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NEW PROOF OF HERMITE-HADAMARD-MERCER TYPE INEQUALITY VIA GREEN FUNCTION IDENTITY

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ABSTRACT. The distinguished Hermite-Hadamard inequality has captivated the attention of several researchers due to its applications and rich geometrical significance. In this paper, we provide a simpler and stronger proof of Hermite-Hadamard-Mercer type inequality given by Öğülmüs and Sarikaya in [20] as

$$\Psi\left(i_1+i_2-\frac{\mu_1+\mu_2}{2}\right) \leq \Psi(i_1)+\Psi(i_2)-\frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^{\alpha}}\left[J^{\alpha}_{\mu_1^+}\Psi(\mu_2)+J^{\alpha}_{\mu_2^-}\Psi(\mu_1)\right]$$
$$\leq \Psi(i_1)+\Psi(i_2)-\Psi\left(\frac{\mu_1+\mu_2}{2}\right),$$

where $\Psi:[i_1,i_2]\to\mathbb{R}$ is convex function, $\mu_1,\mu_2\in[i_1,i_2]$ and $\alpha>0$. To obtain the proposed result we utilize Green function identity and consider $\alpha\geq 1$. We hope that the results in this study will invigorate further research on the topic.

1. INTRODUCTION

The concept of convex convexity was presented over hundered years ago is one of the most useful and applicable area of mathematics. It has various applications in both pure and applied mathematics, economics, physics, statistics, information theory, etc. The convex functions play a central role in mathematical analysis, providing a base in various areas such as optimization, probability theory, and applied mathematics. The mathematical inequalities is a versatile field as it has wide applications in other branches like economics, physics, statistics, nonlinear programming, industry, engineering, optimization theory etc. The class of convex functions and mathematical inequalities are closely related to each other. Many important inequalities such as Hermite-Hadamard's (H-H) inequality, majorization inequality, Ostrowski inequality, Jensen's inequality, Fejer inequality and Hardy-type inequality etc. have been established for convex functions [3, 10, 13, 22].

Key words and phrases. Jensen-Mercer Inequality, Hermite-Hadamard Inequality, Riemann-Liouville fractional integrals, Green functions.

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A function $\Psi : [i_1, i_2] \to \mathbb{R}$ satisfy

$$\Psi(\lambda\mu_1 + (1-\lambda)\mu_2) \leq \lambda\Psi(\mu_1) + (1-\lambda)\Psi(\mu_2)$$

is said to be convex, for all $\mu_1, \mu_2 \in [i_1, i_2]$ and $\lambda \in [0,1]$. This classical definition of a convex function has been modified in several ways to obtain new kinds of convexities such as strongly convexity [1], h-convex [26], $m-$convex functions [19, 24], s-convex [9], 4-convex functions [27], P-convex [23] and coordinate convex [11] etc.

One of the most important integral inequality for convex functions in the literature of applied mathematical inequalities is the H-H inequality. This inequality was first discovered by Hermite in 1883 and J. Hadamard rediscover it independently in 1893 [2, 17, 18].
Let $\Psi : [i_1, i_2] \to \mathbb{R}$ be a convex function, then the inequality

$$\Psi\left(\frac{i_1+i_2}{2}\right) \leq \frac{1}{i_2-i_1}\int_{i_1}^{i_2}\Psi(\lambda)d\lambda \leq \frac{\Psi(i_1)+\Psi(i_2)}{2}$$

is said to be H-H inequality. The H-H inequality is elegant and simple and a lot of literature have been devoted to it providing new proofs, generalizations, extensions, refinements and applications. For more results associated to H-H inequality see [4, 5, 8]

Jensen's inequality is one of the most studied result in literature. Mercer introduced a modification of Jensen's inequality called Jensen-Mercer inequality [14, 15].

Let $\Psi : [i_1, i_2] \to \mathbb{R}$ be a convex function, $u, v \in [i_1, i_2]$, then the inequality

$$\Psi\left(i_1+i_2-\frac{u+v}{2}\right) \leq \Psi(i_1)+\Psi(i_2)-\frac{\Psi(u)+\Psi(v)}{2}$$

is said to be Jensen-Mercer inequality.

In [16], Kian and Moslehian combine the H-H and Jensen-Mercer inequalities.

$$\begin{aligned}
&\Psi\left(i_1+i_2-\frac{u+v}{2}\right) \leq \Psi(i_1)+\Psi(i_2)-\frac{1}{v-u}\int_u^v\Psi(\lambda)d\lambda \\
&\leq \Psi(i_1)+\Psi(i_2)-\Psi\left(\frac{u+v}{2}\right), \\
&\Psi\left(i_1+i_2-\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v\Psi(i_1+i_2-\lambda)d\lambda \\
&\leq \frac{\Psi(i_1+i_2-u)+\Psi(i_1+i_2-v)}{2} \\
&\leq \Psi(i_1)+\Psi(i_2)-\left(\frac{\Psi(i_1)+\Psi(i_2)}{2}\right).
\end{aligned}$$

The fractional integral is an important part of fractional calculus. There are various types of fractional integral operators see [6, 7]. The Riemann-Liouville (RL) fractional integral operators for a function Ψ defined on $[i_1, i_2]$ are:

$$J_{i_1^+}^{a}\Psi(\mu) = \frac{1}{\Gamma(a)}\int_{i_1}^{\lambda}(\mu-\lambda)^{a-1}\Psi(\lambda)d\lambda; \mu > i_1, \tag{1.1}$$

$$J^{a}_{i_2^-}\Psi(\mu)=\frac{1}{\Gamma(a)}\int_{\lambda}^{i_2}(\lambda-\mu)^{a-1}\Psi(\lambda)d\lambda; i_2>\mu. \tag{1.2}$$

Here a is the order of Ψ and Γ represents the gamma function, given by

$$\Gamma(a)=\int_0^{\infty}u^{a-1}e^{-u}du.$$

The operator $J^{a}_{i_1^+}\Psi(u)$, is called left RL integral operator and $J^{a}_{i_2^-}\Psi(u)$ is called right RL fractional integral operator.

Sarikaya et. al. in [21], gave H-H type inequalities containing (1.1) and (1.2). This inequality open a new area for researcher.

Theorem 1.1. *Suppose $\Psi:[i_1,i_2]\to\mathbb{R}$ is a positive convex function such that $\Psi\in L[i_1,i_2]$, then the fractional integrals inequality*

$$\Psi\left(\frac{i_1+i_2}{2}\right)\leq\frac{\Gamma(a+1)}{2(i_2-i_1)^a}\left[J^{a}_{i_1^+}\Psi(i_2)+J^{a}_{i_2^-}\Psi(i_1)\right]\leq\frac{\Psi(i_1)+\Psi(i_2)}{2}$$

holds for all $a>0$.

Tunc et. al. in [25], using Green function and gave the H-H type inequalities for the function whose absolute value of second order derivative is convex. The authors obtained the following result.

$$\left|\frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^{\alpha}}\left[J^{\alpha}_{\mu_1^+}\Psi(\mu_2)+J^{\alpha}_{\mu_2^-}\Psi(\mu_1)\right]-\frac{\Psi(i_1)+\Psi(i_2)}{2}\right|$$
$$\leq\frac{(\mu_2-\mu_1)^2}{(\alpha+1)(\alpha+2)}\left(\frac{|\Psi''(\mu_1)|+|\Psi''(\mu_2)|}{2}\right).$$

In this article, we give H-H-Mercer type inequality, by using Green function defined as [12].

$$G(\mu_1,\mu_2)=\begin{cases}i_1-\mu_1\ , & i_1\leq\mu_1\leq\mu_2;\\ i_1-\mu_2, & \mu_2\leq\mu_1\leq i_2.\end{cases} \tag{1.3}$$

The following integral identity holds for G

$$\Psi(w)=\Psi(i_1)+(w-i_1)\Psi'(i_2)+\int_{i_1}^{i_2}G(w,\lambda)\Psi''(\lambda)d\lambda \tag{1.4}$$

here ψ is twice differentiable function.

2. Main Result

Lemma 2.1. *Let $\Psi:[i_1,i_2]\to R$ be a twice differentiable function and $\mu_1,\mu_2\in[i_1,i_2]$ such that $\mu_1<\mu_2$. Then*

$$\Psi(i_1)+\Psi(i_2)-\frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^{\alpha}}\left(J^{\alpha}_{\mu_1^-}\Psi(\mu_2)+J^{\alpha}_{\mu_2^-}\Psi(\mu_1)\right)=\Psi(i_2)+\left(i_1-\frac{\mu_1+\mu_2}{2}\right)\Psi'(i_2)$$
$$-\frac{\alpha}{2(\mu_2-\mu_1)^{\alpha}}\int_{i_1}^{i_2}\int_{\mu_1}^{\mu_2}\left((\mu_2-w)^{\alpha-1}+(w-\mu_1)^{\alpha-1}\right)G(w,\lambda)\Psi''(\lambda)dwd\lambda. \tag{2.1}$$

Proof. From RL fractional integral, we have

$$J^{\alpha}_{\mu_1^+}\Psi(\mu_2)=\frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\alpha-1}\Psi(w)dw.$$

Substituting the value of $\Psi(w)$ from (1.4), we obtain

$$J^{\alpha}_{\mu_1^+}\Psi(\mu_2)=\frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\alpha-1}\left(\Psi(i_1)+(w-i_1)\Psi'(i_2)+\int_{i_1}^{i_2}G(w,\lambda)\Psi''(\lambda)d\lambda\right)dw,$$

$$\Rightarrow J^{\alpha}_{\mu_1^+}\Psi(\mu_2)=\frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\alpha-1}\Psi(i_1)dw+\frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\alpha-1}(w-i_1)\Psi'(i_2)dw$$
$$+\frac{1}{\Gamma(\alpha)}\int_{i_1}^{i_2}\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\alpha-1}G(w,\lambda)\Psi''(\lambda)dwd\lambda,$$

$$\Rightarrow J^{\alpha}_{\mu_1^+}\Psi(\mu_2)=\frac{1}{\Gamma(\alpha)}\Psi(i_1)I_1+\frac{1}{\Gamma(\alpha)}\Psi'(i_2)I_2+\frac{1}{\Gamma(\alpha)}\int_{i_1}^{i_2}\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\alpha-1}G(w,\lambda)\Psi''(\lambda)dwd\lambda, \tag{2.2}$$

where

$$I_1=\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\alpha-1}dw=\frac{(\mu_2-\mu_1)^{\alpha}}{\alpha}, \tag{2.3}$$

and

$$\begin{aligned}I_2&=\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\alpha-1}(w-\mu_2)\,dw\\&=\frac{(\mu_2-w)^{\alpha}}{-\alpha}(w-i_1)\Big|_{\mu_1}^{\mu_2}+\frac{1}{\alpha}\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\alpha}dw\\&=\frac{(\mu_2-\mu_1)^{\alpha}}{\alpha}\left(\mu_1-i_1+\frac{\mu_2-\mu_1}{\alpha+1}\right).\end{aligned} \tag{2.4}$$

Substituting (2.3) and (2.4) in (2.2), we obtain

$$\begin{aligned}J^{\alpha}_{\mu_1^+}\Psi(\mu_2)&=\frac{(\mu_2-\mu_1)^{\alpha}}{\Gamma(\alpha+1)}\Psi(i_1)+\frac{(\mu_2-\mu_1)^{\alpha}}{\Gamma(\alpha+1)}\left(\mu_1-i_1+\frac{\mu_2-\mu_1}{\alpha+1}\right)\Psi'(i_2)\\&+\frac{1}{\Gamma(\alpha)}\int_{i_1}^{i_2}\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\alpha-1}G(w,\lambda)\Psi''(\lambda)dwd\lambda.\end{aligned} \tag{2.5}$$

Similarly

$$\begin{aligned}J^{\alpha}_{\mu_2^-}\Psi(\mu_1)&=\frac{(\mu_2-\mu_1)^{\alpha}}{\Gamma(\alpha+1)}\Psi(i_1)+\frac{(\mu_2-\mu_1)^{\alpha}}{\Gamma(\alpha+1)}\left(\mu_2-i_1-\frac{\mu_2-\mu_1}{\alpha+1}\right)\Psi'(i_2)\\&+\frac{1}{\Gamma(\alpha)}\int_{i_1}^{i_2}\int_{\mu_1}^{\mu_2}(w-\mu_1)^{\alpha-1}G(w,\lambda)\Psi''(\lambda)dwd\lambda.\end{aligned} \tag{2.6}$$

Adding (2.5) and (2.6), we get

$$\begin{aligned}J^{\alpha}_{\mu_1^-}\Psi(\mu_2)+J^{\alpha}_{\mu_2^-}\Psi(\mu_1)&=\frac{2(\mu_2-\mu_1)^{\alpha}}{\Gamma(\alpha+1)}\Psi(i_1)+\frac{(\mu_2-\mu_1)^{\alpha}}{\Gamma(\alpha+1)}(\mu_1+\mu_2-2i_1)\Psi'(i_2)\\&+\frac{1}{\Gamma(\alpha)}\int_{i_1}^{i_2}\int_{\mu_1}^{\mu_2}\left((\mu_2-w)^{\alpha-1}+(w-\mu_1)^{\alpha-1}\right)G(w,\lambda)\Psi''(\lambda)dwd\lambda.\end{aligned} \tag{2.7}$$

Multiply (2.7) with $-\frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^{\alpha}}$ and adding $\Psi(i_2)$, we obtain (2.1). □

Theorem 2.1. *Suppose $\Psi : [i_1, i_2] \to R$ is convex and twice differentiable function such that $\Psi(i_1) < \Psi(i_2)$ and $\Psi'(i_2) \le 0$. Let $\mu_1, \mu_2 \in [i_1, i_2]$ such that $\mu_1 < \mu_2$. Then the inequalities*

$$\Psi\left(i_1 + i_2 - \frac{\mu_1 + \mu_2}{2}\right) \le \Psi(i_1) + \Psi(i_2) - \frac{\Gamma(\alpha+1)}{2(\mu_2 - \mu_1)^\alpha}\left[J^\alpha_{\mu_1^-}\Psi(\mu_2) + J^\alpha_{\mu_2^-}\Psi(\mu_1)\right]$$
$$\le \Psi(i_1) + \Psi(i_2) - \Psi\left(\frac{\mu_1 + \mu_2}{2}\right) \tag{2.8}$$

hold, where $\alpha \ge 1$.

Proof. Putting $w = i_1 + i_2 - \frac{\mu_1+\mu_2}{2}$ in (1.4), we obtain

$$\Psi(i_1 + i_2 - \frac{\mu_1 + \mu_2}{2}) = \Psi(i_1) + (i_2 - \frac{\mu_1 + \mu_2}{2})\Psi'(i_2) + \int_{i_1}^{i_2} G(i_1 + i_2 - \frac{\mu_1 + \mu_2}{2}, \lambda)\Psi''(\lambda)d\lambda. \tag{2.9}$$

Subtracting (2.1) from (2.9), we get

$$\Psi(i_1 + i_2 - \frac{\mu_1 + \mu_2}{2}) - \left(\Psi(i_1) + \Psi(i_2) - \frac{\Gamma(\alpha+1)}{2\,(\mu_2 - \mu_1)^\alpha}\left(J^\alpha_{\mu_1^-}\Psi(\mu_2) + J^\alpha_{\mu_2^-}\Psi(\mu_1)\right)\right)$$
$$= \Psi(i_1) - \Psi(i_2) + (i_2 - i_1)\Psi'(i_2) + \int_{i_1}^{i_2} G(i_1 + i_2 - \frac{\mu_1 + \mu_2}{2}, \lambda)\Psi''(\lambda)d\lambda$$
$$+ \frac{\alpha}{2\,(\mu_2 - \mu_1)^\alpha}\int_{i_1}^{i_2}\int_{\mu_1}^{\mu_2}\left((\mu_2 - w)^{\alpha-1} + (w - \mu_1)^{\alpha-1}\right)G(w, \lambda)\Psi''(\lambda)dwd\lambda. \tag{2.10}$$

Obviously, the first two terms in the right hand side of (2.10) are non-positive. Also $G \le 0$ and Ψ is convex, therefore the third term is also non-positive. As $w \in [\mu_1, \mu_2]$, therefore

$$\left((\mu_2 - w)^{\alpha-1} + (w - \mu_1)^{\alpha-1}\right) \ge 0.$$

Also $G \le 0$ implies that the fourth term is also non-positive. From (2.10), we have

$$\Psi(i_1 + i_2 - \frac{\mu_1 + \mu_2}{2}) \le \Psi(i_1) + \Psi(i_2) - \frac{\Gamma(\alpha+1)}{2\,(\mu_2 - \mu_1)^\alpha}\left(J^\alpha_{\mu_1^-}\Psi(\mu_2) + J^\alpha_{\mu_2^-}\Psi(\mu_1)\right). \tag{2.11}$$

Now we prove the second part of (2.8). Substituting $w = \frac{\mu_1+\mu_2}{2}$ in (1.4), we obtain

$$\Psi(\frac{\mu_1 + \mu_2}{2}) = \Psi(i_1) + (\frac{\mu_1 + \mu_2}{2} - i_1)\Psi'(i_2) + \int_{i_1}^{i_2} G(\frac{\mu_1 + \mu_2}{2}, \lambda)\Psi''(\lambda)d\lambda.$$

Multiply with -1 and adding $\Psi(i_2)$

$$\Psi(i_1) + \Psi(i_2) - \Psi(\frac{\mu_1 + \mu_2}{2}) = (i_1 - \frac{\mu_1 + \mu_2}{2})\Psi'(i_2) - \int_{i_1}^{i_2} G(\frac{\mu_1 + \mu_2}{2}, \lambda)\Psi''(\lambda)d\lambda. \tag{2.12}$$

Subtracting (2.12) from (2.1)

$$\Psi(i_1) + \Psi(i_2) - \frac{\Gamma(\alpha+1)}{2\,(\mu_2 - \mu_1)^\alpha}\left(J^\alpha_{\mu_1^-}\Psi(\mu_2) + J^\alpha_{\mu_2^-}\Psi(\mu_1)\right) - \left(\Psi(i_1) + \Psi(i_2) - \Psi(\frac{\mu_1 + \mu_2}{2})\right)$$
$$= \int_{i_1}^{i_2}\left(G(\frac{\mu_1 + \mu_2}{2}, \lambda) - \frac{\alpha}{2\,(\mu_2 - \mu_1)^\alpha}\int_{\mu_1}^{\mu_2}\left((\mu_2 - w)^{\alpha-1} + (w - \mu_1)^{\alpha-1}\right)G(w, \lambda)\right)\Psi''(\lambda)dwd\lambda,$$

$$\Rightarrow \Psi(i_1) + \Psi(i_2) - \frac{\Gamma(\alpha+1)}{2\,(\mu_2 - \mu_1)^\alpha}\left(J^\alpha_{\mu_1^-}\Psi(\mu_2) + J^\alpha_{\mu_2^-}\Psi(\mu_1)\right) - \left(\Psi(i_1) + \Psi(i_2) - \Psi(\frac{\mu_1 + \mu_2}{2})\right)$$
$$= \int_{i_1}^{i_2} H(\lambda)\Psi''(\lambda)d\lambda, \tag{2.13}$$

here

$$H(\lambda)=G(\frac{\mu_1+\mu_2}{2},\lambda)-\frac{\alpha}{2\left(\mu_2-\mu_1\right)^{\alpha}}\int_{\mu_1}^{\mu_2}\left(\left(\mu_2-w\right)^{\alpha-1}+\left(w-\mu_1\right)^{\alpha-1}\right)G(w,\lambda)dw. \tag{2.14}$$

Let

$$\begin{aligned}
I(\lambda) &= \int_{\mu_1}^{\mu_2}\left(\left(\mu_2-w\right)^{\alpha-1}+\left(w-\mu_1\right)^{\alpha-1}\right)G(w,\lambda)dw,\\
\Rightarrow I(\lambda) &= \int_{\mu_1}^{\lambda}\left(\left(\mu_2-w\right)^{\alpha-1}+\left(w-\mu_1\right)^{\alpha-1}\right)(\mu_1-w)dw\\
&+\int_{\lambda}^{\mu_2}\left(\left(\mu_2-w\right)^{\alpha-1}+\left(w-\mu_1\right)^{\alpha-1}\right)(\mu_1-\lambda)dw
\end{aligned}$$

$$\begin{aligned}
\Rightarrow I(\lambda) &= \int_{\mu_1}^{\lambda}\left(\mu_2-w\right)^{\alpha-1}(\mu_1-w)dw+\int_{\mu_1}^{\lambda}\left(w-\mu_1\right)^{\alpha-1}(\mu_1-w)dw\\
&+\int_{\lambda}^{\mu_2}\left(\mu_2-w\right)^{\alpha-1}(\mu_1-\lambda)dw+\int_{\lambda}^{\mu_2}\left(w-\mu_1\right)^{\alpha-1}(\mu_1-\lambda)dw.
\end{aligned} \tag{2.15}$$

Now, since

$$\int_{\mu_1}^{\lambda}\left(\mu_2-w\right)^{\alpha-1}(\mu_1-w)dw=-\frac{(\mu_2-\lambda)^{\alpha}(\mu_1-\lambda)}{\alpha}+\frac{(\mu_2-\lambda)^{\alpha+1}}{\alpha(\alpha+1)}-\frac{(\mu_2-\mu_1)^{\alpha+1}}{\alpha(\alpha+1)}, \tag{2.16}$$

$$\int_{\mu_1}^{\lambda}\left(w-\mu_1\right)^{\alpha-1}(\mu_1-w)dw=\frac{(\lambda-\mu_1)^{\alpha}(\mu_1-\lambda)}{\alpha}+\frac{(\lambda-\mu_1)^{\alpha+1}}{\alpha(\alpha+1)}, \tag{2.17}$$

$$\int_{\lambda}^{\mu_2}\left(\mu_2-w\right)^{\alpha-1}(\mu_1-\lambda)dw=\frac{(\mu_1-\lambda)(\mu_2-\lambda)^{\alpha}}{\alpha}, \tag{2.18}$$

$$\int_{\lambda}^{\mu_2}\left(w-\mu_1\right)^{\alpha-1}(\mu_1-\lambda)dw=\frac{(\mu_1-\lambda)(\mu_2-\mu_1)^{\alpha}}{\alpha}-\frac{(\mu_1-\lambda)(\lambda-\mu_1)^{\alpha}}{\alpha}. \tag{2.19}$$

Substituting these values from (2.16), (2.17), (2.18) and (2.19) in (2.15), we obtain

$$I(\lambda)=\frac{(\mu_2-\lambda)^{\alpha+1}}{\alpha(\alpha+1)}+\frac{(\lambda-\mu_1)^{\alpha+1}}{\alpha(\alpha+1)}+\frac{(\mu_1-\lambda)(\mu_2-\mu_1)^{\alpha}}{\alpha}-\frac{(\mu_2-\mu_1)^{\alpha+1}}{\alpha(\alpha+1)}. \tag{2.20}$$

Putting this value from (2.20) in (2.14), we get

$$\begin{aligned}
H(\lambda) &= G(\frac{\mu_1+\mu_2}{2},\lambda)-\frac{1}{2\left(\mu_2-\mu_1\right)^{\alpha}}\left(\frac{(\mu_2-\lambda)^{\alpha+1}}{(\alpha+1)}+\frac{(\lambda-\mu_1)^{\alpha+1}}{(\alpha+1)}\right)\\
&-\frac{1}{2}\left(\mu_1-\lambda-\frac{\mu_2-\mu_1}{\alpha+1}\right).
\end{aligned} \tag{2.21}$$

If $\mu_1\le\lambda\le\frac{\mu_1+\mu_2}{2}$, then from (2.21), we have

$$\begin{aligned}
H(\lambda) &= \mu_1-\lambda-\frac{1}{2\left(\mu_2-\mu_1\right)^{\alpha}}\left(\frac{(\mu_2-\lambda)^{\alpha+1}}{(\alpha+1)}+\frac{(\lambda-\mu_1)^{\alpha+1}}{(\alpha+1)}\right)\\
&-\frac{1}{2}\left(\mu_1-\lambda-\frac{\mu_2-\mu_1}{\alpha+1}\right)\\
\Rightarrow H'(\lambda) &= -\frac{1}{2}-\frac{1}{2\left(\mu_2-\mu_1\right)^{\alpha}}\left((\lambda-\mu_1)^{\alpha}-(\mu_2-\lambda)^{\alpha}\right),
\end{aligned}$$

$$H''(\lambda) = -\frac{\alpha}{2\left(\mu_2-\mu_1\right)^{\alpha}}\Big((\lambda-\mu_1)^{\alpha-1}+(\mu_2-\lambda)^{\alpha-1}\Big) \leq 0.$$

Since $H''(\lambda) \leq 0$, implies that $H'(\lambda)$ is decreasing and $H(\mu_1)=0$, therefore $H(\lambda) \leq 0$.

If $\frac{\mu_1+\mu_2}{2} \leq \lambda \leq \mu_2$, then $G(\frac{\mu_1+\mu_2}{2},\lambda) = \frac{\mu_1-\mu_2}{2}$ and from(2.21), we obtain

$$H(\lambda) = \frac{\mu_1-\mu_2}{2} - \frac{1}{2\left(\mu_2-\mu_1\right)^{\alpha}}\Big(\frac{(\mu_2-\lambda)^{\alpha+1}}{(\alpha+1)} + \frac{(\lambda-\mu_1)^{\alpha+1}}{(\alpha+1)}\Big)$$
$$-\frac{1}{2}\Big(\mu_1-\lambda-\frac{\mu_2-\mu_1}{\alpha+1}\Big),$$
$$\Rightarrow H'(\lambda) = \frac{1}{2} - \frac{1}{2\left(\mu_2-\mu_1\right)^{\alpha}}\Big((\lambda-\mu_1)^{\alpha}-(\mu_2-\lambda)^{\alpha}\Big),$$

$$\Rightarrow H''(\lambda) = -\frac{\alpha}{2\left(\mu_2-\mu_1\right)^{\alpha}}\Big((\lambda-\mu_1)^{\alpha-1}+(\mu_2-\lambda)^{\alpha-1}\Big) \leq 0.$$

Since $H''(\lambda) \leq 0$ implies that $H'(\lambda)$ is decreasing and

$$H(\frac{\mu_1+\mu_2}{2}) = \frac{\mu_2-\mu_1}{2}\left(-\frac{1}{2}+\frac{1}{\alpha+1}(1-\frac{1}{2^{\alpha}})\right)$$

as $\frac{1}{\alpha+1}(1-\frac{1}{2^{\alpha}}) \leq \frac{1}{2}$ for $\alpha \geq 1$, we have

$$H(\frac{\mu_1+\mu_2}{2}) \leq 0.$$

Therefore $H(\lambda) \leq 0$. Hence in both the cases $H(\lambda) \leq 0$, also $\Psi'' \geq 0$. Therefore from (2.13), we have

$$\Psi(i_1)+\Psi(i_2) - \frac{\Gamma(\alpha+1)}{2\left(\mu_2-\mu_1\right)^{\alpha}}\left(J^{\alpha}_{\mu_1^-}\Psi(\mu_2)+J^{\alpha}_{\mu_2^-}\Psi(\mu_1)\right) \leq \Psi(i_1)+\Psi(i_2)-\Psi(\frac{\mu_1+\mu_2}{2}). \tag{2.22}$$

Combining (2.11) and (2.22), we obtain (2.8). □

3. Conclusion

The inequalities involve fractional integral operators gain attention in recent years. Some mathematicians have worked out on Hermite-Hadamard-Mercer type inequalities involving integral operators. In this study, we gave an identity containing Riemann-Liouville fractional integral operators and Green function. From this identity and apply some conditions, we proved Hermite-Hadamard-Mercer type inequality. We hope that many inequalities can be proved by using the idea proposed in this article and open path for further research.

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