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GENERALIZED HERMITE-HADAMARD INTEGRAL INEQUALITIES ON INTERVALS

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ABSTRACT. The aim of this study is to derive numerous new generalized inequalities of trapezoid and midpoint types by dividing the closed interval $[a,b]$ into n equal subintervals and utilizing convex functions in conjunction with the Hermite-Hadamard inequality.

1. INTRODUCTION

The concept of convex functions plays a fundamental role in various branches of mathematics, particularly in optimization theory, functional analysis, and approximation theory. A function $f:[a,b]\to\mathbb{R}$ is called convex if, for all $x,y\in[a,b]$ and $t\in[0,1]$, the following inequality holds:

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y).$$

Convex functions exhibit several intriguing properties, among which the Hermite-Hadamard inequality is a notable result. This inequality provides a connection between the value of a convex function at the midpoint of an interval and the average value of the function over that interval. The inequalities discovered by Hermite and Hadamard for convex functions state that if $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex mapping defined on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave [6]. In recent years, significant progress has been made in the generalization and extension of the Hermite-Hadamard inequality. For further details and comprehensive discussions on these developments, readers may consult [1–19], along with the references cited within those works. These studies explore a variety of approaches to broadening the scope of the inequality.

Key words and phrases. Convex function, Hermite-Hadamard inequalities, trapezoid ve midpoint inequalities.

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The Hermite–Hadamard inequality has been extensively studied in the literature for its numerous applications, particularly its ability to connect the midpoint and trapezoid formulas within a single inequality. In this study, a new inequality is established that integrates the composite trapezoid and composite midpoint formulas, extending the classical Hermite–Hadamard inequality as a specific case of the presented result.

2. Generalized Hermite–Hadamard inequalities

Throughout this study, let $a, b \in I$ with $a < b$. The closed interval $[a, b]$ is partitioned into $\mathbf{n}$ equal subintervals. The set of nodes $\{x_i\}_{i=0}^{n}$ defining these subintervals is given by

$$x_i = a + i\left(\frac{b-a}{n}\right), \qquad i = 0, 1, 2, \ldots, n.$$

Here, $\mathbf{h} = \frac{\mathbf{b-a}}{\mathbf{n}}$ denotes the length of each subinterval. By using convex function, generalized Hermite-Hadamard's inequalities can be represented in the following forms. The generalization of Hermite-Hadamard inequality is given in the succeeding theorem (see [2], Theorem 3, p.3). Let us give this theorem again here with a different proof.

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be convex function on I, where $a, b \in I$ with $a < b$. Then the following inequalities hold:*

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2n}\sum_{i=1}^{n}\left[f(x_i) + f(x_{i-1})\right] \tag{2.1}$$

for $n \in \mathbb{N}$.

Proof. Let $x_i = a + i\frac{b-a}{n}$, $i = 0, 1, 2, ..., n$ dividing the closed interval $[a, b]$ into n equal subintervals, each denoted by $[x_{i-1}, x_i]$. Since the function f is convex on each $[x_{i-1}, x_i] \subset [a, b]$, applying the Hermite-Hadamard inequality on $[x_{i-1}, x_i]$ yields the following result:

$$f\left(\frac{x_i + x_{i-1}}{2}\right) \leq \frac{n}{b-a}\int_{x_{i-1}}^{x_i} f(x)\,dx \leq \frac{f(x_i) + f(x_{i-1})}{2}.$$

Thus, taking the sum over i from 1 to n, we get

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right) \leq \frac{1}{b-a}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\,dx \leq \frac{1}{2n}\sum_{i=1}^{n}\left[f(x_i) + f(x_{i-1})\right]$$

which is completed the inequalities of (2.1). □

Remark 2.1. Under the assumptions of Theorem 2.1
with $n = 1$, the inequalities (2.1) reduces to (1.1),
with $n = 2$, the inequalities (2.1) reduces to

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]$$

which is proved by Tseng et. al in [9],

with $n=3$, the inequalities (2.1) reduces to

$$\begin{aligned} & \frac{1}{3}\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+b}{2}\right)+f\left(\frac{a+5b}{6}\right)\right] \\ \leq & \frac{1}{b-a}\int_a^b f(x)\,dx \\ \leq & \frac{1}{6}\left[f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right]. \end{aligned}$$

Let us present the following two lemmas for the generalized trapezoid and midpoint inequalities:

Lemma 2.1. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\mathring{I}$ such that $f'\in L_1[a,b]$ where $a,b\in I,\ a<b$. Then the following equality holds:*

$$\frac{1}{2n}\sum_{i=1}^{n}\left[f(x_i)+f(x_{i-1})\right]-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{b-a}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left(x-\frac{x_i+x_{i-1}}{2}\right)f'(x)\,dx \tag{2.2}$$

for $n\in\mathbb{N}$.

Proof. Using by partial integration method, we have

$$\begin{aligned} \int_{x_{i-1}}^{x_i}\left(x-\frac{x_i+x_{i-1}}{2}\right)f'(x)\,dx &= \left(x-\frac{x_i+x_{i-1}}{2}\right)f(x)\bigg|_{x_{i-1}}^{x_i}-\int_{x_{i-1}}^{x_i}f(x)\,dx \\ &= \frac{b-a}{2n}\left[f(x_i)+f(x_{i-1})\right]-\int_{x_{i-1}}^{x_i}f(x)\,dx. \end{aligned}$$

Dividing both sides by $b-a$ and taking the sum over i from 1 to n, the following result is obtained:

$$\frac{1}{b-a}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left(x-\frac{x_i+x_{i-1}}{2}\right)f'(x)\,dx=\frac{1}{2n}\sum_{i=1}^{n}\left[f(x_i)+f(x_{i-1})\right]-\frac{1}{b-a}\int_a^b f(x)\,dx$$

which is completed the inequalities of (2.2). □

Remark 2.2. In Lemma 2.1,

taking $n=1$, the equality (2.2) becomes

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx,$$

which is proved by Dragomir and Agarwal in [5].

Taking $n=2$, the equality (2.2) reduces to

$$\begin{aligned} & \frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx \\ = & \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left(x-\frac{3a+b}{4}\right)f'(x)\,dx+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b\left(x-\frac{a+3b}{4}\right)f'(x)\,dx \end{aligned}$$

which is proved by Sarikaya in [16].

Corollary 2.1. *Under the assumptions of Lemma 2.1, if we take $n = 3$, the equality (2.2) reduces to*

$$\begin{aligned}
&\frac{1}{6}\left[f(a)+f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx \\
=\;&\frac{1}{b-a}\int_a^{\frac{2a+b}{3}}\left(x-\frac{5a+b}{2}\right)f'(x)\,dx+\frac{1}{b-a}\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}\left(x-\frac{a+b}{2}\right)f'(x)\,dx \\
&+\frac{1}{b-a}\int_{\frac{a+2b}{3}}^{b}\left(x-\frac{a+5b}{6}\right)f'(x)\,dx.
\end{aligned}$$

Lemma 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathring{I}$ such that $f' \in L_1[a,b]$ where $a, b \in I,\ a < b$. Then the following equality holds:*

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i-1}}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx \qquad (2.3)\\
=\;&\frac{1}{b-a}\sum_{i=1}^{n}\int_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}(x-x_{i-1})\,f'(x)\,dx+\frac{1}{b-a}\sum_{i=1}^{n}\int_{\frac{x_i+x_{i-1}}{2}}^{x_i}(x-x_i)\,f'(x)\,dx
\end{aligned}$$

for $n \in \mathbb{N}$.

Proof. Using by partial integration method, we have

$$\begin{aligned}
&\int_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}(x-x_{i-1})\,f'(x)\,dx+\int_{\frac{x_i+x_{i-1}}{2}}^{x_i}(x-x_i)\,f'(x)\,dx \\
=\;&(x-x_{i-1})\,f(x)\Big|_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}-\int_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}f(x)\,dx+(x-x_i)\,f(x)\Big|_{\frac{x_i+x_{i-1}}{2}}^{x_i}-\int_{\frac{x_i+x_{i-1}}{2}}^{x_i}f(x)\,dx \\
=\;&\frac{b-a}{n}f\left(\frac{x_i+x_{i-1}}{2}\right)-\int_{x_{i-1}}^{x_i}f(x)\,dx.
\end{aligned}$$

Dividing both sides by $b-a$ and taking the sum over i from 1 to n, the following result is obtained:

$$\begin{aligned}
&\frac{1}{b-a}\sum_{i=1}^{n}\int_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}(x-x_{i-1})\,f'(x)\,dx+\frac{1}{b-a}\sum_{i=1}^{n}\int_{\frac{x_i+x_{i-1}}{2}}^{x_i}(x-x_i)\,f'(x)\,dx \\
=\;&\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i-1}}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx
\end{aligned}$$

which is completed the inequalities of (2.3). □

Remark 2.3. In Lemma 2.2,

taking $n = 1$, the equality (2.3) becomes

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx \\
=\;&\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(x-a)\,f'(x)\,dx+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}(x-b)\,f'(x)\,dx
\end{aligned}$$

which is proved by Kirmaci in [8].

Taking $n = 2$, the equality (2.3) reduces to

$$\begin{aligned}
&\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx \\
=\;& \frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(x-a)\,f'(x)\,dx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(x-\frac{a+b}{2}\right)f'(x)\,dx \\
&+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(x-\frac{a+b}{2}\right)f'(x)\,dx+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^{b}(x-b)\,f'(x)\,dx \\
=\;& \frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(x-a)\,f'(x)\,dx+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(x-\frac{a+b}{2}\right)f'(x)\,dx \\
&+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^{b}(x-b)\,f'(x)\,dx
\end{aligned}$$

for $n \in \mathbb{N}$, $n \geq 3$, there is the following relationship

$$a=\frac{a+(n-1)\,a}{n}\leq\frac{b+(n-1)\,a}{n}=\frac{a+(n-1)\,b}{n}+\frac{n-2}{n}(a-b)\leq\frac{a+(n-1)\,b}{n}\leq b.$$

Corollary 2.2. *Under the assumptions of Lemma 2.2 if we take $n = 3$, the equality (2.3) reduces to*

$$\begin{aligned}
&\frac{1}{3}\left[f\left(\frac{5a+b}{2}\right)+f\left(\frac{a+b}{2}\right)+f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx \\
=\;& \frac{1}{b-a}\int_a^{\frac{5a+b}{6}}(x-a)\,f'(x)\,dx+\frac{1}{b-a}\int_{\frac{2a+b}{3}}^{\frac{a+b}{2}}\left(x-\frac{2a+b}{3}\right)f'(x)\,dx \\
&+\frac{1}{b-a}\int_{\frac{a+2b}{3}}^{\frac{a+5b}{6}}\left(x-\frac{a+2b}{3}\right)f'(x)\,dx+\frac{1}{b-a}\int_{\frac{5a+b}{6}}^{\frac{2a+b}{3}}\left(x-\frac{2a+b}{3}\right)f'(x)\,dx \\
&+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{\frac{a+2b}{3}}\left(x-\frac{a+2b}{3}\right)f'(x)\,dx+\frac{1}{b-a}\int_{\frac{a+5b}{6}}^{b}(x-b)\,f'(x)\,dx \\
=\;& \frac{1}{b-a}\int_a^{\frac{5a+b}{6}}(x-a)\,f'(x)\,dx+\frac{1}{b-a}\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left(x-\frac{2a+b}{3}\right)f'(x)\,dx \\
&+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}\left(x-\frac{a+2b}{3}\right)f'(x)\,dx+\frac{1}{b-a}\int_{\frac{a+5b}{6}}^{b}(x-b)\,f'(x)\,dx.
\end{aligned}$$

Now, we give the new following results for generalized Trapezoid type inequality:

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathring{I}$ such that $f' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality*

$$\left|\frac{1}{2n}\sum_{i=1}^{n}\left[f(x_i)+f(x_{i-1})\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{8n^2}\sum_{i=1}^{n}\left[\left|f'(x_{i-1})\right|+\left|f'(x_i)\right|\right] \quad (2.4)$$

for $n \in \mathbb{N}$.

Proof. From Lemma 2.1 and using convexity of $|f'|$, then we have

$$\begin{aligned}
&\left|\frac{1}{2n}\sum_{i=1}^{n}\left[f\left(x_i\right)+f\left(x_{i-1}\right)\right]-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\
\leq\ & \frac{1}{b-a}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left|x-\frac{x_i+x_{i-1}}{2}\right|\left|f'\left(x\right)\right|dx \\
\leq\ & \frac{1}{b-a}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left|x-\frac{x_i+x_{i-1}}{2}\right|\left[\frac{x_i-x}{x_i-x_{i-1}}\left|f'\left(x_{i-1}\right)\right|+\frac{x-x_{i-1}}{x_i-x_{i-1}}\left|f'\left(x_i\right)\right|\right]dx \\
=\ & \frac{1}{b-a}\sum_{i=1}^{n}\frac{\left|f'\left(x_{i-1}\right)\right|}{x_i-x_{i-1}}\int_{x_{i-1}}^{x_i}\left|x-\frac{x_i+x_{i-1}}{2}\right|\left(x_i-x\right)dx \\
&+\frac{1}{b-a}\sum_{i=1}^{n}\frac{\left|f'\left(x_i\right)\right|}{x_i-x_{i-1}}\int_{x_{i-1}}^{x_i}\left|x-\frac{x_i+x_{i-1}}{2}\right|\left(x-x_{i-1}\right)dx.
\end{aligned}$$

If the last two integrals above are computed as follows and substituted into their places, the desired result is obtained.

$$\begin{aligned}
&\int_{x_{i-1}}^{x_i}\left|x-\frac{x_i+x_{i-1}}{2}\right|\left(x_i-x\right)dx \\
=\ & \int_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}\left(\frac{x_i+x_{i-1}}{2}-x\right)\left(x_i-x\right)dx+\int_{\frac{x_i+x_{i-1}}{2}}^{x_i}\left(x-\frac{x_i+x_{i-1}}{2}\right)\left(x_i-x\right)dx \\
=\ & \frac{\left(x_i-x_{i-1}\right)^3}{8}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{x_{i-1}}^{x_i}\left|x-\frac{x_i+x_{i-1}}{2}\right|\left(x-x_{i-1}\right)dx \\
=\ & \int_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}\left(\frac{x_i+x_{i-1}}{2}-x\right)\left(x-x_{i-1}\right)dx+\int_{\frac{x_i+x_{i-1}}{2}}^{x_i}\left(x-\frac{x_i+x_{i-1}}{2}\right)\left(x-x_{i-1}\right)dx \\
=\ & \frac{\left(x_i-x_{i-1}\right)^3}{8}
\end{aligned}$$

which is completed the proof. □

Remark 2.4. In Theorem 2.2, taking $n=1$, the inequality (2.4) reduces to

$$\left|\frac{f\left(a\right)+f(b)}{2}-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|\leq\frac{(b-a)}{8}\left(\frac{\left|f'\left(a\right)\right|+\left|f'\left(b\right)\right|}{2}\right)$$

which is proved by Dragomir and Agarwal in [5].

Corollary 2.3. *Under the assumptions of Theorem 2.2,* *for $n=2$, the inequality (2.4) reduces to*

$$\begin{aligned}&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\ &\frac{(b-a)}{32}\left[|f'(a)|+2\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right]\\ \leq\ &\frac{(b-a)}{16}\left[|f'(a)|+|f'(b)|\right],\end{aligned}$$

for $n=3$, the inequality (2.4) reduces to

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\ &\frac{(b-a)}{72}\left[|f'(a)|+2\left|f'\left(\frac{2a+b}{3}\right)\right|+2\left|f'\left(\frac{a+2b}{3}\right)\right|+|f'(b)|\right].\end{aligned}$$

Now, we give the new following results for generalized Midpoint type inequality:

Theorem 2.3. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\mathring{I}$ such that $f'\in L_1[a,b]$ where $a,b\in I$, $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality*

$$\begin{aligned}&\left|\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i-1}}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \qquad (2.5)\\ \leq\ &\frac{(b-a)^2}{24n^2}\sum_{i=1}^{n}\left[|f'(x_i)|+2\left|f'\left(\frac{x_i+x_{i-1}}{2}\right)\right|+|f'(x_{i-1})|\right]\end{aligned}$$

for $n\in\mathbb{N}$.

Proof. From Lemma 2.2 and using convexity of $|f'|$, then we have

$$\begin{aligned}&\left|\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i-1}}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\ &\frac{1}{b-a}\sum_{i=1}^{n}\int_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}(x-x_{i-1})\,|f'(x)|\,dx+\frac{1}{b-a}\sum_{i=1}^{n}\int_{\frac{x_i+x_{i-1}}{2}}^{x_i}(x_i-x)\,|f'(x)|\,dx\\ \leq\ &\frac{1}{b-a}\sum_{i=1}^{n}\frac{2}{x_i-x_{i-1}}\int_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}(x-x_{i-1})\\ &\times\left[(x-x_{i-1})\left|f'\left(\frac{x_i+x_{i-1}}{2}\right)\right|+\left(\frac{x_i+x_{i-1}}{2}-x\right)|f'(x_{i-1})|\right]dx\\ &+\frac{1}{b-a}\sum_{i=1}^{n}\frac{2}{x_i-x_{i-1}}\int_{\frac{x_i+x_{i-1}}{2}}^{x_i}(x_i-x)\\ &\times\left[\left(x-\frac{x_i+x_{i-1}}{2}\right)|f'(x_i)|+(x_i-x)\left|f'\left(\frac{x_i+x_{i-1}}{2}\right)\right|\right]dx.\end{aligned}$$

The integrals above can be readily computed as outlined below:

$$\begin{aligned}
&\left|f'\left(\frac{x_i+x_{i-1}}{2}\right)\right|\int_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}(x-x_{i-1})^2\,dx \\
&+|f'(x_{i-1})|\int_{x_{i-1}}^{\frac{x_i+x_{i-1}}{2}}(x-x_{i-1})\left(\frac{x_i+x_{i-1}}{2}-x\right)dx \\
=\ &\left|f'\left(\frac{x_i+x_{i-1}}{2}\right)\right|\frac{(x_i-x_{i-1})^3}{24}+\frac{(x_i-x_{i-1})^3}{48}|f'(x_{i-1})|
\end{aligned}$$

and

$$\begin{aligned}
&|f'(x_i)|\int_{\frac{x_i+x_{i-1}}{2}}^{x_i}(x_i-x)\left(x-\frac{x_i+x_{i-1}}{2}\right)dx+\left|f'\left(\frac{x_i+x_{i-1}}{2}\right)\right|\int_{\frac{x_i+x_{i-1}}{2}}^{x_i}(x_i-x)^2\,dx \\
=\ &\frac{(x_i-x_{i-1})^3}{48}|f'(x_i)|+\frac{(x_i-x_{i-1})^3}{24}\left|f'\left(\frac{x_i+x_{i-1}}{2}\right)\right|
\end{aligned}$$

which is completed the proof. □

Remark 2.5. In Theorem 2.3,
for $n=1$, the inequality (2.5) reduces to

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{(b-a)^2}{24}\left[|f'(a)|+2\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right] \\
\le\ &\frac{(b-a)^2}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right)
\end{aligned}$$

which is proved by Kirmaci in [8].

Corollary 2.4. *Under the assumptions of Theorem 2.3,*
for $n=2$, the inequality (2.5) reduces to

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\le\ &\frac{(b-a)^2}{96}\left[|f'(a)|+2\left|f'\left(\frac{a+b}{2}\right)\right|+2\left|f'\left(\frac{3a+b}{4}\right)\right|+2\left|f'\left(\frac{a+3b}{4}\right)\right|+|f'(b)|\right] \\
\le\ &\frac{(b-a)^2}{96}\left[|f'(a)|+3\left|f'\left(\frac{3a+b}{4}\right)\right|+3\left|f'\left(\frac{a+3b}{4}\right)\right|+|f'(b)|\right],
\end{aligned}$$

for $n=3$, the inequality (2.5) reduces to

$$\begin{aligned}
&\left|\frac{1}{3}\left[f\left(\frac{5a+b}{2}\right)+f\left(\frac{a+b}{2}\right)+f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\le\ &\frac{(b-a)^2}{216}\left[|f'(a)|+2\left|f'\left(\frac{2a+b}{3}\right)\right|+2\left|f'\left(\frac{5a+b}{6}\right)\right|\right. \\
&\left.+2\left|f'\left(\frac{a+2b}{3}\right)\right|+2\left|f'\left(\frac{a+b}{2}\right)\right|+2\left|f'\left(\frac{a+5b}{6}\right)\right|+|f'(b)|\right].
\end{aligned}$$

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