac{2k(\phi+2)}{\phi^{3k/4}} \\
&< \frac{9.1}{\phi^r} + \frac{3.62\phi^s}{\phi^r} + \frac{3.62k}{\phi^{k/2}} + \frac{2}{\phi^{k/4}} + \frac{7.24k}{\phi^{3k/4}} \\
&< \frac{0.05}{\phi^{k/4}} + \frac{0.05}{\phi^{k/4}} + \frac{0.05}{\phi^{k/4}} + \frac{2}{\phi^{k/4}} + \frac{0.05}{\phi^{k/4}} = \frac{2.20}{\phi^{k/4}},
\end{aligned} \tag{3.5}$$

for all $k > 1200$. By Theorem 2.1, we take

$$(\gamma_1, b_1) := (2, s), \quad (\gamma_2, b_2) := (\phi, -r), \quad (\gamma_3, b_3) := (\phi + 2, 1).$$

Note that for the number field $\mathbb{L} := \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3)$, we have $D := [\mathbb{L}, \mathbb{Q}] = 2$. Let

$$\Lambda_2 := 2^s\phi^{-r}(\phi + 2) - 1.$$

To show that $\Lambda_2 \neq 0$, assume for contradiction that $\Lambda_2 = 0$. Then $2^s = \frac{\phi^r}{\phi+2}$, which is impossible, since the right-hand side is an irrational number, whereas 2^s is rational number. Thus,

$$h(\gamma_1) \leq \log 2, \quad h(\gamma_2) < \frac{\log \phi}{2}, \quad h(\gamma_3) < \frac{\log \phi}{2} + 2\log 2 < 2.$$

We set $A_1 := 2\log 2, A_2 := \log \phi$, and $A_3 := 4$. Since $s < r$, we choose $B := r$. By Theorem 2.1, we obtain

$$\frac{2.20}{\phi^{k/4}} > |\Lambda_2| > \exp\{-G(1 + \log r)(2\log 2)(\log \phi)(4)\},$$

where $G = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)$. This yields

$$k < 2.4 \cdot 10^{13} \log r.$$

By Lemma 3.1, we have

$$\log r < \log\left(2.2 \cdot 10^{14} k^4 (\log k)^3\right) < 33 + 4\log k + 3\log(\log k) < 11\log k,$$

for all $k > 1200$. Therefore, $k < 2.7 \cdot 10^{14} \log k$. Applying Lemma 2.3 simplifies this to $k < 1.8 \cdot 10^{16}$. By inequality (3.2), we obtain

$$r < 1.3 \cdot 10^{84}.$$

□

3.2.4. *Reducing the upper bound for $k > 1200$.* To reduce the upper bound, we define

$$\Gamma_2 := s\log 2 - r\log \phi + \log(\phi + 2).$$

Note that for all $k \geq 1200$ in (3.5), we have $|\Lambda_2| = \left|e^{\Gamma_2} - 1\right| < 2.20/\phi^{k/4} < 3/4$. By Lemma 2.4, we obtain

$$|\Gamma_2| < \frac{4.5}{\phi^{k/4}}.$$

To estimate a lower bound for $|\Gamma_2|$, we consider the lattice $\mathcal{L}$ generated by the matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lfloor C\log 2 \rceil & \lfloor C\log \phi \rceil & \lfloor C\log(\phi + 2) \rceil \end{pmatrix},$$

with $x_1 = s, x_2 = -r, x_3 = 1, C := 10^{255}$, and $y := (0, 0, 0)$. By Theorem 2.3, we get

$$\ell(\mathcal{L}, y)^2 \geq c_1^{-1} \|b_1\|^2 := 3.3 \cdot 10^{169} := c_4^2.$$

By Lemma 2.2, we have $S := 2.5 \cdot 10^{168}$ and $T := 1.1 \cdot 10^{84}$. Since $c_4^2 \geq T^2 + S$, setting $c_2 := 4.5$ and $c_3 := \log \phi$, and after simplifying the computations, we obtain $k/4 < 818.2804475$. This implies $k < 3273$.

Repeating the process with the new upper bounds $r < 1.3 \cdot 10^{31}, C := 10^{95}, c_4^2 := 1.43 \cdot 10^{63}$, $T := 1.1 \cdot 10^{31}$, and $S := 2.5 \cdot 10^{62}$, we obtain $k < 1230$.

In the third process, taking $r < 1.8 \cdot 10^{29}, C := 10^{90}, c_4^2 := 1.2 \cdot 10^{59}, T := 1.52 \cdot 10^{29}$, and $S := 4.8 \cdot 10^{58}$, we finally obtain $k/4 < 294.7851583$. Thus, $k < 1179$, which contradicts our initial assumption.

3.2.5. *Reducing the upper bound for* $3 \leq k \leq 1200$. By Lemma 3.1, we have $r < 1.6 \cdot 10^{29}$ for all $k \in [3, 1200]$. Let

$$\Gamma_1 := s \log 2 - (r-1) \log \xi - \log \left(q_k(\xi) \right).$$

From (3.3), assuming $r > 25$, we obtain $|\Lambda_1| < 3/4$. By Lemma 2.4, we deduce that $|\Gamma_1| < 29.2/\xi^{(r-1)/2}$.

To estimate a lower bound for $|\Gamma_1|$, we consider the lattice $\mathcal{L}$ generated by the matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lfloor C \log 2 \rceil & \lfloor C \log \xi \rceil & \lfloor C \log \left(q_k(\xi) \right) \rceil \end{pmatrix},$$

where $x_1 = s, x_2 = -(r-1)$, and $x_3 = -1$, with $C := 10^{90}$ and $y := (0,0,0)$. Applying Theorem 2.3 again yields

$$\ell(\mathcal{L}, y)^2 \geq c_1^{-1} \|b_1\|^2 := 7.2 \cdot 10^{58} := c_4^2.$$

Using Lemma 2.2, we choose $S := 3.8 \cdot 10^{58}$ and $T := 1.35 \cdot 10^{29}$. Since $c_4^2 \geq T^2 + S$, setting $c_2 := 29.2$ and $c_3 := \log \xi$, and after simplifying the calculations, we obtain $(r-1)/2 < 300.3807465$. Hence, $r < 601$.

Given that $r \geq k+2$, this implies $k \leq 599$. We performed an exhaustive computational search for solutions to the Diophantine inequality (1.2) with

$$k \in [3, 599], \quad r \in [k+2, 601] \quad \text{and} \quad s \in [1, r).$$

This computation was performed using Maple for an extended period. The only solutions obtained are those presented in Theorem 1.1, which completes the proof of Theorem 1.1.

Acknowledgements. The authors express their sincere appreciation to the editor and the referees for their insightful comments and suggestions, which have substantially enhanced the quality of this paper.

References

[1] A. Açikel, N. Irmak, L. Szalay, *The k-generalized Lucas numbers close to a power of 2*, Math. Slovaca, **73**(4) (2023), 871–882.

[2] M. Bachabi, A. Togbé, *On k-Pell numbers close to power of 2*, Rev. Colomb. Mat., **58**(1) (2024), 67–80.

[3] M. Bachabi, B. Kourouma, A. Togbé, *On k-Pell-Lucas numbers close to a power of 2*, Integers, **25** (2025), A67, 1–14.

[4] A. Baker, *Transcendental number theory*, Cambridge university press, Cambridge, 2022.

[5] J.J. Bravo, C.A. Gómez, J.L. Herrera, *k-Fibonacci numbers close to a power of 2*, Quaest. Math., **44**(12) (2021), 1681–1690.

[6] J.J. Bravo, J.L. Herrera, *Generalized Padovan sequences*, Commun. Korean Math. Soc., **37** (2022), 977–988.

[7] Y. Bugeaud, M. Mignotte, S. Siksek, *Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers*, Ann. Math., **163** (2006), 969–1018.

[8] S. Chern, A. Cui, *Fibonacci numbers close to a power of 2*, Fibonacci Q., **52**(4) (2014), 344–348.

[9] A. Gaha, S. Mezroui, *On the Fibonacci numbers and their sums which are close to a power of 3*, Gulf J. Math. **19**(2) (2025), 47–60.

[10] E. Hasanalizade, *Sums of Fibonacci numbers close to a power of 2*, Bol. Soc. Mat. Mex., **29**(1) (2023), 19.

[11] E.M. Matveev, *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II*, Izv. Math., **64** (2000), 1217–1269.

[12] A.K. Lenstra, H.W. Lenstra Lovász, *Factoring polynomials with rational coefficients*, Math. ann., **261**(4) (1982), 515–534.

[13] E. Özkan, E. Eser, M. Uysal, *On a generalization of the Gadovan numbers*, Math. Morav., **28**(1) (2024), 109–119.

[14] F. Pomeo, J.J. Bravo, *Pell numbers close to Fibonacci numbers*, Indian J. Pure Appl. Math., (2024).

[15] S.G. Sanchez, F. Luca, *Linear combinations of factorials and S-units in a binary recurrence sequence*, Ann. Math. Que., **38** (2014), 169–188.

[16] N.P. Smart, *The algorithmic resolution of Diophantine equations a computational cookbook*, (Vol. 41), Cambridge university press, Cambridge, 1998.

[17] H.S. Taher, S.K. Dash, *On k-generalized Padovan numbers which are repdigits in base η*, JP J. Algebra Number Theory Appl., **64**(4) (2025), 395–416.

[18] B.P. Tripathy, P.K. Patel, *Sums of three Fibonacci numbers close to a power of 2*, Integers, **23** (2023), A4.

[19] M. Uysal, E. Özkan, *Padovan hybrid quaternions and some properties*, J. Sci. Arts, **22**(1) (2022), 121–132.

[20] M. Waldschmidt, *Diophantine approximation on linear algebraic groups: transcendence properties of the exponential function in several variables*, Grundlehren der mathematischen Wissenschaften (Vol. 326), Springer-Verlag Berlin Heidelberg, 2000.

[1]Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Chennai,
600127, Tamil Nadu, India.
Email address: `hunarsherzad.taher2022@vitstudent.ac.in`
Email address: `sarojkumar.dash@vit.ac.in`