Turkish J. Ineq., 9 (2) (2025), Pages 79 –89 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

SOME NEW ESTIMATIONS FOR DIFFERENT KINDS OF CONVEX FUNCTIONS VIA KATUGAMPOLA FRACTIONAL OPERATOR

JAVANSHIR ZEYNALOV[1], YETER ERDAŞ[2], GÜLTAÇ ZEYNELZADE[1], ZÜMRÜD NAĞIYEVA[1], AND AHMET OCAK AKDEMIR[3]

Abstract. The main motivation of this study is to present new Hermite-Hadamard (HH) type inequalities via a certain fractional operators. We have used an integral identity and give new estimations of HH- type inequalities for differentiable $m-$convex and exponentially convex mappings via Katugampola-fractional operator. Main findings of this study would provide elegant connections and general variants of well known results established recently.

1. Introduction

Convexity is a very functional concept in programming, statistics and numerical analysis as in many different branches of mathematics. In theory of inequality, the concept of convexity exists in the proof of many classical inequalities, but has been a source of inspiration for many new and useful inequalities.

Definition 1.1. [14]. The function $f:[c_1,c_2]\to\Re$, is said to be convex, if we have

$$f\left(t\kappa+(1-t)\tau\right)\leq tf(\kappa)+(1-t)\,f(\tau)$$

for all $\kappa,\tau\in[c_1,c_2]$ and $t\in[0,1]$.

The definition of $m-$convex function, which is one of these general forms, is given as follows.

Definition 1.2. [33] The mapping $f:[0,b]\to\mathbb{R}$ is said to be m-convex $m\in[0,1]$, if for every $x_1,x_2\in[0,b]$ and $\tau\in[0,1]$, we have

$$f\left(\tau x_1+m(1-\tau)x_2\right)\leq\tau f\left(x_1\right)+m(1-\tau)f(x_2).$$

Key words and phrases. Convex functions, Hermite-Hadamard type inequalities, fractional integral operators, generalized integral operators.

2010 *Mathematics Subject Classification.* Primary: 26D15. Secondary: 26A51.

Received: 11/10/2025 *Accepted*: 23/12/2025.

Cite this article as: J. Zeynalov, Y. Erdaş, G. Zeynelzede, Z. Nağiyeva, A.O. Akdemir, Some new estimations for different kinds of convex functions via Katugampola fractional operator, Turkish Journal of Inequalities, 9(2) (2025), 79-89.

Definition 1.3. [4] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be exponentially convex function, if

$$f\left((1-\xi)\varrho_1 + \xi\varrho_2\right) \leq (1-\xi)\frac{f(\varrho_1)}{e^{\alpha\varrho_1}} + \xi\frac{f(\varrho_2)}{e^{\alpha\varrho_2}}$$

for all $\varrho_1, \varrho_2 \in I, \alpha \in \mathbb{R}$ and $\xi \in [0,1]$.

For related results on convex functions and inequalities, see the papers ([10,25,29,30,36]). In addition to the use of convex functions in many fields, inequality has increased its reputation in theory with the Hermite-Hadamard inequality (See [14]). This celebrated inequality can be stated as:
If a mapping $f : J \subseteq \Re \to \Re$ is a convex function on J and $r, s \in J$, $r < s$, then

$$f\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s f(\lambda)d\lambda \leq \frac{f(r)+f(s)}{2}.$$

Fractional calculus is a good expansion of the concept of derivative operator from integer order n to arbitrary order a. Fractional derivative operators are accepted as the inverse of fractional integral operators. Recently, the multiplicity of applications in many fields of engineering, physics, statistics and mathematics has led to the study of fractional integrals by many researchers. The fact that they are a more effective tool than the results in classical analysis has resulted in more use of these operators on real world problems.
Since the definition of the convex functions has been given as an inequality, this concept has established a powerful link between convexity and inequalities. It is now become a trending aspect of mathematical research to generalize classical known results via fractional integral operator. Although fractional analysis is basically a generalization of classical analysis, it has developed rapidly with the concepts of fractional order operators. Fractional analysis has recently become a popular topic with its applications in many fields such as modeling, physics, approximation theory, engineering, control theory and mathematical biology, based on applied mathematics problems (see[1–3,5–9,11–13,15,16,20–22,24,26–28,31,32,34,35,37,38].

Recently in [17], the author introduced a new concept to unify Riemann-Liouville and Hadamard fractional integral operators which a certain general form for fractional integral operators. Also the conditions are given so that the operator is bounded in an extended Lebesgue measurable space. The corresponding fractional derivative approach to this new generalized operator can be seen in [18]. Moreover, Katugampola worked for the Mellin transforms of the fractional integrals and derivatives (see [19]).

Definition 1.4. [17] Let $[\kappa, \tau] \subset \Re$ be a finite interval. Then, the left-sided and right-sided Katugampola fractional integrals of order $\xi > 0$ of $f \in X_c^\nu(\kappa^\nu, \tau^\nu)$ are defined as follows:

$$({}^\nu I_{\kappa+}^\xi f)(x) = \frac{\nu^{1-\xi}}{\Gamma(\xi)}\int_\kappa^x \frac{f(\lambda)}{(x-\lambda)^{1-\xi}}\lambda^{\nu-1}d\lambda,\ x > \kappa$$

and

$$({}^\nu I_{\tau-}^\xi f)(x) = \frac{\nu^{1-\xi}}{\Gamma(\xi)}\int_x^\tau \frac{f(\lambda)}{(\lambda-x)^{1-\xi}}\lambda^{\nu-1}d\lambda,\ x < \tau,$$

with $\kappa < x < \tau$ and $\nu > 0$, if the integrals exist.

Theorem 1.1. [17] *If $\xi > 0$ and $\nu > 0$, then for $x > \kappa$*

$$1) \lim_{\nu\to 1} ({}^{\nu}I^{\xi}_{\kappa^+}f)(x) = (J^{\xi}_{\kappa^+}f)(x)$$

$$2) \lim_{\nu\to (0^+)} ({}^{\nu}I^{\xi}_{\kappa^+}f)(x) = (H^{\xi}_{\kappa^+}f)(x).$$

The main motivation point of the study is to prove the HH type inequalities with specific and general forms for the functions whose absolute values of derivatives are $m-$convex and exponentially convex functions with the help of the fractional integral operator, which has a general kernel structure. The main results are reduced to the results available in the literature in some special cases, as well as giving new approximations and estimates for differentiable and $m-$convex and exponentially convex functions. To obtain our results, we used some known proof methods alongside classical inequalities such as the Hölder inequality, Power mean inequality and Young inequality.

2. Hermite-Hadamard Type inequalities for Katugampola-Fractional Integrals

We will start with the following identity that will be useful to prove our main findings via Katugampola fractional integral operator (see [23]):

Lemma 2.1. *Let $\xi \in (0,1)$ and $\nu > 0$ and $f : [\kappa^{\nu}, \tau^{\nu}] \to \Re$ be a twice differentiable mapping on $(\kappa^{\nu}, \tau^{\nu})$ with $0 < \kappa^{\nu} < \tau^{\nu}$. Then, the following equality holds for Katugampola fractional integral operator:*

$$\begin{aligned} A &= \frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{(\tau^{\nu}-\kappa^{\nu})^{\xi}} \\ &\times \left[\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}} \right) f(\tau^{\nu}) + \left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}} \right) f(\kappa^{\nu}) \right] - f\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right) \\ &= \frac{(\tau^{\nu}-\kappa^{\nu})}{4}\left[\int_0^1 t^{\nu\xi+\nu-1} f'\left(\frac{t^{\nu}}{2}\kappa^{\nu} + \frac{2-t^{\nu}}{2}\tau^{\nu}\right) dt \right. \\ &\left. + \int_0^1 t^{\nu\xi+\nu-1} f'\left(\frac{t^{\nu}}{2}\tau^{\nu} + \frac{2-t^{\nu}}{2}\kappa^{\nu}\right) dt \right]. \end{aligned}$$

Theorem 2.1. *Suppose that $f : [\kappa^{\nu}, \tau^{\nu}] \to \Re$ be a differentiable function on $(\kappa^{\nu}, \tau^{\nu})$ with $0 \leq \kappa < \tau$. If $|f'|$ is $m-$convex function, then we have the following inequality for Katugampola fractional integral operator:*

$$\begin{aligned} &\left| \frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{(\tau^{\nu}-\kappa^{\nu})^{\xi}} \left[\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}} \right) f(\tau^{\nu}) + \left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}} \right) f(\kappa^{\nu}) \right] - f\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right) \right| \\ \leq\; &\frac{(\tau^{\nu}-\kappa^{\nu})}{4(2\nu\xi+4\nu)} \left(\left|f'(\kappa^{\nu})\right| + \frac{m(\nu\xi+3\nu)}{\nu\xi+\nu} \left(\left|f'\left(\frac{\tau^{\nu}}{m}\right)\right| + \left|f'\left(\frac{\kappa^{\nu}}{m}\right)\right| \right) + \left|f'(\tau^{\nu})\right| \right) \end{aligned}$$

for $m \in (0,1]$.

Proof. By using right hand side of the Lemma (2.1), we can write

$$\begin{aligned} |A| &\leq \frac{(\tau^\nu-\kappa^\nu)}{4}\left[\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|dt\right. \\ &\quad \left.+\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)\right|dt\right]. \end{aligned}$$

By making use of the necessary calculations, we get

$$|A| \leq \frac{(\tau^\nu-\kappa^\nu)}{4(2\nu\xi+4\nu)}\left(\left|f'(\kappa^\nu)\right|+\frac{m(\nu\xi+3\nu)}{\nu\xi+\nu}\left(\left|f'\left(\frac{\tau^\nu}{m}\right)\right|+\left|f'\left(\frac{\kappa^\nu}{m}\right)\right|\right)+\left|f'(\tau^\nu)\right|\right).$$

Which completes the proof. □

Theorem 2.2. *Suppose that $f:[\kappa^\nu,\tau^\nu]\to\Re$ be a differentiable function on (κ^ν,τ^ν) with $0\leq\kappa<\tau$. If $|f'|^q$ is $m-$convex function, then we have the following inequality for Katugampola fractional integral operator:*

$$\begin{aligned} &\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{(\tau^\nu-\kappa^\nu)^\xi}\left[\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right)f(\tau^\nu)+\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right)f(\kappa^\nu)\right]-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)\right| \\ &\leq \frac{(\tau^\nu-\kappa^\nu)}{4}\left(\frac{1}{\nu\xi p+\nu p-p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2\nu+2}\right)^{\frac{1}{q}} \\ &\quad\times\left[\left(\left|f'(\kappa^\nu)\right|^q+m(2\nu+1)\left|f'\left(\frac{\tau^\nu}{m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'(\tau^\nu)\right|^q+m(2\nu+1)\left|f'\left(\frac{\kappa^\nu}{m}\right)\right|^q\right)^{\frac{1}{q}}\right] \end{aligned}$$

for $p>1$, $m\in(0,1]$ and $p^{-1}+q^{-1}=1$.

Proof. From the right hand side of Lemma (2.1), we have

$$\begin{aligned} &|A| \\ &\leq \frac{(\tau^\nu-\kappa^\nu)}{4}\left[\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|dt\right. \\ &\quad \left.+\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)\right|dt\right]. \end{aligned}$$

By using the Hölder inequality, we get

$$\begin{aligned} |A| &\leq \frac{(\tau^\nu-\kappa^\nu)}{4}\left(\int_0^1 t^{\nu\xi p+\nu p-p}\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\quad+\frac{(\tau^\nu-\kappa^\nu)}{4}\left(\int_0^1 t^{\nu\xi p+\nu p-p}\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned}$$

Thus, we provide

$$\begin{aligned} |A| &\leq \frac{(\tau^\nu-\kappa^\nu)}{4}\left(\frac{1}{\nu\xi p+\nu p-p+1}\right)^{\frac{1}{p}}\times\left(\frac{1}{2\nu+2}\left|f'(\kappa^\nu)\right|^q+\frac{m(2\nu+1)}{2\nu+2}\left|f'\left(\frac{\tau^\nu}{m}\right)\right|^q\right)^{\frac{1}{q}} \\ &\quad+\frac{(\tau^\nu-\kappa^\nu)}{4}\left(\frac{1}{\nu\xi p+\nu p-p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2\nu+2}\left|f'(\tau^\nu)\right|^q+\frac{m(2\nu+1)}{2\nu+2}\left|f'\left(\frac{\kappa^\nu}{m}\right)\right|^q\right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof. □

Theorem 2.3. *If $f:[\kappa^\nu,\tau^\nu]\to\Re$ be differentiable function on (κ^ν,τ^ν) with $\kappa^\nu<\tau^\nu$ and $f'\in L_1[\kappa^\nu,\tau^\nu]$. If $|f'|^q$ is a $m-$convex function, then we have the following inequality for Katugampola fractional integral operator:*

$$\begin{aligned}&|A|\\ \leq\;& \frac{(\tau^\nu-\kappa^\nu)}{4}\left(\frac{1}{\nu\xi+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{2\nu\xi+4\nu}\right)^{\frac{1}{q}}\\ &\times\left[\left(\left|f'(\kappa^\nu)\right|^q+\frac{m(\nu\xi+3\nu)}{(\nu\xi+\nu)}\left|f'\left(\frac{\tau^\nu}{m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'(\tau^\nu)\right|^q+\frac{m(\nu\xi+3\nu)}{(\nu\xi+\nu)}\left|f'\left(\frac{\kappa^\nu}{m}\right)\right|^q\right)^{\frac{1}{q}}\right]\end{aligned}$$

where $q\geq 1$ and $m\in(0,1]$.

Proof. From Lemma 2.1, we have

$$\begin{aligned}&\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{(\tau^\nu-\kappa^\nu)^\xi}\left[\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right)f(\tau^\nu)+\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right)f(\kappa^\nu)\right]-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)\right|\\ \leq\;& \frac{(\tau^\nu-\kappa^\nu)}{4}\left[\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|dt+\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)\right|dt\right].\end{aligned}$$

By applying Power-mean inequality, we get

$$\begin{aligned}|A|\leq\;&\frac{(\tau^\nu-\kappa^\nu)}{4}\left(\int_0^1 t^{\nu\xi+\nu-1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\frac{(\tau^\nu-\kappa^\nu)}{4}\left(\int_0^1 t^{\nu\xi+\nu-1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

By using $m-$convexity of $|f'|^q$ and making some simple computations, we get

$$\begin{aligned}|A|\;\leq\;&\frac{(\tau^\nu-\kappa^\nu)}{4}\left(\frac{1}{\nu\xi+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{2\nu\xi+4\nu}\right)^{\frac{1}{q}}\\ &\times\left[\left(\left|f'(\kappa^\nu)\right|^q+\frac{m(\nu\xi+3\nu)}{(\nu\xi+\nu)}\left|f'\left(\frac{\tau^\nu}{m}\right)\right|^q\right)^{\frac{1}{q}}\left(\left|f'(\tau^\nu)\right|^q+\frac{m(\nu\xi+3\nu)}{(\nu\xi+\nu)}\left|f'\left(\frac{\kappa^\nu}{m}\right)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Which completes the proof. □

Theorem 2.4. *Suppose that $f:[\kappa^\nu,\tau^\nu]\to\Re$ be a differentiable function on (κ^ν,τ^ν) with $0\leq\kappa<\tau$. If $|f'|^q$ is $m-$convex function, then we have the following inequality for Katugampola fractional integral operator:*

$$\begin{aligned}&\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{(\tau^\nu-\kappa^\nu)^\xi}\left[\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right)f(\tau^\nu)+\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right)f(\kappa^\nu)\right]-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)\right|\\ \leq\;&\frac{(\tau^\nu-\kappa^\nu)}{4}\left[\frac{2}{\nu\xi p^2+\nu p^2-p^2+p}\right.\\ &\left.+\frac{\left|f'(\kappa^\nu)\right|^q+m(2\nu+1)\left(\left|f'\left(\frac{\tau^\nu}{m}\right)\right|^q+\left|f'\left(\frac{\kappa^\nu}{m}\right)\right|^q\right)+\left|f'(\tau^\nu)\right|^q}{2\nu q+2q}\right]\end{aligned}$$

for $p,q>1$ and $m\in(0,1]$.

Proof. From the right hand side of Lemma (2.1), we have

$$\begin{aligned}
&\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{(\tau^\nu-\kappa^\nu)^\xi}\left[\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right)f(\tau^\nu)+\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right)f(\kappa^\nu)\right]-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)\right|\\
&\leq \frac{(\tau^\nu-\kappa^\nu)}{4}\left[\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|dt\right.\\
&\quad\left.+\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)\right|dt\right].
\end{aligned}$$

By using the Young inequality, we get

$$\begin{aligned}
|A| &\leq \frac{(\tau^\nu-\kappa^\nu)}{4}\int_0^1\left(\frac{t^{(\nu\xi+\nu-1)p}}{p}+\frac{\left|f'\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|^q}{q}dt\right)\\
&\quad+\frac{(\tau^\nu-\kappa^\nu)}{4}\int_0^1\left(\frac{t^{(\nu\xi+\nu-1)p}}{p}+\frac{\left|f'\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)\right|^q}{q}dt\right)
\end{aligned}$$

Thus, we can conclude

$$\begin{aligned}
&\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{(\tau^\nu-\kappa^\nu)^\xi}\left[\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right)f(\tau^\nu)+\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right)f(\kappa^\nu)\right]-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)\right|\\
&\leq \frac{(\tau^\nu-\kappa^\nu)}{4}\left[\frac{2}{\nu\xi p^2+\nu p^2-p^2+p}\right.\\
&\quad\left.+\frac{\left|f'(\kappa^\nu)\right|^q+m(2\nu+1)\left(\left|f'\left(\frac{\tau^\nu}{m}\right)\right|^q+\left|f'\left(\frac{\kappa^\nu}{m}\right)\right|^q\right)+\left|f'(\tau^\nu)\right|^q}{2\nu q+2q}\right].
\end{aligned}$$

This completes the proof. ∎

Theorem 2.5. *Suppose that $f:[\kappa^\nu,\tau^\nu]\to\Re$ be a differentiable function on (κ^ν,τ^ν) with $0\leq\kappa<\tau$. If $|f'|$ is exponentially convex function, then one has the following result for Katugampola fractional integral operator:*

$$\begin{aligned}
&\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{(\tau^\nu-\kappa^\nu)^\xi}\left[\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right)f(\tau^\nu)+\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right)f(\kappa^\nu)\right]-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)\right|\\
&\leq \frac{(\tau^\nu-\kappa^\nu)}{4(\nu\xi+\nu)}\left[\frac{\left|f'(\kappa^\nu)\right|}{e^{\alpha\kappa^\nu}}+\frac{\left|f'(\tau^\nu)\right|}{e^{\alpha\tau^\nu}}\right]
\end{aligned}$$

for $\alpha\in\mathbb{R}$.

Proof. By using the integral identity in Lemma (2.1), we have

$$\begin{aligned}
|A| &\leq \frac{(\tau^\nu-\kappa^\nu)}{4}\left[\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|dt\right.\\
&\quad\left.+\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)\right|dt\right].
\end{aligned}$$

From the definition of exponentially convex functions, we obtain

$$\begin{aligned} |A| &\leq \frac{(\tau^\nu - \kappa^\nu)}{4}\left[\int_0^1 t^{\nu\xi+\nu-1}\left[\frac{t^\nu}{2}\frac{\left|f'(\kappa^\nu)\right|}{e^{\alpha\kappa^\nu}} + \frac{2-t^\nu}{2}\frac{\left|f'(\tau^\nu)\right|}{e^{\alpha\tau^\nu}}\right]dt \right. \\ &\quad \left. + \int_0^1 t^{\nu\xi+\nu-1}\left[\frac{t^\nu}{2}\frac{\left|f'(\tau^\nu)\right|}{e^{\alpha\tau^\nu}} + \frac{2-t^\nu}{2}\frac{\left|f'(\kappa^\nu)\right|}{e^{\alpha\kappa^\nu}}\right]dt\right]. \end{aligned}$$

By making the necessary calculations, we get

$$\begin{aligned} |A| &\leq \frac{(\tau^\nu - \kappa^\nu)}{4}\left[\frac{\left|f'(\kappa^\nu)\right|}{e^{\alpha\kappa^\nu}}\int_0^1 t^{\nu\xi+\nu-1}\frac{t^\nu}{2}dt + \frac{\left|f'(\tau^\nu)\right|}{e^{\alpha\tau^\nu}}\int_0^1 t^{\nu\xi+\nu-1}\frac{2-t^\nu}{2}dt \right. \\ &\quad \left. + \frac{\left|f'(\tau^\nu)\right|}{e^{\alpha\tau^\nu}}\int_0^1 t^{\nu\xi+\nu-1}\frac{t^\nu}{2} + \frac{\left|f'(\kappa^\nu)\right|}{e^{\alpha\kappa^\nu}}\int_0^1 t^{\nu\xi+\nu-1}\frac{2-t^\nu}{2}dt\right] \\ &= \frac{(\tau^\nu - \kappa^\nu)}{4}\left[\frac{\left|f'(\kappa^\nu)\right|}{e^{\alpha\kappa^\nu}}\int_0^1 t^{\nu\xi+\nu-1}dt + \frac{\left|f'(\tau^\nu)\right|}{e^{\alpha\tau^\nu}}\int_0^1 t^{\nu\xi+\nu-1}dt\right] \\ &= \frac{(\tau^\nu - \kappa^\nu)}{4(\nu\xi+\nu)}\left[\frac{\left|f'(\kappa^\nu)\right|}{e^{\alpha\kappa^\nu}} + \frac{\left|f'(\tau^\nu)\right|}{e^{\alpha\tau^\nu}}\right]. \end{aligned}$$

Which completes the proof. □

Theorem 2.6. *Suppose that $f : [\kappa^\nu, \tau^\nu] \to \Re$ be a differentiable function on (κ^ν, τ^ν) with $0 \leq \kappa < \tau$. If $|f'|^q$ is exponentially convex function, then one can obtain the following inequality for Katugampola fractional integral operator:*

$$\begin{aligned} &\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{(\tau^\nu-\kappa^\nu)^\xi}\left[\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right) f(\tau^\nu) + \left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right) f(\kappa^\nu)\right] - f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)\right| \\ &\leq \frac{(\tau^\nu - \kappa^\nu)}{4}\left(\frac{1}{\nu\xi p + \nu p - p + 1}\right)^{\frac{1}{p}} \\ &\times \left[\left(\frac{\left|f'(\kappa^\nu)\right|^q}{2e^{\alpha\kappa^\nu}(v+1)} + \frac{(2v-1)\left|f'(\tau^\nu)\right|^q}{2e^{\alpha\tau^\nu}(v+1)}\right)^{\frac{1}{q}} + \left(\frac{\left|f'(\tau^\nu)\right|^q}{2e^{\alpha\tau^\nu}(v+1)} + \frac{(2v-1)\left|f'(\kappa^\nu)\right|^q}{2e^{\alpha\kappa^\nu}(v+1)}\right)^{\frac{1}{q}}\right] \end{aligned}$$

for $p > 1$, $\alpha \in \mathbb{R}$ and $p^{-1} + q^{-1} = 1$.

Proof. From the right hand side of Lemma (2.1), we have

$$\begin{aligned} |A| &\leq \frac{(\tau^\nu - \kappa^\nu)}{4}\left[\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\right)\right|dt \right. \\ &\quad \left. + \int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\right)\right|dt\right]. \end{aligned}$$

By using the Hölder inequality, we can write

$$|A| \leq \frac{(\tau^{\nu}-\kappa^{\nu})}{4}\left(\int_0^1 t^{\nu\xi p+\nu p-p}\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{t^{\nu}}{2}\kappa^{\nu}+\frac{2-t^{\nu}}{2}\tau^{\nu}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(\tau^{\nu}-\kappa^{\nu})}{4}\left(\int_0^1 t^{\nu\xi p+\nu p-p}\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Thus, by using the definition of exponentially convexity, we provide

$$\begin{aligned}|A| \leq\ & \frac{(\tau^{\nu}-\kappa^{\nu})}{4}\left(\frac{1}{\nu\xi p+\nu p-p+1}\right)^{\frac{1}{p}} \\ & \times\left(\frac{\left|f'(\kappa^{\nu})\right|^q}{e^{\alpha\kappa^{\nu}}}\int_0^1\frac{t^{\nu}}{2}dt+\frac{\left|f'(\tau^{\nu})\right|^q}{e^{\alpha\tau^{\nu}}}\int_0^1\frac{2-t^{\nu}}{2}dt\right)^{\frac{1}{q}} \\ & +\frac{(\tau^{\nu}-\kappa^{\nu})}{4}\left(\frac{1}{\nu\xi p+\nu p-p+1}\right)^{\frac{1}{p}} \\ & \times\left(\frac{\left|f'(\tau^{\nu})\right|^q}{e^{\alpha\tau^{\nu}}}\int_0^1\frac{t^{\nu}}{2}dt+\frac{\left|f'(\kappa^{\nu})\right|^q}{e^{\alpha\kappa^{\nu}}}\int_0^1\frac{2-t^{\nu}}{2}dt\right)^{\frac{1}{q}}.\end{aligned}$$

After necessary computations, we have

$$\begin{aligned}|A| \leq\ & \frac{(\tau^{\nu}-\kappa^{\nu})}{4}\left(\frac{1}{\nu\xi p+\nu p-p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(\kappa^{\nu})\right|^q}{2e^{\alpha\kappa^{\nu}}(v+1)}+\frac{(2v-1)\left|f'(\tau^{\nu})\right|^q}{2e^{\alpha\tau^{\nu}}(v+1)}\right)^{\frac{1}{q}} \\ & +\frac{(\tau^{\nu}-\kappa^{\nu})}{4}\left(\frac{1}{\nu\xi p+\nu p-p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(\tau^{\nu})\right|^q}{2e^{\alpha\tau^{\nu}}(v+1)}+\frac{(2v-1)\left|f'(\kappa^{\nu})\right|^q}{2e^{\alpha\kappa^{\nu}}(v+1)}\right)^{\frac{1}{q}}.\end{aligned}$$

This completes the proof. □

Theorem 2.7. *Suppose that $f:[\kappa^{\nu},\tau^{\nu}]\to\Re$ be a differentiable function on $(\kappa^{\nu},\tau^{\nu})$ with $0\leq\kappa<\tau$. If $|f'|^q$ is exponentially convex function, then we have the following inequality for Katugampola fractional integral operator:*

$$\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{(\tau^{\nu}-\kappa^{\nu})^{\xi}}\left[\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}}\right)f(\tau^{\nu})+\left({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}}\right)f(\kappa^{\nu})\right]-f\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)\right|$$
$$\leq \frac{\nu(\tau^{\nu}-\kappa^{\nu})}{4}\left[\frac{2}{\nu\xi p^2+\nu p^2-p^2+p}+\frac{v\left|f'(\tau^{\nu})\right|^q}{qe^{\alpha\tau^{\nu}}(v+1)}+\frac{v\left|f'(\kappa^{\nu})\right|^q}{qe^{\alpha\kappa^{\nu}}(v+1)}\right]$$

for $p,q>1$ and $\alpha\in\mathbb{R}$.

Proof. From Lemma (2.1), we can write

$$\begin{aligned}|A| \leq\ & \frac{(\tau^{\nu}-\kappa^{\nu})}{4}\left[\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^{\nu}}{2}\kappa^{\nu}+\frac{2-t^{\nu}}{2}\tau^{\nu}\right)\right|dt\right. \\ & \left.+\int_0^1 t^{\nu\xi+\nu-1}\left|f'\left(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\right)\right|dt\right].\end{aligned}$$

By using the well-known Young inequality, we have

$$|A| \leq \frac{(\tau^{\nu}-\kappa^{\nu})}{4}\int_0^1 \left(\frac{t^{(\nu\xi+\nu-1)p}}{p} + \frac{\left| f'\left(\frac{t^{\nu}}{2}\kappa^{\nu}+\frac{2-t^{\nu}}{2}\tau^{\nu}\right)\right|^q}{q} dt \right) + \frac{(\tau^{\nu}-\kappa^{\nu})}{4}\int_0^1 \left(\frac{t^{(\nu\xi+\nu-1)p}}{p} + \frac{\left| f'\left(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\right)\right|^q}{q} dt \right).$$

Therefore, by taking into account exponentially convexity of $|f'|^q$, we obtain

$$|A| \leq \frac{\nu(\tau^{\nu}-\kappa^{\nu})}{4}\left[\frac{2}{\nu\xi p^2+\nu p^2-p^2+p} + \frac{v\left|f'(\tau^{\nu})\right|^q}{qe^{\alpha\tau^{\nu}}(v+1)} + \frac{v\left|f'(\kappa^{\nu})\right|^q}{qe^{\alpha\kappa^{\nu}}(v+1)} \right].$$

This completes the proof. □

3. Conclusion

In the literature, there are many studies of different researchers that include Katugampola integral operators for functions whose absolute values of first derivatives are convex. The main motivation point of the study is to obtain the inequalities with the help of Katugampola integral operators for the functions whose absolute value of the derivatives are $m-$convex and exponentially convex functions. In this sense, the findings contribute to the improvement in convex analysis and take the discussion one step further. In addition, Hölder's inequality is used to prove the main results and new approaches are obtained. Several special cases of our main findings can be found by selecting different values of m, α, v and ξ.

Recently, researchers working in the field of inequalities frequently use fractional integral operators and thus obtain new generalizations associated with the certain types of inequalities. Katugampola integral operators structurally combine Riemann-Liouville and Hadamard fractional integral operators and contribute to the effectiveness of the results with its generalized kernel structure. The results can be performed for different kinds of convexity and operators. These results can be applied in convex analysis, optimization and different areas of pure and applied sciences. The authors hope that these results will serve as a motivation for future work in this fascinating area.

References

[1] A.O. Akdemir, E. Deniz, E. Yüksel, *Some new generalizations for $m-$convexity via new conformable fractional integral operators,* Mathematica Moravica, **23**(2) (2019), 69–77.

[2] F.A. Aliev, N.A. Aliev, N.A. Safarova, *Transformation of the Mittag-Leffler function to an exponential function and some of its applications to problems with a fractional derivative,* Appl. Comput. Math., **18**(3) (2019), 316–325.

[3] S. Aslan, A.O., Akdemir, J., Zeynalov, *New integral inequalities on the co-ordinates for geometrically exponentially s-convex functions in the second sense*, Filomat, **39**(27) (2025), 9717–9728.

[4] M.U. Awan, M.A., Noor, K.I., Noor, *Hermite-Hadamard Inequalities for Exponentially Convex Functions,* Appl. Math. Inf. Sci., **12**(2) (2018), 405–409.

[5] S.I. Butt, S. Yousaf, A.O. Akdemir, M.A. Dokuyucu, *New Hadamard-type integral inequalities via a general form of fractional integral operators,* Chaos, Solitons and Fractals, **148** (2021), 111025.

[6] S.I. Butt, J. Nasir, S. Qaisar, K.M. Abualnaja, *$k-$Fractional variants of Hermite-Mercer-type inequalities via $s-$convexity with Applications,* J. Funct. Spaces, **2021** (2021), Article ID 5566360, 15 pages.

[7] S.I. Butt, M. Tariq, A. Aslam, H. Ahmad, T.A. Nofal, *Hermite-Hadamard type inequalities via generalized harmonic exponential convexity and applications*, J. Funct. Spaces, **2021** (2021), Article ID 5533491, 12 pages.

[8] S.I. Butt, A.O. Akdemir, E. Gül, N. Nadeem, A. Yalçın, *Some novel results for Chebyshev type inequalities via generalized proportional fractional integral operators,* Turkish J. Sci., **8**(3) (2023), 114–123.

[9] S.I. Butt, I. Hira, M.A. Dokuyucu, *New fractal Simpson estimates for twice local differentiable generalized convex mappings,* Appl. Comput. Math., **23**(4) (2024), 474–503.

[10] H. Chen, U.N. Katugampola, *Hermite-Hadamard and Hermite-Hadamar-Fejer type inequalityies for generalized fractional integrals* , J. Math. Anal. Appl., **26** (2013), 742–753.

[11] B. Çelik, A.O. Akdemir, E. Set, S. Aslan, *Ostrowski-Mercer type integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators,* TWMS J. Pure Appl. Math., **15**(2) (2024), 269–285.

[12] B. Çelik, A.O. Akdemir, E. Set, M.E. Özdemir, *Novel generalizations for Grüss type inequalities pertaining to the constant proportional fractional integrals,* Appl. Comput. Math., **22**(2) (2023), 275–291.

[13] S.S. Dragomir, R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula,* Appl. Math. Lett., **11**(5) (1998), 91–95.

[14] S.S. Dragomir, C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[15] A. Ekinci, M.E. Özdemir, *Some new integral inequalities via Riemann Liouville integral operators,* Appl. Comput. Math., **18**(3) (2019), 288–295.

[16] R. Gorenflo, *Fractional calculus: Some numerical methods. In: A. Carpinteri, F. Mainardi (Eds), Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, Wien- New York, 1997, 277-290.

[17] U.N. Katugampola , *New approach to a generalized fractional integral*, Appl. Math. Comput., **218**(3) (2011), 860–865.

[18] U.N. Katugampola , *A new approach to a generalized fractional derivatives*, Bull. Math. Anal. Appl., **6**(4) (2014), 1–15.

[19] U.N. Katugampola , *Mellin transforms of the generalized fractional integrals and derivatives* , Appl. Math. Comput., **257** (2015), 566–580.

[20] S. Kermausuor, E.R. Nwaeze, A.M. Tameru, *New integral inequalities via the Katugampola fractional integrals for functions whose second derivatives are strongly η-convex*, Mathematics, **7**(2) (2019), 183.

[21] A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier B.V., Amsterdam, Netherlands, 2006.

[22] K.S. Miller, B. Ross, *An introduction to fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, John Wiley and Sons, Inc., New York, 1993.

[23] J. Nasir, S.I. Butt, M.A. Dokuyucu, A.O. Akdemir, *New variants of Hermite-Hadamard type inequalities via generalized fractional operator for differentiable functions,* Turkish J. Sci., **7**(3) (2022), 185–201.

[24] M. E. Özdemir, A. Ekinci, A.O. Akdemir, *Some new integral inequalities for functions whose derivatives of absolute values are convex and concave,* TWMS J. Pure Appl. Math., 2(10) (2019), 212–224.

[25] J.E. Pečarić, D.S. Mitrinovic, A.M. Fink, *Classical and New Inequalities in Analysis*, 1993.

[26] S. Rashid, A.O. Akdemir, F. Jarad, M.A. Noor, K.I. Noor, *Simpson's type integral inequalities for κ-fractional integrals and their applications,* AIMS Math., **4**(4) (2019), 1087–1100.

[27] S. Rashid, Z. Hammouch, H. Kalsoom, R. Ashraf, Y.M. Chu, *New investigations on the generalized $\mathcal{K}$-fractional integral operators*, Frontiers in Physics, **8** (2020), 25.

[28] S. Rashid, H. Kalsoom, Z. Hammouch, R. Ashraf, D. Baleanu, Y.M. Chu, *New multi-parametrized estimates having pth$-$order differentiability in fractional calculus for predominating $h-$convex functions in Hilbert space*, Symmetry, **12**(2) (2020), 222.

[29] M. Z. Sarıkaya, E. Set, H. Yaldız, N. Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model., **57** (2013), 2403–2407.

[30] M.Z. Sarıkaya, H. Yıldırım, *On Hermite-Hadamard type inequalities for Riemann Liouville fractional integrals*, Miskolc Math. Notes, **17**(2) (2016), 1049–1059.

[31] M.Z. Sarıkaya, N. Alp, *On Hermite-Hadamard-Fejer type integral inequalities for generalized convex functions via local fractional integrals,* Open J. Math. Sci., **3**(1) (2019), 273–284.

[32] E. Set, A.O. Akdemir, F. Özata, *Grüss type inequalities for fractional integral operator involving the extended generalized Mittag-Leffler function,* Appl. Comput. Math., **19**(3) (2020), 402–414.
[33] G. Toader, *Some generalizations of the convexity. In: Proceedings of the Colloquium on Approximation and Optimization,* Cluj-Napoca, Romania, 1985, 329–338.
[34] M. Vivas-Cortez, A. Kashuri, S.I. Butt, M. Tariq, J. Nasir, *Exponential type* $p-$*convex function with some related inequalities and their applications*, Appl. Math., **15**(3) (2021), 253–261.
[35] A. Yalçın, E. Gül, A.O. Akdemir, *Hermite-Hadamard type inequalities for co-ordinated convex functions with variable-order fractional integrals,* Appl. Comput. Math., **24**(2) (2025), 326–343.
[36] H. Yaldız, A.O. Akdemir, *Katugampola fractional integrals within the class of* $s-$*convex function*, Turkish J. Sci., **3**(1) (2018), 40–50.
[37] E. Yüksel, *Inequalities for strongly s-convex functions via Atangana-Baleanu fractional integral operators,* Turkish J. Nature Sci., **13**(2) (2024), 49–60.
[38] E. Yüksel, *On Ostrowski-Mercer type inequalities for twice differentiable convex functions,* Filomat, **38**(19) (2024), 6945–6955.

[1]Nakhcivan State University,
Nakhcivan-Azerbaijan
Email address: cavansirzeynalov@ndu.edu.az
Email address: gultac.z@ndu.edu.az
Email address: z.nagiyeva@ndu.edu.az

[2]Department of Mathematics,
Turkey
Email address: yeterrerdass@gmail.com

[3]Department of Mathematics,
Ağri İbrahim Çeçen University,
Ağri, Turkey
Email address: aocakakdemir@gmail.com