Turkish J. Ineq., 9 (2) (2025), Pages 11 –24 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

SOME BI-UNIVALENT FUNCTION SUBFAMILIES ESTABLISHED BY IMAGINARY ERROR FUNCTIONS LINKED TO BERNOULLI POLYNOMIALS

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Abstract. Using special functions, many researchers have studied several subfamilies of the bi-univalent function family. This article presents and examines two subfamilies of bi-univalent functions that are governed by Bernoulli polynomials defined by imaginary error functions in the open unit disk. We obtain limits on initial coefficients for functions in the specified subfamilies. The Fekete-Szegö problem is also addressed for the elements of the subfamilies that have been defined. We also present some new and intriguing results.

1. Introduction

Geometric Function Theory (GFT) is a fruitful branch of mathematics within complex analysis. This sub-branch has been successful in attracting the attention of researchers in recent years. Let $\mathfrak{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. The family of holomorphic functions ϕ in $\mathfrak{U}$ of the form

$$\phi(\varsigma) = \varsigma + d_2\varsigma^2 + d_3\varsigma^3 + \cdots = \varsigma + \sum_{j=2}^{\infty} d_j\varsigma^j, \quad \varsigma \in \mathfrak{U}, \tag{1.1}$$

is identified by $\mathcal{A}$. Let $\mathcal{S}$ be the subset of $\mathcal{A}$ defined by $\mathcal{S}=\{\phi \in \mathcal{A} : \phi$ is univalent in $\mathfrak{U}\}$. Bieberbach conjectured in [4] that for every function $\phi \in \mathcal{S}$, $|d_j| \leq j, j \geq 2$. Many new subfamilies of $\mathcal{S}$ were defined, and several results were established, in order to resolve the Bieberbach conjecture. Branges finally resolved this hypothesis for each $j \geq 2$ in [5] after years of research into its proof. For every function $\phi \in \mathcal{S}$ [21], Fekete-Szegö Functional (FSF) $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ is another GFT problem. Researchers have published a large number of papers on the above problem for functions that are members of subsets of $\mathcal{S}$. The bi-univalent function class σ is one of the most notable subclasses of $\mathcal{S}$. Levin introduced the concept

Key words and phrases. Holomorphic functions, Imaginary error functions, Bi-univalent functions, Bernoulli polynomials.

2010 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 05A15, 11B39.

Received: 25/04/2025 *Accepted*: 15/08/2025.

Cite this article as: S.R. Swamy, P. Kumar, N.M. Geethalakshmi, Some bi-univalent function subfamilies established by imaginary error functions linked to Bernoulli polynomials, Turkish Journal of Inequalities, 9(2) (2025), 11-24.

of σ of bi-univalent functions in [29] and is defined by σ=$\{\phi \in \mathcal{A} : \phi$ and $\phi^{-1} = \psi$ are both univalent in $\mathfrak{U}\}$. The well-known Koebe theorem (see[19]) says that the inverse of $\phi \in \mathcal{S}$ of the form (1.1) is given by

$$\phi^{-1}(w) = w - d_2w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2d_3 + d_4)w^4 + \cdots = \psi(w) \tag{1.2}$$

obeying $\varsigma = \psi(\phi(\varsigma))$, $\varsigma \in \mathfrak{U}$, and $w = \phi(\psi(w))$, $|w| < r_0(\phi)$, $1/4 \leq r_0(\phi)$, $w \in \mathfrak{U}$. Since the functions $\frac{1}{2}\log\left(\frac{1+\varsigma}{1-\varsigma}\right)$, $\frac{\varsigma}{1-\varsigma}$, and $-\log(1-\varsigma)$ are members of the σ family, the class σ is not a null set. $\frac{2\varsigma-\varsigma^2}{2}$, $\frac{\varsigma}{1-\varsigma^2}$, and the Koebe function $\frac{\varsigma}{(1-\varsigma)^2}$, despite being in $\mathcal{S}$, are not elements of σ. For a brief analysis and to learn about some of the traits of the σ family, see [6,7,32,43]. Research on the family of bi-univalent functions have recently gained momentum thanks to Srivastava and his co-authors for an article[38]. Since this article revived the topic, numerous researchers have looked into a number of fascinating special σ families; see [11, 15, 16, 22, 23, 27, 44] as well as the citation provided in these articles.

Certain polynomials such as Bernoulli, Fibonacci, Gegenbauer, Faber, Horadam, Lucas-Balancing, Lucas-Lehmer, (m, n)-Lucas, and their extensions are essential in a wide range of disciplines, including combinatorics, computer science, engineering, number theory, numerical analysis, and physics. Because of their extensive use in the applied sciences, some extensions of these polynomials have been described in the literature. Recently, researchers have concentrated on a particular class of polynomials known as Bernoulli polynomials. The ability of this family of polynomials to capture complex behavior within a finite set of terms makes them an intriguing basis set for function approximation, especially when working with fractional derivatives, where the derivative is taken to a non-integer power. By applying the concepts of integration and differentiation to non-integer orders, the area fractional calculus makes it possible to model systems with memory effects or anomalous diffusion more precisely. The Bernoulli polynomials have been applied in new ways to numerically resolve Lane-Emden type fractional-order differential equations, a new approximation method based on orthonormal polynomials has been created in [36], whereas, equations involving Fredholm fractional integro-differentials and right-sided Caputo derivatives with multi-fractional orders are numerically resolved using Bernoulli polynomials in Loh and Phang [30].

The Bernoulli polynomials $B_j(x), x \in \mathbb{R}$, and j is non-negative integer, are frequently specified (see, [31]) using the generating function:

$$\mathfrak{B}(x,\varsigma) = \frac{\varsigma e^{x\varsigma}}{e^{\varsigma} - 1} = B_0(x) + B_1(x)\varsigma + B_2(x)\frac{\varsigma^2}{2!} + \cdots .|\varsigma| < 2|\pi. \tag{1.3}$$

With the following recursion, the Bernoulli polynomials can be easily calculated:

$$\sum_{n=0}^{j-1} \binom{j}{n} B_n(x) = jx^{j-1}, \quad j = 2, 3, 4, \cdots,$$

with the initial condition $B_0(x) = 1$. The following are the first few Bernoulli polynomials:

$$B_1(x) = \frac{2x-1}{2}, B_2(x) = x^2 - x + \frac{1}{6}, \cdots . \tag{1.4}$$

Several scientific domains, such as statistics, probability, computer science, and numerous engineering issues, depend on the error function. Mathematicians have therefore thought about it a great deal. The error function was the subject of several important inequalities and related subjects; for instances, see [12, 13, 20]. The error function

$$erf(\varsigma) = \frac{2}{\sqrt{\pi}} \int_0^{\varsigma} e^{-y^2} dy = \frac{2}{\sqrt{\pi}} \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa} \varsigma^{2\kappa+1}}{(2\kappa+1)\kappa!}, \ \varsigma \in \mathbb{C}. \tag{1.5}$$

and its approximations are commonly used to forecast events that have a high or low probability of occurrin. Using (1.5), Ramachandran et al. [35] examined the normalized regular error function given by

$$nerf(\varsigma) = \frac{\sqrt{\pi\varsigma}}{2} erf(\sqrt{\varsigma}) = \varsigma + \sum_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1} \varsigma^{\kappa}}{(2\kappa-1)(\kappa-1)!}, \ \varsigma \in \mathbb{C}. \tag{1.6}$$

By writing the integrand e^{-y^2} as a Maclaurins series and integrating term by term, the imaginary error function, represented by ierf, where "i" stands for the imaginary unit. This allows for the inclusion of oscillatory components in solutions, which can be crucial when modeling wave-like phenomena. As explained in ([1, 14]), the Maclaurins series can be obtained as indicated below.

$$ierf(\varsigma) = \frac{2}{\sqrt{\pi}} \int_0^{\varsigma} e^{-y^2} dy = \frac{2}{\sqrt{\pi}} \sum_{\kappa=0}^{\infty} \frac{\varsigma^{2\kappa+1}}{(2\kappa+1)\kappa!}, \ \varsigma \in \mathbb{C}. \tag{1.7}$$

Using (1.7), the imaginary error function in the normalized form is represented by nier and is is defined by

$$nierf(\varsigma) = \frac{\sqrt{\pi\varsigma}}{2} ierf(\sqrt{\varsigma}) = \varsigma + \sum_{\kappa=2}^{\infty} \frac{\varsigma^{\kappa}}{(2\kappa-1)(\kappa-1)!}, \ \varsigma \in \mathbb{C}. \tag{1.8}$$

and utilizing the convolution product represented by $\star$, we define

$$I\phi(\varsigma) = (nierf \star \phi)(\varsigma) = \varsigma + \sum_{\kappa=2}^{\infty} \frac{d_{\kappa}}{(2\kappa-1)(\kappa-1)!} \varsigma^{\kappa}, \tag{1.9}$$

where $\phi \in \mathcal{S}$ is of the form (1.1).

For $\mathfrak{a}_1$, $\mathfrak{a}_2 \in \mathcal{A}$ holomorphic in $\mathfrak{U}$, $\mathfrak{a}_1$ is subordinate to $\mathfrak{a}_2$, if there is a Schwarz function $\varphi(\varsigma)$ that is holomorphic in $\mathfrak{U}$ with $\varphi(0) = 0$ and $|\varphi(\varsigma)| < 1$, such that $\mathfrak{a}_1(\varsigma) = \mathfrak{a}_2(\varphi(\varsigma))$, $\varsigma \in \mathfrak{U}$. This is symbolized as

$$\mathfrak{a}_1 \prec \mathfrak{a}_2 \ or \ \mathfrak{a}_1(\varsigma) \prec \mathfrak{a}_2(\varsigma).$$

Further, if $\mathfrak{a}_2 \in \mathcal{S}$, then

$$\mathfrak{a}_1(\varsigma) \prec \mathfrak{a}_2(\varsigma) \ \Leftrightarrow \ \mathfrak{a}_1(0) = \mathfrak{a}_2(0) \quad and \quad \mathfrak{a}_1(\mathfrak{U}) \subset \mathfrak{a}_2(\mathfrak{U}).$$

Bernoulli polynomials is a family of polynomials with special properties, GFT is a sub-branch of complex analysis that focuses on the geometric properties of regular functions, and the imaginary error function is a particular complex function. These three concepts are related but represent different ideas in mathematics. When combined, they can be used to study complex analytic functions and their geometric behavior, especially in relation to conformal mappings and univalent functions. We may direct readers to [14, 35] for certain investigations that combine the ideas of GFT and the error function. For some studies that integrate the concepts of GFT and Bernoulli polynomials, we might refer readers to [8,9,28]. In [2,3], intriguing investigations are conducted by fusing the concepts of Bernoulli polynomials, GFT, and the imaginary function.

In recent years, a large number of studies have been carried out on functions that belong to a certain σ subfamily and are governed by known polynomials. For members of σ subfamilies that are linked to special polynomials, many researchers have discovered coefficient estimates and the FSF $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$ (See [10,17,18,24–26,33,34,37,39–42,45]).

Using the ideas of Bernoulli polynomials, GFT, and the imaginary function, we identify two Bernoulli polynomials-governed subfamilies of σ: $\mathfrak{W}_\sigma(\beta,\tau,x)$and $\mathfrak{P}_\sigma(\beta,\tau,x)$ This research is inspired by the work done in[2,3].

This paper employs the function $\mathfrak{B}(x,\varsigma)$ as in(1.3), $\varsigma \in \mathfrak{U}$, and $\psi(w) = \phi^{-1}(w)$ as in (1.2), $w \in \mathfrak{U}$, unless otherwise noted.

Definition 1.1. Let $0 \le \beta \le 1$, *and* $\tau \ge 1$. If $\phi \in \sigma$ satisfies

$$\frac{\varsigma((I\phi(\varsigma))')^\tau}{(1-\beta)\varsigma + \beta(I\phi(\varsigma))} \prec \mathfrak{B}(x,\varsigma), \tag{1.10}$$

and

$$\frac{w((I\psi(w))')^\tau}{(1-\beta)w + \beta(I\psi(w))} \prec \mathfrak{B}(x,w), \tag{1.11}$$

then we say that $\phi \in \mathfrak{W}_\sigma(\beta,\tau,x)$.

Definition 1.2. Let $0 \le \beta \le 1$, *and* $\tau \ge 1$. If $\phi \in \sigma$ satisfies

$$\frac{[(\varsigma(I\phi(\varsigma))')']^\tau}{1-\beta+\beta(I\phi(\varsigma))'} \prec \mathfrak{B}(x,\varsigma), \tag{1.12}$$

and

$$\frac{[(w(I\psi(w))')']^\tau}{1-\beta+\beta(I\psi(w))'} \prec \mathfrak{B}(x,w), \tag{1.13}$$

then we say that $\phi \in \mathfrak{P}_\sigma(\beta,\tau,x)$.

The following is the structure of the article's content. For functions in the families $\mathfrak{W}_\sigma(\beta,\tau,x)$ and $\mathfrak{P}_\sigma(\beta,\tau,x)$, the estimates for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ are found in Section 2. In Section 3, we highlight relevant instances of our primary findings, which were demonstrated in Section 2. In Section 4, we conclude the study with some observations.

2. Principal findings

For functions in the families $\mathfrak{W}_\sigma(\beta,\tau,x)$ and $\mathfrak{P}_\sigma(\beta,\tau,x)$, Section 2 starts with bounds for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$.

Theorem 2.1. *If $\phi \in \sigma$ is a member of $\mathfrak{W}_\sigma(\beta,\tau,x)$, $0 \le \beta \le 1, \tau \ge 1$, then*

$$|d_2| \le \frac{3|2x-1|\sqrt{|2x-1|}}{\sqrt{|9(\frac{1}{5}(3\tau-\beta)+\frac{2}{9}(2\tau(\tau-1)+\beta^2-2\tau\beta))(2x-1)^2 - 4(2\tau-\beta)^2(x^2-x+\frac{1}{6})|}}, \tag{2.1}$$

$$|d_3| \leq \frac{9(2x-1)^2}{4(2\tau-\beta)^2} + \frac{5|2x-1|}{3\tau-\beta}, \tag{2.2}$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{3\tau-\beta} & ; |1-\xi| \leq \Upsilon \\ \frac{9|2x-1|^3\,|1-\xi|}{|9(\frac{1}{5}(3\tau-\beta)+\frac{2}{9}(2\tau(\tau-1)+\beta^2-2\tau\beta))(2x-1)^2-4(2\tau-\beta)^2(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \Upsilon, \end{cases} \tag{2.3}$$

where

$$\Upsilon = \frac{5}{9}\left|\frac{9(\frac{1}{5}(3\tau-\beta)+\frac{2}{9}(2\tau(\tau-1)+\beta^2-2\tau\beta))(2x-1)^2-4(2\tau-\beta)^2(x^2-x+\frac{1}{6})}{(3\tau-\beta)(2x-1)^2}\right|. \tag{2.4}$$

Proof. Let $\phi \in \mathfrak{W}_\sigma(\beta,\tau,x)$ Then, from subordinations (1.10) and (1.11), we can write

$$\frac{\varsigma((I\phi(\varsigma))')^\tau}{(1-\beta)\varsigma+\beta(I\phi(\varsigma))} = \mathfrak{B}(x,\mathfrak{l}(\varsigma)), \tag{2.5}$$

and

$$\frac{w((I\psi(w))')^\tau}{(1-\beta)w+\beta(I\psi(w))} = \mathfrak{B}(x,\mathfrak{m}(w)), \tag{2.6}$$

where Schwarz functions $\mathfrak{l}(\varsigma) = \mathfrak{l}_1\varsigma + \mathfrak{l}_2\varsigma^2 + \cdots$ and $\mathfrak{m}(w) = \mathfrak{m}_1 w + \mathfrak{m}_2 w^2 + \cdots$ satisfy

$$|\mathfrak{l}_j| \leq 1, \; \textit{and} \; |\mathfrak{m}_j| \leq 1 \,(j \in \mathbb{N}). \tag{2.7}$$

(See[19]). The following are the representation of equations (2.5) and (2.6) using some fundamental mathematical methods:

$$\begin{aligned}\frac{\varsigma((I\phi(\varsigma))')^\tau}{(1-\beta)\varsigma+\beta(I\phi(\varsigma))} &= 1 + \frac{1}{3}(2\tau-\beta)d_2\varsigma \\ &+ \left(\frac{1}{10}(3\tau-\beta)d_3 + \frac{1}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))d_2^2\right)\varsigma^2 + \cdots,\end{aligned} \tag{2.8}$$

$$\mathfrak{B}(x,\mathfrak{l}(\varsigma)) = 1 + B_1(x)\mathfrak{l}_1\varsigma + \left(B_1(x)\mathfrak{l}_2 + \frac{B_2(x)}{2!}\mathfrak{l}_1^2\right)\varsigma^2 + \cdots, \tag{2.9}$$

and

$$\begin{aligned}\frac{w((I\psi(w))')^\tau}{(1-\beta)w+\beta(I\psi(w))} &= 1 - \frac{1}{3}(2\tau-\beta)d_2 w + \\ &\left(\frac{1}{10}(3\tau-\beta)(2d_2^2-d_3) + \frac{1}{9}(2\tau(\tau-1)-\varrho(2\tau-\beta))d_2^2\right)w^2 + \cdots,\end{aligned} \tag{2.10}$$

$$\mathfrak{B}(x,\mathfrak{m}(w)) = 1 + B_1(x)\mathfrak{m}_1 w + \left(B_1(x)\mathfrak{m}_2 + \frac{B_2(x)}{2!}\mathfrak{m}_1^2\right)w^2 + \cdots. \tag{2.11}$$

Due to (2.5), we compare terms of the same degree in (2.8) and (2.9) and arrive at the following relations:

$$\frac{1}{3}(2\tau-\beta)d_2 = B_1(x)\mathfrak{l}_1, \tag{2.12}$$

and

$$\frac{1}{10}(3\tau-\beta)d_3 + \frac{1}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))d_2^2 = B_1(x)\mathfrak{l}_2 + \frac{B_2(x)}{2!}\mathfrak{l}_1^2. \tag{2.13}$$

Similar to this, we compare terms of the same degree in (2.10) and (2.11) due to equality (2.6), and arrive at the following expressions:

$$-\frac{1}{3}(2\tau-\beta)d_2 = B_1(x)\mathfrak{m}_1, \tag{2.14}$$

and

$$\frac{1}{10}(3\tau-\beta)(2d_2^2-d_3)+\frac{1}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))d_2^2 = B_1(x)\mathfrak{m}_2+\frac{B_2(x)}{2!}\mathfrak{m}_1^2. \tag{2.15}$$

From equations (2.25) and (2.14), we get

$$\mathfrak{l}_1 = -\mathfrak{m}_1, \tag{2.16}$$

and

$$\frac{2}{9}(2\tau-\beta)^2 d_2^2 = (\mathfrak{l}_1^2+\mathfrak{m}_1^2)B_1^2(x). \tag{2.17}$$

Addition of equations (2.13) and (2.15) yields

$$\left(\frac{1}{5}(3\tau-\beta)+\frac{2}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))\right)d_2^2 = B_1(x)(\mathfrak{l}_2+\mathfrak{m}_2)+\frac{B_2(x)}{2}(\mathfrak{l}_1^2+\mathfrak{m}_1^2). \tag{2.18}$$

Replacing $\mathfrak{l}_1^2+\mathfrak{m}_1^2$ from equation (2.17) into equation (2.18), we get:

$$d_2^2 = \frac{9B_1^3(x)(\mathfrak{l}_2+\mathfrak{m}_2)}{9\left(\frac{1}{5}(3\tau-\beta)+\frac{2}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))\right)B_1^2(x)-(2\tau-\beta)^2B_2(x)}. \tag{2.19}$$

The ineaquality (2.1) is obtained by using equation (1.4) for $B_1(x), B_2(x)$ and applying equation (2.7) to $\mathfrak{l}_2, \mathfrak{m}_2$.

The bound on $|d_3|$ is obtained by subtracting (2.15) from (2.13):

$$d_3 = d_2^2+\frac{5B_1(x)(\mathfrak{l}_2-\mathfrak{m}_2)}{3\tau-\beta}. \tag{2.20}$$

When d_2^2 is substituted from equation (2.17) into equation (2.20), we obtain

$$d_3 = \frac{9B_1^2(x)(\mathfrak{l}_1^2+\mathfrak{m}_2^2)}{2(2\tau-\beta)^2}+\frac{5B_1(x)(\mathfrak{l}_2-\mathfrak{m}_2)}{3\tau-\beta}. \tag{2.21}$$

Using (1.4) and (2.7), we derive (2.2) from (2.21). Lastly, we use the value of d_2^2 from (2.19) in (2.20) to compute the bound on $|d_3-\xi d_2^2|$. As a result, we have

$$|d_3-\xi d_2^2| = |B_1(x)|\left|\left(\frac{5}{3\tau-\beta}+\mathcal{V}_2(\xi,x)\right)\mathfrak{l}_2-\left(\frac{5}{3\tau-\beta}-\mathcal{V}_2(\xi,x)\right)\mathfrak{m}_2\right|,$$

where

$$\mathcal{V}_2(\xi,x) = \frac{9(1-\xi)B_1^2(x)}{9\left(\frac{1}{5}(3\tau-\beta)+\frac{2}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))\right)B_1^2(x)-(2\tau-\beta)^2B_2(x)}.$$

Clearly

$$|d_3-\xi d_2^2| \le \begin{cases}\frac{10|B_1(x)|}{3\tau-\beta} & ; \ |\mathcal{V}_2(\xi,x)| \le \frac{5}{3\tau-\beta}\\ 2|B_1(x)||\mathcal{V}_2(\xi,x)| & ; \ |\mathcal{V}_2(\xi,x)| \ge \frac{5}{3\tau-\beta}.\end{cases} \tag{2.22}$$

From (2.22), we derive (2.3), where Υ is identical to that in (2.4). □

By taking $\xi = 1$ in the above theorem, we get the below inequality.

Corollary 2.1. *If $\phi \in \sigma$ is a member of $\mathfrak{W}_\sigma(\beta, \tau, x)$, then* $|d_3 - d_2^2| \leq \frac{5|2x-1|}{3\tau-\beta}$.

Theorem 2.2. *If $\phi \in \sigma$ is a member of $\mathfrak{P}_\sigma(\beta, \tau, x)$, $0 \leq \beta \leq 1, \tau \geq 1$, then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{|2x-1|}}{\sqrt{|9(\frac{3}{5}(3\tau-\beta) + \frac{8}{9}(2\tau(\tau-1) - \beta(2\tau-\beta)))(2x-1)^2 - 16(2\tau-\beta)^2(x^2 - x + \frac{1}{6})|}}, \tag{2.23}$$

$$|d_3| \leq \frac{9(2x-1)^2}{16(2\tau-\beta)^2} + \frac{5|2x-1|}{3(3\tau-\beta)}, \tag{2.24}$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{3(3\tau-\beta)} & ; |1-\xi| \leq \daleth \\ \frac{9|2x-1|^3\,|1-\xi|}{|9(\frac{3}{5}(3\tau-\beta)+\frac{8}{9}(2\tau(\tau-1)-\beta(2\tau-\beta)))(2x-1)^2-16(2\tau-\beta)^2(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \daleth, \end{cases} \tag{2.25}$$

where

$$\daleth = \frac{5}{27}\left|\frac{9(\frac{3}{5}(3\tau-\beta) + \frac{8}{9}(2\tau(\tau-1) - \beta(2\tau-\beta)))(2x-1)^2 - 16(2\tau-\beta)^2(x^2 - x + \frac{1}{6})}{(3\tau-\beta)(2x-1)^2}\right|. \tag{2.26}$$

Proof. Let $\phi \in \mathfrak{W}_\sigma(\beta, \tau, x)$. Then, from subordinations (1.10) and (1.11), we can write

$$\frac{[(\varsigma(I\phi(\varsigma))')']^\tau}{1-\beta+\beta(I\phi(\varsigma))'} = \mathfrak{B}(x, \mathfrak{l}(\varsigma)), \tag{2.27}$$

and

$$\frac{[(w(I\psi(w))')']^\tau}{1-\beta+\beta(I\psi(w))'} = \mathfrak{B}(x, \mathfrak{m}(w)), \tag{2.28}$$

where Schwarz functions $\mathfrak{l}(\varsigma) = \sum_{j=1}^{\infty} \mathfrak{l}_j \varsigma^j$, *and* $\mathfrak{m}(w) = \sum_{j=1}^{\infty} \mathfrak{m}_j w^j$, $\varsigma, w \in \mathfrak{U}$ satisfy the property (2.7) (See[19]). Equation (2.27) can be written as follows by employing a few basic mathematical techniques:

$$\frac{[(\varsigma(I\phi(\varsigma))')']^\tau}{1-\beta+\beta(I\phi(\varsigma))'} = 1 + \frac{2}{3}(2\tau-\beta)d_2\varsigma + \left(\frac{3}{10}(3\tau-\beta)d_3 + \frac{4}{9}(2\tau(\tau-1) - \beta(2\tau-\beta))d_2^2\right)\varsigma^2 + \cdots, \tag{2.29}$$

$$\mathfrak{B}(x, \mathfrak{l}(\varsigma)) = 1 + B_1(x)\mathfrak{l}_1\varsigma + \left(B_1(x)\mathfrak{l}_2 + \frac{B_2(x)}{2!}\mathfrak{l}_1^2\right)\varsigma^2 + \cdots, \tag{2.30}$$

and equation (2.28) can be written as follows by employing a few basic mathematical techniques:

$$\frac{[(w(I\psi(w))')']^\tau}{1-\beta+\beta(I\psi(w))'} = 1 - \frac{2}{3}(2\tau-\beta)d_2w + \left(\frac{3}{10}(3\tau-\beta)(2d_2^2 - d_3) + \frac{4}{9}(2\tau(\tau-1) - \beta(2\tau-\beta))d_2^2\right)w^2 + \cdots, \tag{2.31}$$

$$\mathfrak{B}(x, \mathfrak{m}(w)) = 1 + B_1(x)\mathfrak{m}_1 w + \left(B_1(x)\mathfrak{m}_2 + \frac{B_2(x)}{2!}\mathfrak{m}_1^2\right)w^2 + \cdots. \tag{2.32}$$

Due to equation (2.27), we arrive at the following result by comparing the terms in equations (2.29) and (2.30):

$$\frac{2}{3}(2\tau-\beta)d_2 = B_1(x)\mathfrak{l}_1, \tag{2.33}$$

and

$$\frac{3}{10}(3\tau-\beta)d_3 + \frac{4}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))d_2^2 = B_1(x)\mathfrak{l}_2 + \frac{B_2(x)}{2!}\mathfrak{l}_1^2. \tag{2.34}$$

Similarly, due to equation (2.28), we arrive at the following result by comparing the terms in equations (2.31) and (2.32):

$$-\frac{2}{3}(2\tau-\beta)d_2 = B_1(x)\mathfrak{m}_1, \tag{2.35}$$

and

$$\frac{3}{10}(3\tau-\beta)(2d_2^2-d_3) + \frac{4}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))d_2^2 = B_1(x)\mathfrak{m}_2 + \frac{B_2(x)}{2!}\mathfrak{m}_1^2. \tag{2.36}$$

From (2.33) and (2.35), we get

$$\mathfrak{l}_1 = -\mathfrak{m}_1, \tag{2.37}$$

and

$$\frac{8}{9}(2\tau-\beta)^2 d_2^2 = (\mathfrak{l}_1^2+\mathfrak{m}_1^2)B_1^2(x). \tag{2.38}$$

Addition of equations (2.34) and (2.36) yields

$$\left(\frac{3}{5}(3\tau-\beta) + \frac{8}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))\right)d_2^2 = B_1(x)(\mathfrak{l}_2+\mathfrak{m}_2) + \frac{B_2(x)}{2}(\mathfrak{l}_1^2+\mathfrak{m}_1^2). \tag{2.39}$$

Replacing $\mathfrak{l}_1^2+\mathfrak{m}_1^2$ from equation (2.38) into equation (2.39), we get:

$$d_2^2 = \frac{9B_1^3(x)(\mathfrak{l}_2+\mathfrak{m}_2)}{9\left(\frac{3}{5}(3\tau-\beta)+\frac{8}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))\right)B_1^2(x) - 4(2\tau-\beta)^2 B_2(x)}. \tag{2.40}$$

Applying equation (2.7) to $\mathfrak{l}_2, \mathfrak{m}_2$, and using equation (1.4) for $B_1(x), B_2(x)$ yields (2.23).

The bound on $|d_3|$ is obtained by subtracting (2.36) from (2.34):

$$d_3 = d_2^2 + \frac{5B_1(x)(\mathfrak{l}_2-\mathfrak{m}_2)}{3(3\tau-\beta)}. \tag{2.41}$$

If we replace d_2^2 using equation (2.38) into equation (2.41), we obtain:

$$d_3 = \frac{9B_1^2(x)(\mathfrak{l}_1^2+\mathfrak{m}_2^2)}{8(2\tau-\beta)^2} + \frac{5B_1(x)(\mathfrak{l}_2-\mathfrak{m}_2)}{3(3\tau-\beta)}. \tag{2.42}$$

We deduce equation (2.24) from equation (2.42) by applying equations (1.4) and (2.7). Finally, we compute the bound on $|d_3 - \xi d_2^2|$ using the value of d_2^2 from (2.40) in (2.41). Consequently, we have

$$|d_3-\xi d_2^2| = |B_1(x)|\left|\left(\frac{5}{3(3\tau-\beta)} + \mathcal{V}_2(\xi,x)\right)\mathfrak{l}_2 - \left(\frac{5}{3(3\tau-\beta)} - \mathcal{V}_2(\xi,x)\right)\mathfrak{m}_2\right|,$$

where

$$\mathcal{V}_2(\xi,x) = \frac{9(1-\xi)B_1^2(x)}{9\left(\frac{3}{10}(3\tau-\beta)+\frac{4}{9}(2\tau(\tau-1)-\beta(2\tau-\beta))\right)B_1^2(x) - 4(2\tau-\varrho)^2 B_2(x)}.$$

Clearly

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{10|B_1(x)|}{3(3\tau-\beta)} & ; \ |\mathcal{V}_2(\xi,x)| \leq \frac{5}{3(3\tau-\beta)} \\ 2|B_1(x)||\mathcal{V}_2(\xi,x)| & ; \ |\mathcal{V}_2(\xi,x)| \geq \frac{5}{3(3\tau-\beta)}. \end{cases} \tag{2.43}$$

We derive (2.25) from (2.43), with Υ is the same as in (2.26). □

By taking $\xi = 1$ in the above theorem, we get the below inequality.

Corollary 2.2. *If $\phi \in \sigma$ is a member of $\mathfrak{P}_\sigma(\beta,\tau,x)$, then $|d_3 - d_2^2| \leq \frac{5|2x-1|}{3(3\tau-\beta)}$.*

3. Specific instances

We derive the following instances through the specialization of the parameters β, and τ in $\mathfrak{W}_\sigma(\beta,\tau,x)$.

Example 3.1. Letting $\tau = 1$ in $\mathfrak{W}_\sigma(\beta,\tau,x)$, we get $\mathfrak{K}_\sigma(\beta,x) \equiv \mathfrak{W}_\sigma(\beta,1,x)$ a subfamily of elements $\phi \in \sigma$ satisfying

$$\frac{\varsigma(I\phi(\varsigma))'}{(1-\beta)\varsigma + \beta I\phi(\varsigma)} \prec \mathfrak{B}(x,\varsigma), \ \textit{and} \ \frac{w(I\psi(w))'}{(1-\beta)w + \beta I\psi(w)} \prec \mathfrak{B}(x,w),$$

where $0 \leq \beta \leq 1$.

According to Theorem 2.1, the following result holds when $\tau = 1$:

Corollary 3.1. *If $\phi \in \sigma$ is an element of $\mathfrak{K}_\sigma(\beta,x)$, $0 \leq \beta \leq 1$, then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{|2x-1|}}{\sqrt{|9(\frac{1}{5}(3-\beta) - \frac{2\beta}{9}(2-\beta))(2x-1)^2 - 4(2-\beta)^2(x^2-x+\frac{1}{6})|}},$$

$$|d_3| \leq \frac{9(2x-1)^2}{4(2-\beta)^2} + \frac{5|2x-1|}{3-\beta},$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{3-\beta} & ; \ |1-\xi| \leq \Upsilon_1 \\ \frac{9|2x-1|^3 \ |1-\xi|}{|9(\frac{1}{5}(3-\beta) - \frac{2\beta}{9}(2-\beta))(2x-1)^2 - 4(2-\beta)^2(x^2-x+\frac{1}{6})|} & ; \ |1-\xi| \geq \Upsilon_1, \end{cases}$$

where

$$\Upsilon_1 = \frac{5}{9}\left|\frac{9(\frac{1}{5}(3-\beta) + \frac{2\beta}{9}(2-\beta))(2x-1)^2 - 4(2-\beta)^2(x^2-x+\frac{1}{6})}{(3-\beta)(2x-1)^2}\right|.$$

Example 3.2. Letting $\beta = 1$ in $\mathfrak{W}_\sigma(\beta,\tau,x)$, we get $\mathfrak{M}_\sigma(\tau,x) \equiv \mathfrak{W}_\sigma(1,\tau,x)$ a subclass of functions $\phi \in \sigma$ satisfying

$$\frac{\varsigma((I\phi(\varsigma))')^\tau}{I\phi(\varsigma)} \prec \mathfrak{B}(x,\varsigma), \ \textit{and} \ \frac{w((I\psi(w))')^\tau}{I\psi(w)} \prec \mathfrak{B}(x,w),$$

where $\tau \geq 1$.

The following is the outcome of Theorem 2.1 when $\beta = 1$:

Corollary 3.2. *If $\phi \in \sigma$ is an element of $\mathfrak{M}_\sigma(\tau, x)$, $\tau \geq 1$, then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{|2x-1|}}{\sqrt{|9(\frac{1}{5}(3\tau-1)+\frac{2}{9}(2\tau^2-4\tau+1)(2x-1)^2-4(2\tau-1)^2(x^2-x+\frac{1}{6})|}},$$

$$|d_3| \leq \frac{9(2x-1)^2}{4(2\tau-1)^2} + \frac{5|2x-1|}{3\tau-1},$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{3\tau-1} & ; |1-\xi| \leq \Upsilon_2 \\ \frac{9|2x-1|^3\,|1-\xi|}{|9(\frac{1}{5}(3\tau-1)+\frac{2}{9}(2\tau^2-4\tau+1))(2x-1)^2-4(2\tau-1)^2(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \Upsilon_2, \end{cases}$$

where

$$\Upsilon_2 = \frac{5}{9}\left|\frac{9(\frac{1}{5}(3\tau-1)+\frac{2}{9}(2\tau^2-4\tau+1))(2x-1)^2-4(2\tau-1)^2(x^2-x+\frac{1}{6})}{(3\tau-1)(2x-1)^2}\right|.$$

Example 3.3. Letting $\beta = 0$ in $\mathfrak{W}_\sigma(\beta, \tau, x)$, we get a subclass $\mathfrak{Y}_\sigma(\tau, x) \equiv \mathfrak{W}_\sigma(0, \tau, x)$ of functions $\phi \in \sigma$ satisfying

$$((I\phi(\varsigma))')^\tau \prec \mathfrak{B}(x, \varsigma), \ \textit{and} \ ((I\psi(w))')^\tau \prec \mathfrak{B}(x, w),$$

where $\tau \geq 1$.

The outcome of Theorem 2.1 is as follows when $\beta = 0$:

Corollary 3.3. *If $\phi \in \sigma$ is an element of $\mathfrak{Y}_\sigma(\tau, x)$, $\tau \geq 1$, then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{|2x-1|}}{\sqrt{|\tau(4\tau+\frac{7}{5})(2x-1)^2-16\tau^2(x^2-x+\frac{1}{6})|}},$$

$$|d_3| \leq \frac{9(2x-1)^2}{16\tau^2} + \frac{5|2x-1|}{3\tau},$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{3\tau} & ; |1-\xi| \leq \Upsilon_3 \\ \frac{9|2x-1|^3\,|1-\xi|}{|\tau(4\tau+\frac{7}{5})(2x-1)^2-16\tau^2(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \Upsilon_3, \end{cases}$$

where

$$\Upsilon_3 = \frac{5}{27}\left|\frac{(4\tau+\frac{7}{5})(2x-1)^2-16\tau(x^2-x+\frac{1}{6})}{(2x-1)^2}\right|.$$

We derive the following instances by specializing the parameters β and τ in $\mathfrak{P}_\sigma(\beta, \tau, x)$.

Example 3.4. Letting $\tau = 1$ in $\mathfrak{P}_\sigma(\beta, \tau, x)$, we get a family $\mathfrak{Q}_\sigma(\beta, x) \equiv \mathfrak{P}_\sigma(\beta, 1, x)$ of functions $\phi \in \sigma$ satisfying

$$\frac{(\varsigma(I\phi(\varsigma))')'}{1-\beta+\beta(I\phi(\varsigma))'} \prec \mathfrak{B}(x, \varsigma), \ \textit{and} \ \frac{(w(I\psi(w))')'}{1-\beta+\beta(I\psi(w))'} \prec \mathfrak{B}(x, w),$$

where $0 \leq \beta \leq 1$.

According to Theorem 2.2, the following result holds when $\tau = 1$:

Corollary 3.4. *If $\phi \in \sigma$ is an member of $\mathfrak{Q}_\sigma(\beta, x)$, $0 \leq \beta \leq 1$, then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{|2x-1|}}{\sqrt{|9(\frac{3}{5}(3-\beta) - \frac{8\beta}{9}(2-\beta))(2x-1)^2 - 16(2-\beta)^2(x^2-x+\frac{1}{6})|}},$$

$$|d_3| \leq \frac{9(2x-1)^2}{16(2-\beta)^2} + \frac{5|2x-1|}{3(3-\beta)},$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{3(3-\beta)} & ; |1-\xi| \leq \daleth_1 \\ \frac{9|2x-1|^3\,|1-\xi|}{|9(\frac{3}{5}(3-\beta)-\frac{8\beta}{9}(2-\beta))(2x-1)^2-16(2-\beta)^2(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \daleth_1, \end{cases}$$

where

$$\daleth_1 = \frac{5}{27}\left|\frac{9(\frac{3}{5}(3-\beta) + \frac{8\beta}{9}(2-\beta))(2x-1)^2 - 16(2-\beta)^2(x^2-x+\frac{1}{6})}{(3-\beta)(2x-1)^2}\right|.$$

Example 3.5. Let $\beta = 1$ in $\mathfrak{P}_\sigma(\beta, \tau, x)$.Then we get $\mathfrak{R}_\sigma(\tau, x) \equiv \mathfrak{P}_\sigma(1, \tau, x)$ a subclass of members $\phi \in \sigma$ satisfying

$$\frac{[(\varsigma(I\phi(\varsigma))')']^\tau}{(I\phi(\varsigma))'} \prec \mathfrak{B}(x, \varsigma), \ and \ \frac{[(w(I\psi(w))')']^\tau}{(I\psi(w))'} \prec \mathfrak{B}(x, w),$$

where $\tau \geq 1$.

The following is the outcome of Theorem 2.2 when $\beta = 1$:

Corollary 3.5. *If $\phi \in \sigma$ is an element of $\mathfrak{R}_\sigma(\tau, x)$, $\tau \geq 1$, then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{|2x-1|}}{\sqrt{|9(\frac{3}{5}(3\tau-1) + \frac{8}{9}(2\tau^2-4\tau+1)(2x-1)^2 - 16(2\tau-1)^2(x^2-x+\frac{1}{6})|}},$$

$$|d_3| \leq \frac{9(2x-1)^2}{16(2\tau-1)^2} + \frac{5|2x-1|}{3(3\tau-1)},$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{3(3\tau-1)} & ; |1-\xi| \leq \daleth_2 \\ \frac{9|2x-1|^3\,|1-\xi|}{|9(\frac{3}{5}(3\tau-1)+\frac{8}{9}(2\tau^2-4\tau+1))(2x-1)^2-16(2\tau-1)^2(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \daleth_2, \end{cases}$$

where

$$\daleth_2 = \frac{5}{27}\left|\frac{9(\frac{3}{5}(3\tau-1) + \frac{8}{9}(2\tau^2-4\tau+1))(2x-1)^2 - 16(2\tau-1)^2(x^2-x+\frac{1}{6})}{(3\tau-1)(2x-1)^2}\right|.$$

Example 3.6. Let $\beta = 0$ in $\mathfrak{P}_\sigma(\beta, \tau, x)$. Then we get $\mathfrak{S}_\sigma(\tau, x) \equiv \mathfrak{P}_\sigma(0, \tau, x)$ a subclass of elements $\phi \in \sigma$ satisfying

$$(\varsigma(I\phi(\varsigma)')')^\tau \prec \mathfrak{B}(x, \varsigma), \ and \ (w(I\psi(w)')')^\tau \prec \mathfrak{B}(x, w),$$

where $\tau \geq 1$.

The outcome of Theorem 2.2 would be as follows, if $\beta = 0$.

Corollary 3.6. *If $\phi \in \sigma$ is an element of $\mathfrak{S}_\sigma(\tau, x)$, $\tau \geq 1$, then*

$$|d_2| \leq \frac{3|2x-1|\sqrt{|2x-1|}}{\sqrt{|\tau(16\tau+\frac{1}{5})(2x-1)^2 - 64\tau^2(x^2-x+\frac{1}{6})|}},$$

$$|d_3| \leq \frac{9(2x-1)^2}{64\tau^2} + \frac{5|2x-1|}{9\tau},$$

and for $\xi \in \mathbb{R}$

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{5|2x-1|}{9\tau} & ; |1-\xi| \leq \daleth_3 \\ \frac{9|2x-1|^3\,|1-\xi|}{|\tau(16\tau+\frac{1}{5})(2x-1)^2 - 64\tau^2(x^2-x+\frac{1}{6})|} & ; |1-\xi| \geq \daleth_3, \end{cases}$$

where

$$\daleth_3 = \frac{5}{81}\left|\frac{(16\tau+\frac{1}{15})(2x-1)^2 - 64\tau(x^2-x+\frac{1}{6})}{(2x-1)^2}\right|.$$

4. Conclusion

In this presentation, we've established two subfamilies of regular and bi-univalent functions linked to Bernoulli polynomials denoted by $\mathfrak{W}_\sigma(\beta, \tau, x)$ and $\mathfrak{P}_\sigma(\beta, \tau, x)$. Maclaurin coefficients $|d_2|$ and $|d_3|$ have been estimated for functions that are members of the defined σ subfamilies. For functions in these subfamilies,We have also ascertained the FSF $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$. As discussed in Section 2, specialized parameters applied to our findings result in intested outcomes. For readers who are interested, we conclude by pointing out that the defined subfamilies can be examined for Hankel determinant problems of higher order. It is possible to introduce numerous known subfamilies of the σ family that are subordinate to Bernoulli polynomials. When it comes to functions that are part of the new subfamilies of the σ family connected to Bernoulli polynomials, the FSF $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ and the estimates of the coefficients $|d_2|$, $|d_3|$, and the Hankel determinant problems of higher order can be found.

In essence, you get a powerful mathematical tool for approximating and solving complex problems involving fractional derivatives when you combine the ideas of the imaginary error function, Bernoulli polynomials, and fractional calculus. These concepts are frequently used in modeling phenomena with memory effects or non-integer order dynamics, where the imaginary error function adds a complex component to the solution that enables more nuanced analysis of oscillatory behaviors, and the Bernoulli polynomials provides a basis for function representation.

Also, combining the imaginary error function, Bernoulli polynomials, and q-calculus entails investigating the mathematical characteristics and connections among these seemingly separate ideas. This frequently involves number theory, complex analysis, and a particular type of calculus called q-calculus, in which derivatives are defined using a "q" parameter, producing intriguing extensions of standard calculus results.

Acknowledgements. The authors express their gratitude to the reviewers of this article for their insightful feedback, which helped them refine and enhance the paper's presentation. The authors made necessary changes in response to comments of the reviewers.

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