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SOME NEW BOUNDS FOR NIELSON'S BETA FUNCTION

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ABSTRACT. In this paper, we introduce some inequalities for the Nielson's Beta function $\beta(y)$ and its derivatives. To achieve this, we study the completely monotonicity of some functions containing $\beta(y)$ and the Psi function $\psi(y)$ and we study their completely monotonic degrees. Also, we present some sharp bounds for the function $\beta(y)$. These new bounds are superior to some recent results.

1. INTRODUCTION

In both theory and practice, inequalities play a central role in nearly every area of mathematics. Although widely used, the field saw a major shift in the 1930s with the publication of the first systematic study on the topic [5]. This pivotal work organized a previously scattered collection of inequalities into a unified field of study. Important contributions followed in [3] and [12]. Over the past few decades, inequality theory has evolved into a vibrant and independent research area, leading to the creation of several journals dedicated specifically to inequalities and their applications. This paper focuses on inequalities involving special functions, with particular attention given to the Nielsen Beta function.

The Psi function is given by [1]:

$$\psi(y) = \int_0^\infty \left(\frac{1/t}{e^t} - \frac{e^{(1-y)t}}{e^t - 1} \right) dt, \qquad y \in (0, \infty) \tag{1.1}$$

and satisfies the following relation [1]:

$$\psi^{(s)}(1+y) = \frac{(-1)^s \, s!}{y^{1+s}} + \psi^{(s)}(y), \qquad s = 0, 1, 2, \cdots . \tag{1.2}$$

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The function $\psi(y)$ has the following asymptotic expansion:

$$\psi(y) \sim -1/2y + \ln y - \sum_{r=1}^{\infty} \frac{B_{2r}}{2r\ y^{2r}}, \qquad y \to \infty \tag{1.3}$$

where $B_i{}'s$ are the Bernoulli numbers. The Polygamma functions $\psi^{(s)}(y)$ are given by [1]:

$$\psi^{(s)}(y) = \int_0^{\infty} \frac{(-1)^{s-1}\ t^s\ e^{(-y)t}}{1-e^{-t}} dt, \qquad s = 1, 2, \cdots, \quad y > 0.$$

The Nielsen's Beta function is given by [20]:

$$\beta(y) = \frac{1}{2}\Big(\psi\left((1+y)/2\right) - \psi\left(y/2\right)\Big), \qquad y \neq 0, -1, -2, \cdots$$

and has the integral representation [20]:

$$\beta(y) = \int_0^{\infty} \frac{e^{(1-y)t}}{e^t+1} dt, \qquad y \in (0, \infty) \tag{1.4}$$

with the functional relations [20]:

$$\beta(1+y) = 1/y - \beta(y), \tag{1.5}$$

$$\beta(1-y) = \frac{\pi}{\sin(\pi y)} - \beta(y). \tag{1.6}$$

In particular, $\beta(\frac{1}{2}) = \frac{\pi}{2}$. And

$$\beta(y) - \beta(y+2) = \frac{1}{y(y+1)} \tag{1.7}$$

The function $\beta(y)$ is useful in evaluating some hypergeometric functions [7]:

$${}_2F_1(1, y; y+1; -1) = y\ \beta(y), \qquad y \neq 0, -1, -2, \cdots$$

and in evaluating some alternating series [21]:

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i\ a + c} = \frac{1}{a}\beta\left(\frac{c}{a}\right), \qquad c \neq 0, -a, -2a, \cdots.$$

In 2016, Mahmoud and Agarwal [8] deduced the following asymptotic expansion for Bateman's $G-$function [2], which is $G(y) = 2\beta(y)$:

$$2\beta(y) \sim \frac{1}{y} + \sum_{s=1}^{\infty} \frac{(4^s-1)B_{2s}}{s\ y^{2s}}, \qquad y \to \infty \tag{1.8}$$

and the following inequality:

$$\frac{1}{2y^2+1.5} < 2\beta(y) - \frac{1}{y} < \frac{1}{2y^2}, \qquad y \in (0, \infty). \tag{1.9}$$

In 2017, Mahmoud et al. [9] introduced the next inequality for $\beta(y)$:

$$\ln\left(\frac{\frac{4}{e^2-4}+y+1}{\frac{4}{e^2-4}+y}\right) + \frac{2/y}{y+1} < 2\beta(y) < \ln\left(\frac{y+2}{y+1}\right) + \frac{2/y}{y+1}, \qquad y > 0. \tag{1.10}$$

In 2018, Nantomah [16] deduced some useful relations for $\beta'(y)$ such as:

$$\beta'(\frac{1}{2}) = \sum_{k=0}^{\infty} \frac{4}{(4k+3)^2} - \sum_{k=0}^{\infty} \frac{4}{(4k+1)^2} = -4G,$$

where the Catalan's constant $G = 0.915965594177\cdots$, (see [4]). In 2019, Hegazi et al. [6] refined the inequality (1.10) by:

$$\ln\left(\frac{y+\sqrt{2/3}+1}{y+\sqrt{2/3}}\right)+\frac{\sqrt{2/3}+1}{y(y+1)} < 2\beta(y) < \ln\left(\frac{y-\sqrt{2/3}+1}{y-\sqrt{2/3}}\right)+\frac{1-\sqrt{2/3}}{y(y+1)}. \tag{1.11}$$

Nantomah [17] presented the following inequalities:

$$\frac{(y+1)^{-2}}{4} < \beta(y)-\frac{1}{2y} < \frac{y^{-2}}{2}, \qquad y>0 \tag{1.12}$$

and

$$-1 < y^2\,\beta'(y) < \frac{-1}{2}, \qquad y\in(0,\infty). \tag{1.13}$$

After that, Nantomah [18] deduced the next inequality for $\beta(y)$:

$$-\ln(2) < \beta(y)-\frac{1}{y} < 0, \qquad y>0 \tag{1.14}$$

In 2021, Mahmoud et al. [10] refined (1.9) by:

$$1/(2y^2+1) < 2\beta(y)-\frac{1}{y} < \frac{1}{2y^2}, \qquad y\in(0,\infty). \tag{1.15}$$

In 2025, Moustafa and Al Sayed [14] presented for

$$\rho=\sqrt{15+2\sqrt{57}}, \quad \sigma=\frac{6-31\rho+\rho^3}{12}\simeq 0.0884 \text{ and } \xi=\frac{(\sqrt{57}-7)\rho}{6}\simeq 0.503 \tag{1.16}$$

the following inequality:

$$-2\rho\Big(\psi(y-\xi)-\ln(y+\sigma-1)\Big) < 2\beta(y) < 2\rho\Big(\psi(y+\xi)-\ln(y-\sigma)\Big), \tag{1.17}$$

which refines the inequality (1.11) and they also, deduced

$$2\rho\Big(\psi'(y+\xi)-\frac{1}{y-\sigma}\Big) < 2\beta'(y) < -2\rho\Big(\psi'(y-\xi)-\frac{1}{y+\sigma-1}\Big). \tag{1.18}$$

For more information about Nielsen's Beta function, see [19] and [15].

An infinitely differentiable function $T(y)$ on $0<y<\infty$, is completely monotonic (CM) if

$$(-1)^i T^{(i)}(y)\geq 0, \qquad y>0;\ i\geq 0.$$

The necessary and sufficient condition [23] for the function $T(y)$ to be CM on $(0,\infty)$ is the convergence of the following integral:

$$T(y)=\int_0^\infty e^{-yt}dv(t), \qquad 0<y<\infty$$

where $v(t)$ is non-negative measure on $[0,\infty)$. Suppose $T(y)$ is a CM function for $y>0$ and consider that $T(\infty)=\lim_{y\to\infty}T(y)$. If $y^\epsilon[T(y)-T(\infty)]$ is a CM function for $y>0$ if and only if $\epsilon\in[0,\eta]$, then the number $\eta\in\mathbb{R}^+$ is called the CM degree of $T(y)$ for $y>0$ and denoted by $deg^y_{CM}[T(y)]=\eta$. See [11] and [13], for more detailed information on this topic.

The outline of this paper is as follows:
In Section 1, we define Nielsen's Beta function along with some of its known inequalities.

In Section 2, we present some preliminary functions and investigate their monotonicity. These functions were used to prove that our results improve upon some recent findings.
In Section 3, we investigate the complete monotonicity (CM) of a function containing $\beta(y)$ and $\psi(y)$, and we derive several inequalities for $\beta(y)$ and its derivatives that refine recently published results.
In Section 4, we demonstrate that the function

$$\chi(y) = e^{\psi\left(y+\frac{1}{2}\right)-\frac{1}{6}\beta(y)+\frac{1}{12y}} - y, \quad y \geq \frac{1}{2}$$

is strictly decreasing and convex with the sharp bounds $0 < \chi(y) < e^{\frac{1}{6}-\gamma-\frac{\pi}{12}} - \frac{1}{2}$. As a result, we deduce the following sharp bounds for $\beta(y)$:

$$\frac{1}{2y} - 6\left(\ln\left(y + e^{\frac{1}{6}-\gamma-\frac{\pi}{12}} - \frac{1}{2}\right) - \psi\left(y+\frac{1}{2}\right)\right) < \beta(y) < 1/(2y) - 6\left(\ln(y) - \psi\left(y+\frac{1}{2}\right)\right), \qquad y \geq \frac{1}{2}.$$

2. Auxiliary Results

The following result [22] will be useful in this paper:

Corollary 2.1. *Let $\alpha \in (0,\infty)$ and S be a real-valued function defined on $y > y_0$, $y_0 \in (-\infty,\infty)$ with $\lim_{y\to\infty} S(y) = 0$. Then $S(y) > 0$, if $S(y+\alpha) < S(y)$ for every $y \in (y_0,\infty)$ and $S(y) < 0$, if $S(y+\alpha) > S(y)$ for every $y \in (y_0,\infty)$.*

Lemma 2.1. *The following functions are negative on the indicated intervals.*

(a) *For the values of ρ, σ and ξ in (1.16), the following statement hold*

$$\Upsilon_1(y) = -6\Big(\ln(y) - \psi(y+\frac{1}{2})\Big) + \frac{1}{2y} - \rho\Big(\psi(y+\xi) - \ln(y-\sigma)\Big) < 0, \qquad y \in (\sigma,\infty).$$

(b)

$$\Upsilon_2(y) = -9\ln(y) + 6\psi(y+\frac{1}{2}) + 3\psi(y) + \frac{3}{2y} < 0, \qquad y \in (0,\infty).$$

(c)

$$\Upsilon_3(y) = -9\ln(y) + 6\psi(y+\frac{1}{2}) + 3\psi(y) + \frac{3}{2y} + \frac{1}{10y^5} < 0, \qquad y \in [8,\infty).$$

(d)

$$\Upsilon_4(y) = -6\ln(y) + 6\psi(y+\frac{1}{2}) - \frac{1}{4y^2} < 0, \qquad y \in (0,\infty).$$

(e)

$$\Upsilon_5(y) = -6\Big(\ln(y) - \psi(y+\frac{1}{2})\Big) + \frac{1}{2y} - \frac{1}{2}\ln\left(1 + \frac{1}{y-\sqrt{\frac{2}{3}}}\right) - \frac{\left(1-\sqrt{\frac{2}{3}}\right)}{2y(y+1)} < 0, \qquad y \in \left(\sqrt{\frac{2}{3}},\infty\right).$$

(f)

$$\Upsilon_6(y) = 3\Big(\ln(y) - \psi(y)\Big) - \frac{1}{y} - \frac{1}{2}\ln\left(1+\frac{1}{y+1}\right) - \frac{1}{y(y+1)} < 0, \qquad y \in [4,\infty).$$

(g)

$$\Upsilon_7(y) = 3\Big(\ln(y) - \psi(y)\Big) - \frac{3}{2y} - \frac{1}{4y^2} < 0, \qquad y \in (0,\infty).$$

(h)

$$\Upsilon_8(y) = 3\Big(\ln(y) - \psi(y)\Big) - \frac{3}{2y} - \frac{1}{2y^2} < 0, \qquad y \in (0,\infty).$$

(i)
$$\Upsilon_9(y) = 3\Big(\psi'(y) - \frac{1}{y}\Big) - \frac{2}{y^2} < 0, \qquad y \in (1, \infty)\,.$$

(j)
$$\Upsilon_{10}(y) = -6\Big(\ln(y) - \psi(y + \frac{1}{2})\Big) - \frac{1}{2y} < 0, \qquad y \in \Big[\frac{4}{5}, \infty\Big)\,.$$

(k)
$$\Upsilon_{11}(y) = 6\Big(\ln(y + e^{\frac{1}{6}-\gamma-\frac{\pi}{12}} - \frac{1}{2}) - \psi(y + \frac{1}{2})\Big) + \frac{1}{2y} - \ln(2) < 0, \qquad y \in \Big[\frac{3}{5}, \infty\Big)\,.$$

Proof. (a) The first derivative of $\Upsilon_1(y)$ is
$$\Upsilon_1'(y) = -6\Big(\frac{1}{y} - \psi'(y + \frac{1}{2})\Big) - \frac{1}{2y^2} - \rho\Big(\psi'(y + \xi) - \frac{1}{y - \sigma}\Big)$$
and we use (1.2) to get
$$\Upsilon_1'(1 + y) - \Upsilon_1'(y) = \frac{-l(y)}{2592y^2(1 + y)^2(1 + 2y)^2(y - \delta + 1)(y - \sigma)(y + c)^2},$$
where
$$\begin{aligned} l(y) \;\; = \;\; & 48(-7 + \sqrt{57}) + 24\Bigg[36\Big(- 7 + \sqrt{57}\Big) + \Big(- 39 + 5\sqrt{57}\Big)\rho\Bigg]y \\ & +144\Bigg[- 65 + 8\sqrt{57} + \Big(- 111 + 15\sqrt{57}\Big)\rho\Bigg]y^2 + 48\Bigg[- 351 + \Big(- 84 + 20\sqrt{57}\Big)\rho\Bigg]y^4 \\ & +48\Bigg[8\Big(- 61 + \sqrt{57}\Big) + \Big(- 369 + 60\sqrt{57}\Big)\rho\Bigg]y^3 > 0, \qquad y > 0. \end{aligned}$$

($l(y)$ is a polynomial of degree 4 with positive coefficients). Then $\Upsilon_1'(1 + y) < \Upsilon_1'(y)$ for $y > \sigma$ and by using (1.3), we get $\lim_{y\to\infty} \Upsilon_1'(y) = 0$. We use Corollary 2.1 to get that $\Upsilon_1'(y) > 0$ for $y > \sigma$ and then $\Upsilon_1(y)$ is increasing on (σ, ∞) with $\lim_{y\to\infty} \Upsilon_1(y) = 0$. Hence $\Upsilon_1(y) < 0$ for all $y \in (\sigma, \infty)$.
In the same way, we will prove the result for the other functions.

(b)
$$\Upsilon_2'(y + 1) - \Upsilon_2'(y) = \frac{-3}{2y^2(y + 2)^2(1 + 2y)^2} < 0, \qquad y > 0$$
with $\lim_{y\to\infty} \Upsilon_2'(y) = 0$ and then $\Upsilon_2(y) < 0$ on $(0, \infty)$.

(c) We have
$$\Upsilon_3'(y + 1) - \Upsilon_3'(y) = \frac{-L(y - 8)}{2y^6(1 + y)^6(1 + 2y)^2} < 0, \qquad y \geq 8$$
where
$$L(y) = 2794735 + 12259142y + 8960533y^2 + 3018136y^3 + 576248y^4 + 66586y^5 + 4638y^6 + 180y^7 + 3y^8$$
with $\lim_{y\to\infty} \Upsilon_3'(y) = 0$ and then $\Upsilon_3(y) < 0$ on $[8, \infty)$.

(d)
$$\Upsilon_4'(y + 1) - \Upsilon_4'(y) = \frac{-(1 + 7y + 7y^2)}{2y^2(y + 2)^2(1 + 2y)^2} < 0, \qquad y > 0$$
with $\lim_{y\to\infty} \Upsilon_4'(y) = 0$ and then $\Upsilon_4(y) < 0$ on $(0, \infty)$.

(e)

$$\Upsilon_5'(y+1)-\Upsilon_5'(y)=\frac{\frac{3h(y)}{2(-6+\sqrt{6}-3y)(-3+\sqrt{6}-3y)(\sqrt{6}-3y)}}{y^2(1+y)^2(2+y)^2(1+2y)^2}<0,\quad y>\sqrt{\frac{2}{3}}$$

where

$$h(y)=-8(-8+3\sqrt{6}))+8(-34+21\sqrt{6})y+2(-619+432\sqrt{6})y^2+2(-756+635\sqrt{6})y^3$$
$$+(-723+757\sqrt{6})y^4+(-117+160\sqrt{6})y^5>0,\quad y\geq 0$$

($h(y)$ is a polynomial of degree 5 with positive coefficients) with $\lim_{y\to\infty}\Upsilon_5'(y)=0$ and then $\Upsilon_5(y)<0$ on $\left(\sqrt{\frac{2}{3}},\infty\right)$.

(f)

$$\Upsilon_6'(y+1)-\Upsilon_6'(y)=\frac{-2\Big((y-4)^2+6(y-4)+2\Big)}{y^2(1+y)^2(2+y)^2(3+y)}<0,\quad y\geq 4$$

with $\lim_{y\to\infty}\Upsilon_6'(y)=0$ and then $\Upsilon_6(y)<0$ on $[4,\infty)$

(g)

$$\Upsilon_7'(y+1)-\Upsilon_7'(y)=\frac{-1}{2y^3(y+1)^3}<0,\quad y>0$$

with $\lim_{y\to\infty}\Upsilon_7'(y)=0$ and then $\Upsilon_7(y)<0$ on $(0,\infty)$.

(h)

$$\Upsilon_8'(y+1)-\Upsilon_8'(y)=\frac{-(2+3y+3y^2)}{2y^3(y+1)^3}<0,\quad y>0$$

with $\lim_{y\to\infty}\Upsilon_8'(y)=0$ and then $\Upsilon_8(y)<0$ on $(0,\infty)$.

(i)

$$\Upsilon_9(y+1)-\Upsilon_9(y)=\frac{y-1}{y^2(y+1)^2}>0,\quad y>1$$

with $\lim_{y\to\infty}\Upsilon_9'(y)=0$ and then $\Upsilon_9(y)<0$ for $y>1$.

(j)

$$\Upsilon_{10}'(y+1)-\Upsilon_{10}'(y)=\frac{-\frac{1}{125}\Big(37+1170(y-\frac{4}{5})+2400(y-\frac{4}{5})^2+1000(y-\frac{4}{5})^3\Big)}{2y^2(y+2)^2(1+2y)^2}<0$$

with $\lim_{y\to\infty}\Upsilon_{10}'(y)=0$ and then $\Upsilon_{10}(y)<0$ on $\left[\frac{4}{5},\infty\right)$.

(k)

$$\Upsilon_{11}'(y+1)-\Upsilon_{11}'(y)=\frac{\frac{1}{3125}H\Big(y-\frac{3}{5}\Big)}{2y^2(1+y)^2(1+2y)^2\Big(2e^{\frac{1}{6}}-e^{\gamma+\frac{\pi}{12}}+2e^{\gamma+\frac{\pi}{12}}\,y\Big)\Big(2e^{\frac{1}{6}}+e^{\gamma+\frac{\pi}{12}}+2e^{\gamma+\frac{\pi}{12}}\,y\Big)}$$

where

$$\begin{aligned}H(y) &= 686060e^{\frac{1}{3}}+823272e^{\frac{1}{6}+\gamma+\frac{\pi}{12}}-593615e^{\frac{\pi}{6}+2\gamma}\\
&\quad+\Big(2897400e^{\frac{1}{3}}+4849000e^{\frac{1}{6}+\gamma+\frac{\pi}{12}}-3141150e^{\frac{\pi}{6}+2\gamma}\Big)y\\
&\quad+\Big(4386000e^{\frac{1}{3}}+11058000e^{\frac{1}{6}+\gamma+\frac{\pi}{12}}-6391000e^{\frac{\pi}{6}+2\gamma}\Big)y^2\\
&\quad+\Big(2740000e^{\frac{1}{3}}+12060000e^{\frac{1}{6}+\gamma+\frac{\pi}{12}}-6190000e^{\frac{\pi}{6}+2\gamma}\Big)y^3\\
&\quad+\Big(600000e^{\frac{1}{3}}+6200000e^{\frac{1}{6}+\gamma+\frac{\pi}{12}}-2850000e^{\frac{\pi}{6}+2\gamma}\Big)y^4\end{aligned}$$

$$+\Big(1200000e^{\frac{1}{6}+\gamma+\frac{\pi}{12}}-500000e^{\frac{\pi}{6}+2\gamma}\Big)y^5>0,\qquad y\geq 0$$

then $\Upsilon_{11}'(y+1)-\Upsilon_{11}'(y)>0$ for $y\geq\frac{3}{5}$ with $\lim\limits_{y\to\infty}\Upsilon_{11}'(y)=0$ and we use Corollary 2.1 to get $\Upsilon_{11}'(y)<0$ for $y\geq\frac{3}{5}$. Thus $\Upsilon_{11}(y)$ is decreasing on $y\geq\frac{3}{5}$ and then $\Upsilon_{11}(y)\leq\Upsilon_{11}(\frac{3}{5})\simeq -0.278065<0$ on $\left[\frac{3}{5},\infty\right)$.

□

3. Main Results

In the next theorem, we investigate the complete monotonicity of a function containing $\beta(y)$ and $\psi(y)$.

Theorem 3.1.

$$W_\delta(y)=-\beta(y)+3\Big(\ln(y+\delta)-\psi(y)\Big)-\frac{1}{y}$$

is CM on $(0,\infty)$ if and only if $\delta\geq 0$. Also, the function $W_0(y)$ satisfies $1\leq\deg_{CM}^{y}\left[W_0(y)\right]<2$.

Proof. Using(1.1), (1.4) and the identities $\frac{1}{y^k}=\frac{1}{(k-1)!}\int_0^\infty t^{k-1}e^{-yt}dt$, and $\ln\left(\frac{a}{b}\right)=\int_0^\infty\frac{e^{-bt}-e^{-at}}{t}dt,\ \ a,b>0$ (see [1]), we have

$$W_\delta(y)=\int_0^\infty\frac{\varphi_\delta(t)\ e^{-yt}}{t(e^t-1)(e^t+1)}dt,$$

where

$$\varphi_\delta(t)=-3\ e^{-\delta t}\Big(e^{2t}-1\Big)+t\Big[e^{2t}+4e^t+1\Big].$$

Assuming that $\delta\geq 0$, then we get

$$\varphi_\delta(t)\geq -3\Big(e^{2t}-1\Big)+t\Big[e^{2t}+4e^t+1\Big]=\frac{t^5}{30}+\sum_{r=5}^{\infty}\frac{f(r)}{(r+1)!}t^{r+1},$$

where

$$f(r)=-3(2^{r+1})+(r+1)(2^r+4)=2^r(r-5)+4(r+1)>0,\qquad r\geq 5.$$

Consequently, $W_\delta(y)$ is completely monotonic on $(0,\infty)$ for $\delta\geq 0$. On the other side, if $W_\delta(y)$ is CM, then we have

$$y\ W_\delta(y)=3y\Big(\ln(y+\delta)-\psi(y)\Big)-y\ \beta(y)-1>0\quad\ y>0.\tag{3.1}$$

Using (1.8), we have $\lim\limits_{y\to\infty}y\ \beta(y)=\frac{1}{2}$ and the asymptotic (1.3), we get $\lim\limits_{y\to\infty}y\Big[\ln(y+\delta)-\psi(y)\Big]=\delta+\frac{1}{2}$. From (3.1), we conclude that $3\ \delta\geq 0$ and then $\delta\geq 0$. And also $W_0(\infty)=\lim\limits_{y\to\infty}W_0(y)=0$.

Next,

$$y\ W_0(y)=\int_0^\infty\frac{\Delta_1(t)}{t^2(e^t-1)^2(e^t+1)^2}\ e^{-yt}dt,\qquad y\in\mathbb{R}^+$$

where

$$\begin{aligned}\Delta_1(t)&=3\Big(1-2e^{2t}+e^{4t}\Big)-4t^2\Big(e^t+e^{2t}+e^{3t}\Big)\\&=\frac{t^6}{5}+\frac{2t^7}{5}+\frac{269t^8}{630}+\frac{101t^9}{315}+\frac{2039t^{10}}{10800}+\sum_{p=9}^{\infty}\frac{b_p}{(p+2)!}t^{p+2}>0,\qquad p\geq 9\end{aligned}$$

with

$$\begin{aligned}b_p&=3\Big(4^{p+2}-2^{p+3}\Big)-4(p+2)(p+1)\Big(3^p+2^p+1\Big)\\&=3\Bigg(3^{p+2}-2^{p+3}+(p+2)3^{p+1}+(p+2)(p+1)3^p\Big(\frac{1}{2}+\frac{p}{18}+\frac{p(p-1)}{216}+\frac{p(p-1)(p-2)}{3240}\Big)\end{aligned}$$

$$+\sum_{s=6}^{p+2}\binom{p+2}{s}3^{p+2-s}\Bigg)-4(p+2)(p+1)\Big(3^p+2^p+1\Big).$$

Then

$$\begin{aligned}\frac{b_p}{3^p} &= 3\left(9-8\Big(\frac{2}{3}\Big)^p+3(p+2)+\frac{(p+2)(p+1)}{3240}\Big(1620+167p+12p^2+p^3\Big)\right.\\ &\left.+\sum_{s=6}^{p+2}\binom{p+2}{s}3^{2-s}\right)-4(p+2)(p+1)\left(1+\Big(\frac{2}{3}\Big)^p+\Big(\frac{1}{3}\Big)^p\right).\end{aligned}$$

Since the sequence $a_p = 9-8\left(\frac{2}{3}\right)^p$ is increasing for $p \geq 9$. Then $a_p \geq a_9 \simeq 8.78 > 0$ and hence

$$\begin{aligned}\frac{b_p}{3^p(p+2)(p+1)} &> \frac{1620+167p+12p^2+p^3}{1080}-4\left(\Big(\frac{2}{3}\Big)^p+\Big(\frac{1}{3}\Big)^p\right)-4\\ &> \frac{67}{15}-4\left(\Big(\frac{2}{3}\Big)^9+\Big(\frac{1}{3}\Big)^9\right)-4\simeq 0.362>0, \quad p\geq 9.\end{aligned}$$

Then, $1 \leq \deg^y_{CM}[W_0(y)]$. But,

$$y^2\, W_0(y) = \int_0^\infty \frac{2\,\Delta_2(t)}{t^3(e^t-1)^3(e^t+1)^3}\, e^{-yt}dt, \qquad y\in\mathbb{R}^+$$

where

$$\Delta_2(t) = 3\Big(1-e^{6t}\Big)+2t^3\Big(1+6e^{2t}+e^{4t}\Big)e^t+e^{2t}(-9+4t)+e^{4t}(9+4t^3)$$

with $\Delta_2(3.04)\simeq 248854.094$ and $\Delta_2(3.05)\simeq -103877$. Hence $y^2\, W_0(y)$ is not CM on $\mathbb{R}^+$ and then $\deg^l_{CM}[W_0(y)] < 2$. □

From Theorem 3.1, we obtain the next two corollaries:

Corollary 3.1. *For $y > 0$ and $\theta \geq 0$, we have*

$$\beta(y) < 3\Big(\ln(y+\theta)-\psi(y)\Big)-\frac{1}{y}, \tag{3.2}$$

where $\theta = 0$ being the best.

Proof. The inequality (3.2) is deduced from $y\, W_\theta(y) > 0$ which gives $\theta \geq 0$ as we have discussed in proving Theorem 1.1. Since $\ln(y)$ is increasing on $y > 0$, we get for $\theta \geq 0$, that $\ln(y) \leq \ln(y+\theta)$ which proves that $\theta = 0$ is the sharpest in (3.2). □

Remark 3.1. (a) Since $\Upsilon_6(y) < 0$ on $[4,\infty)$, we get that

$$3\Big(\ln(y)-\psi(y)\Big)-\frac{1}{y} < 1/2\ln\left(\frac{2+y}{1+y}\right)+\frac{1}{y(1+y)}$$

which implies that the upper of (3.2) refines the upper of (1.10) for all $y \geq 4$.

(b) Since $\Upsilon_7(y) < 0$ on $y > 0$, we get that

$$3\Big(\ln(y)-\psi(y)\Big)-\frac{1}{y} < \frac{1}{2y}+\frac{1}{4y^2}$$

which implies that the upper of (3.2) refines the upper of (1.9) for all $y > 0$.

(c) Since $\Upsilon_8(y) < 0$ on $y > 0$, we get that

$$3\Big(\ln(y) - \psi(y)\Big) - \frac{1}{y} < \frac{1}{2y} + \frac{1}{2y^2}$$

which implies that the upper of (3.2) refines the upper of (1.12) for all $y > 0$.

Corollary 3.2. *For $y > 0, \theta \geq 0$ and $s \in \mathbb{N}$, we have*

$$3\Big((-1)^s\ \psi^{(s)}(y) + \frac{(s-1)!}{(y+\theta)^s}\Big) + \frac{s!}{y^{s+1}} < (-1)^{s+1}\beta^{(s)}(y) \tag{3.3}$$

with the best constant $\theta = 0$.

Proof. The inequality (3.3) is deduced from $y^{s+1}\ (-1)^s W_\theta^{(s)}(y) > 0$ for $y > 0$. Then

$$\begin{aligned}\lim_{y\to\infty} y^{s+1}(-1)^s W_\theta^{(s)}(y) &= 3\lim_{y\to\infty} y^{s+1}\Big[(-1)^{s+1}\psi^{(s)}(y) - \frac{(s-1)!}{(y+\theta)^s}\Big] \\ &+(-1)^{s+1}\lim_{s\to\infty} y^{s+1}\beta^{(s)}(y) - s! \geq 0.\end{aligned} \tag{3.4}$$

Using the asymptotic (1.8), we have $(-1)^{s+1}\lim_{s\to\infty} y^{s+1}\beta^{(s)}(y) = \frac{-s!}{2}$ and by using the asymptotic (1.3), we have

$$\lim_{y\to\infty} y^{s+1}\Big[(-1)^{s+1}\psi^{(s)}(y) - \frac{(s-1)!}{(y+\theta)^s}\Big] = s!\ \Big(\theta + \frac{1}{2}\Big).$$

From (3), we conclude that $3\ s!\ \theta \geq 0$ and then $\theta \geq 0$. Since $\frac{1}{y^s}$ is strictly decreasing function on $(0,\infty)$ for $s = 1, 2, ...$ and then $\theta = 0$ is the sharpest in (3.3). ☐

Remark 3.2. Letting $s = 1$, and $\theta = 0$ in (3.3), we get

$$3\Big(\frac{1}{y} - \psi'(y)\Big) + \frac{1}{y^2} < \beta'(y) \tag{3.5}$$

Since $\Upsilon_9(y) < 0$ on $y > 1$, we get that

$$3\Big(1/y - \psi'(y)\Big) + 1/y^2 > \frac{-1}{y^2}$$

which implies that the lower of (3.5) refines the lower of (1.13) for all $y > 1$.

In the following Proposition, we will refine the inequality (3.2)

Proposition 3.1. *For $y \geq 11/5$, we have*

$$\beta(y) < 3\Big(\ln(y) - \psi(y)\Big) - \frac{1}{y} - \frac{1}{10y^5}, \tag{3.6}$$

Proof. Let $\Lambda(y) = \beta(y) - 3\Big(\ln(y) - \psi(y)\Big) + \frac{1}{y} + \frac{1}{10y^5}$ and then $\Lambda'(y) = \beta'(y) + 3\psi'(y) - \frac{3}{y} - \frac{1}{y^2} - \frac{1}{2y^6}$. By using the relation (1.2) and (1.7), we get

$$\Lambda'(y+2) - \Lambda'(y) = \frac{-2\ f\left(y - \frac{11}{5}\right)}{y^6(1+y)^2(2+y)^6} < 0, \quad y \geq \frac{11}{5}$$

where

$$\begin{aligned}390625f(y) &= 319140802 + 4885742080y + 9308991400y^2 + 7990092125y^3 + 3852587500y^4 \\ &+1117693750y^5 + 194625000y^6 + 18828125y^7 + 781250y^8 > 0, \quad y \geq 0\end{aligned}$$

then $\Lambda'(y+2) - \Lambda'(y) < 0$ for $y \geq \frac{11}{5}$ with $\lim_{y\to\infty}\Lambda'(y) = 0$ and we use Corollary 2.1 to get $\Lambda'(y) > 0$ for $y \geq \frac{11}{5}$. Then $\Lambda(y)$ is increasing for $y \geq \frac{11}{5}$ with $\lim_{y\to\infty}\Lambda(y) = 0$ and hence $\Lambda(y) < 0$ for $y \geq \frac{11}{5}$. ☐

Remark 3.3. The upper bound of (3.6) improves its counterparts of (3.2) for $y \geq 2.2$.

4. Some sharp bounds for the Nielson's $\beta-$function

In the following Proposition, we introduce some sharp bounds for $\beta(y)$.

Proposition 4.1. *For $y \geq \frac{1}{2}$, the function*

$$\chi(y) = e^{\psi\left(y+\frac{1}{2}\right)-\frac{1}{6}\beta(y)+\frac{1}{12y}} - y \tag{4.1}$$

is strictly decreasing and convex, and as consequence

$$\frac{1}{2y} - 6\left(\ln\left(y + e^{\frac{1}{6}-\gamma-\frac{\pi}{12}} - \frac{1}{2}\right) - \psi(y+1/2)\right) < \beta(y) < \frac{1}{2y} - 6\left(\ln(y+0) - \psi(y+1/2)\right), \quad y \geq \frac{1}{2} \tag{4.2}$$

where the constants 0 *and* $e^{\frac{1}{6}-\gamma-\frac{\pi}{12}} - \frac{1}{2} \approx 0.0105083$ *are the best possible.*

Proof.

$$\chi'(y) = \left(\psi'(y+1/2) - \frac{1}{6}\beta'(y) - \frac{1}{12y^2}\right) e^{\psi\left(y+\frac{1}{2}\right)-\frac{1}{6}\beta(y)+\frac{1}{12y}} - 1,$$

and

$$\frac{1}{e^{\psi(y+1/2)-\frac{1}{6}\beta(y)+\frac{1}{12y}}}\chi''(y) = \left(\psi'(y+\frac{1}{2}) - \frac{1}{6}\beta'(y) - \frac{1}{12y^2}\right)^2 + \psi''(y+\frac{1}{2}) - \frac{1}{6}\beta''(y) + \frac{1}{6y^3} \triangleq \Omega_1(y).$$

Now, by using (1.7), we get

$$\begin{aligned}
\Omega_1(y+2) - \Omega_1(y) &= \frac{-\psi'(y+1/2)}{3y^2(y+1)^2(2+y)^2(1+2y)^2(3+2y)^2}\Big(18 + 150y + 1451y^2 + 5216y^3 \\
&+ 9176y^4 + 9024y^5 + 5088y^6 + 1536y^7 + 192y^8\Big) \\
&+ \frac{\beta'(y)}{18y^2(y+1)^2(2+y)^2(1+2y)^2(3+2y)^2}\Big(18 + 150y + 1451y^2 + 5216y^3 \\
&+ 9176y^4 + 9024y^5 + 5088y^6 + 1536y^7 + 192y^8\Big) \\
&+ \frac{1}{36y^4(y+1)^4(2+y)^4(1+2y)^4(3+2y)^4}\Big(972 + 23976y + 320382y^2 \\
&+ 2430162y^3 + 13132626y^4 + 52868532y^5 + 156888155y^6 + 343105960y^7 \\
&+ 558868304y^8 + 686121152y^9 + 640000400y^{10} + 454398592y^{11} + 244100672y^{12} \\
&+ 97624320y^{13} + 28179456y^{14} + 5548032y^{15} + 666624y^{16} + 36864y^{17}\Big)
\end{aligned}$$

Also, let

$$\Omega_2(y) = \frac{y^2(y+1)^2(2+y)^2(1+2y)^2(3+2y)^2\Big(\Omega_1(y+2) - \Omega_1(y)\Big)}{18 + 150y + 1451y^2 + 5216y^3 + 9176y^4 + 9024y^5 + 5088y^6 + 1536y^7 + 192y^8},$$

then

$$\Omega_2(y+2) - \Omega_2(y) = \frac{\frac{A\left(y-\frac{1}{2}\right)}{C(y)\,D(y)}}{18y^2(y+1)^2(2+y)^2(3+y)^2(4+y)^2(1+2y)^2(3+2y)^2(5+2y)^2(7+2y)^2}$$

with

$$\begin{aligned}
C(y) &= 1054826 + 2945730y + 3572267y^2 + 2457888y^3 + 1049816y^4 + 285120y^5 + 48096y^6 \\
&+ 4608y^7 + 192y^8,
\end{aligned}$$

$$D(y) = 18 + 150y + 1451y^2 + 5216y^3 + 9176y^4 + 9024y^5 + 5088y^6 + 1536y^7 + 192y^8,$$

and

$$
\begin{aligned}
8\ A(y) &= 593509809240204966 + 17929659735419721030y + 194244817332074601681y^2 \\
&+ 1175888223519972411240y^3 + 4708482753046300072749y^4 \\
&+ 13612042463141357093790y^5 + 29958147851848857500986y^6 \\
&+ 51989396109982858532240y^7 + 72917363584170434755612y^8 \\
&+ 84150002031089589265200y^9 + 80978394666244288508208y^{10} \\
&+ 65629513144336823425920y^{11} + 45127855796051686180864y^{12} \\
&+ 26466688547015594384000y^{13} + 13285793637587407254400y^{14} \\
&+ 5719287968655745474560y^15 + 2112239570722290238464y^{16} \\
&+ 668522306559352565760y^{17} + 180837257603564720128y^{18} \\
&+ 41619950636315033600y^{19} + 8095855103353987072y^{20} \\
&+ 1318551079216250880y^{21} + 177482786245410816y^{22} \\
&+ 19389436809707520y^{23} + 1675217716641792y^{24} \\
&+ 110113865072640y^{25} + 5172133625856y^{26} + 154581073920y^{27} + 2208301056y^{28}.
\end{aligned}
$$

Using $A(y) > 0, C(y) > 0, D(y) > 0$ for all $y \geq 0$, then we obtain $\Omega_2(y+2) - \Omega_2(y) > 0$ for all $y \geq \frac{1}{2}$. Using the asymptotic expansions (1.3) and (1.8) and their derivatives, we have

$$
\begin{aligned}
& \lim_{y\to\infty} \Omega_2(y) \\
= & \lim_{y\to\infty} \Big[\frac{\frac{1}{18+150y+1451y^2+5216y^3+9176y^4+9024y^5+5088y^6+1536y^7+192y^8}}{36y^2(y+1)^2(2+y)^2(3+2y)^2} \\
\times & \Big(324 + 11880y + 124416y^2 + 644244y^3 + 2043301y^4 + 4367224y^5 + 6562372y^6 \\
+ & 7041384y^7 + 5395392y^8 + 2916048y^9 + 1081536y^{10} + 261120y^{11} + 36864y^{12} + 2304y^{13}\Big) \\
+ & O(y^{-4})\Big] = 0
\end{aligned}
$$

so, $\Omega_2(y) < 0$ for all $y \geq \frac{1}{2}$. Now, $\Omega_1(y+2) - \Omega_1(y) < 0$ and

$$
\lim_{y\to\infty} \Omega_1(y) = \lim_{y\to\infty} \Big[\frac{1 + 8y + 24y^2 + 128y^3 + 496y^4 + 768y^5 + 960y^6}{144y^6(1+2y)^4} + O(y^{-5})\Big] = 0
$$

so, $\Omega_1(y) > 0$ for all $y \geq \frac{1}{2}$. Then $\chi''(y) > 0$ for all $y \geq \frac{1}{2}$ and hence the function $\chi(y)$ is convex for $y \in [\frac{1}{2}, \infty)$. Also,

$$
\lim_{y\to\infty} \chi'(y) = \lim_{y\to\infty} \Big[\frac{-(1 + 10y + 48y^2 + 144y^3 + 192y^4 + 960y^5 + 2112y^6 + 1152y^7)}{576y^5(1+2y)^4} + O(y^{-3})\Big] = 0
$$

and thus $\chi'(y) < 0$ all $y \geq \frac{1}{2}$. Hence the function $\chi(y)$ is decreasing on $[\frac{1}{2}, \infty)$ with $\chi(\frac{1}{2}) = e^{\frac{1}{6}-\gamma-\frac{\pi}{12}} - \frac{1}{2}$ and

$$
\lim_{y\to\infty} \chi(y) = \lim_{y\to\infty} \left(\frac{1}{24(y+\frac{1}{2})} - \frac{(y+\frac{1}{2})}{24y^2} + \frac{1}{24(y+\frac{1}{2})^2} + \frac{1}{96y^2} + O(y^{-3})\right) = 0.
$$

Then, for all $y \geq \frac{1}{2}$, we have

$$
0 < e^{\psi\left(y+\frac{1}{2}\right)-\frac{1}{6}\beta(y)+\frac{1}{12y}} - y < e^{\frac{1}{6}-\gamma-\frac{\pi}{12}} - \frac{1}{2},
$$

where the constants 0 and $e^{\frac{1}{6}-\gamma-\frac{\pi}{12}} - \frac{1}{2}$ are the best possible. $\square$

Remark 4.1. (a) Since $\Upsilon_1(y) < 0$ on (σ, ∞), we get that

$$-6\Big(\ln(y) - \psi(y+1/2)\Big) + \frac{1}{2y} < \rho\Big(\psi(y+\xi) - \ln(y-\sigma)\Big), \qquad y > \sigma$$

which implies that the upper of (4.2) refines the upper of (1.17) for all $y \geq \frac{1}{2}$.

(b) Since $\Upsilon_2(y) < 0$ on $\left[\frac{1}{2}, \infty\right)$, we get that

$$-6\Big(\ln(y) - \psi(y+1/2)\Big) + \frac{1}{2y} < 3\Big(\ln(y) - \psi(y)\Big) - \frac{1}{y}$$

which implies that the upper of (4.2) refines the upper of (3.2) at $\theta = 0$ for all $y \geq \frac{1}{2}$.

(c) Since $\Upsilon_3(y) < 0$ on $[8, \infty)$, we get that

$$-6\Big(\ln(y) - \psi(y+1/2)\Big) + \frac{1}{2y} < 3\Big(\ln(y) - \psi(y)\Big) - \frac{1}{y} - \frac{1}{10y^5}$$

which implies that the upper of (4.2) refines the upper of (3.6) for all $y \geq 8$.

(d) Since $\Upsilon_4(y) < 0$ on $\left[\frac{1}{2}, \infty\right)$, we get that

$$-6\Big(\ln(y) - \psi(y+\frac{1}{2})\Big) + \frac{1}{2y} < \frac{1}{2y} + \frac{1}{4y^2},$$

which implies that the upper of (4.2) refines the upper of (1.15) for all $y \geq \frac{1}{2}$.

(e) Since $\Upsilon_5(y) < 0$ on $\left(\sqrt{\frac{2}{3}}, \infty\right)$, we get that

$$-6\Big(\ln(y) - \psi(y+1/2)\Big) + \frac{1}{2y} < 1/2\ln\left(\frac{y - \sqrt{\frac{2}{3}} + 1}{y - \sqrt{\frac{2}{3}}}\right) + \frac{1 - \sqrt{\frac{2}{3}}}{2y(y+1)},$$

which implies that the upper of (4.2) refines the upper of (1.11) for all $y > \sqrt{\frac{2}{3}}$.

(f) Since $\Upsilon_{10}(y) < 0$ on $\left[\frac{4}{5}, \infty\right)$, we get that

$$-6\Big(\ln(y) - \psi(y+\frac{1}{2})\Big) + \frac{1}{2y} < \frac{1}{y},$$

which implies that the upper of (4.2) refines the upper of (1.14) for all $y \geq \frac{4}{5}$.

(g) Since $\Upsilon_{11}(y) < 0$ on $\left[\frac{3}{5}, \infty\right)$, we get that

$$-6\left(\ln\left(y + e^{\frac{1}{6} - \gamma - \frac{\pi}{12}} - \frac{1}{2}\right) - \psi\left(y + \frac{1}{2}\right)\right) + \frac{1}{2y} > \frac{1}{y} - \ln(2),$$

which implies that the lower of (4.2) refines the lower of (1.14) for all $y \geq \frac{3}{5}$.

In the following Proposition, we will refine the inequality (4.2).

Proposition 4.2. *For* $y \geq \frac{9}{4}$, *we have*

$$\beta(y) < -6\Big(\ln(y) - \psi(y+\frac{1}{2})\Big) + \frac{1}{2y} - \frac{13}{160y^5} \tag{4.3}$$

Proof. Let $S(y) = \beta(y) + 6\Big(\ln(y) - \psi(y+\frac{1}{2})\Big) - \frac{1}{2y} + \frac{13}{160y^5}$ and then $S'(y) = \beta'(y) - 6\psi'(y+\frac{1}{2}) + \frac{6}{y} + \frac{1}{2y^2} - \frac{13}{32y^6}$. By using the relation (1.2) and (1.7), we get

$$S'(y+2) - S'(y) = \frac{-v\left(y-\frac{9}{4}\right)}{8y^6(y+1)^2(2+y)^6(1+2y)^2(3+2y)^2} < 0, \quad y \geq \frac{9}{4}$$

where

$$\begin{aligned}524288\, v(y) &= 700649248491 + 140729760216552y + 461141774060608y^2 + 689388303437184y^3 \\ &+617757770387200y^4 + 368300208001024y^5 + 153333113815040y^6 + 45480722759680y^7 \\ &+9610982916096y^8 + 1418145038336y^9 + 139179589632y^{10} + 8178892800y^{11} + 218103808y^{12}\end{aligned}$$

then $S'(y+2) - S'(y) < 0$ for $y \geq \frac{9}{4}$ with $\lim_{y\to\infty} S'(y) = 0$ and we use Corollary 2.1 to get $S'(y) > 0$ for $y \geq \frac{9}{4}$. Then $S(y)$ is increasing for $y \geq \frac{9}{4}$ with $\lim_{y\to\infty} S(y) = 0$ and hence $S(y) < 0$ for $y \geq \frac{9}{4}$. □

Remark 4.2. The upper of (4.3) refines the upper of (4.2) for all $y \geq \frac{9}{4}$.

In the following, we will introduce an inequality containing $\beta'(y)$ which will refine (3.5):

Proposition 4.3. *For $y \geq \frac{1}{5}$, we have*

$$6\Big(\psi'(y+1/2) - 1/y\Big) - \frac{1}{2y^2} < \beta'(y). \tag{4.4}$$

Proof. Let $F(y) = -\beta'(y) + 6\psi'\Big(y+\frac{1}{2}\Big) - \frac{6}{y} - \frac{1}{2y^2}$. By using the relation (1.2) and (1.7), we get

$$F(y+2) - F(y) = \frac{u\left(y-\frac{1}{5}\right)}{y^2(y+1)^2(2+y)^2(1+2y)^2(3+2y)^2} > 0, \quad 5y \geq 1$$

where

$$625u(y) = 4117 + 118140y + 236425y^2 + 156000y^3 + 32500y^4 > 0, \quad y \geq 0$$

then $F(y+2) - F(y) > 0$ for $y \geq \frac{1}{5}$ with $\lim_{y\to\infty} F(y) = 0$ and we use Corollary 2.1, we get $F(y) < 0$ for $y \geq \frac{1}{5}$. □

Remark 4.3. (a) Since $\Upsilon_1'(y) > 0$ on (σ, ∞), we get that

$$6\Big(\psi'(y+1/2) - 1/y\Big) - \frac{1}{2y^2} > \sigma\Big(\psi'(y+\xi) - \frac{1}{y-\sigma}\Big), \qquad y > \sigma$$

which implies that the lower of (4.4) refines the lower of (1.18) for all $y \geq \frac{1}{5}$.

(b) Using Since $\Upsilon_2'(y) > 0$ on $(0, \infty)$, we get that

$$6\Big(\psi'(y+1/2) - 1/y\Big) - \frac{1}{2y^2} > 3\Big(1/y - \psi'(y)\Big) + \frac{1}{y^2},$$

which implies that the lower of (4.4) refines the lower of (3.5) for all $y \geq \frac{1}{5}$

5. Conclusion

The primary findings of this paper are presented in Corollaries 3.1 and 3.2, and Proposition 4.1. Specifically, the author examined two approximations for Nielsen's Beta function. As a result, the new inequalities for $\beta(y)$ refine several recent results. These findings also provide sharper bounds for various alternating series, generalized hypergeometric functions, and related functions.

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