Turkish J. Ineq., 9 (2) (2025), Pages 1 –10 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

A NOTE ON THE ANALYTIC BI-UNIVALENT FUNCTIONS ASSOCIATED WITH VAN DER POL NUMBERS

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Abstract. In this work, we introduce a new general subclass of analytic and bi-univalent functions associated with Van der Pol numbers and determine estimates for the general Taylor-Maclaurin coefficients of the functions in this class by using the Faber polynomial expansions. Also, we obtain initial coefficient bounds and investigate Fekete-Szegö problem.

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, and consider the classes $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{S}$ defined by

$$\begin{aligned} \mathcal{P} &= \{p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \ (z \in \mathbb{U})\}, \\ \mathcal{A} &= \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}, \\ \mathcal{S} &= \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}, \end{aligned}$$

respectively.

For two functions $\mathfrak{f}, \mathfrak{g} \in \mathcal{H}$, we say that the function $\mathfrak{f}$ is subordinate to $\mathfrak{g}$ in $\mathbb{U}$, and write

$$\mathfrak{f}(z) \prec \mathfrak{g}(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \ (z \in \mathbb{U})\},$$

such that

$$\mathfrak{f}(z) = \mathfrak{g}(\omega(z)) \quad (z \in \mathbb{U}).$$

It is clear that a function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}). \tag{1.1}$$

Key words and phrases. Analytic function, bi-univalent function, coefficient estimates, Faber polynomials, Van der Pol numbers.

2010 *Mathematics Subject Classification.* Primary: 30C45.

Received: 17/04/2025 *Accepted*: 26/07/2025.

Cite this article as: S. Bulut, A note on the analytic bi-univalent functions associated with Van der Pol numbers, Turkish Journal of Inequalities, 9(2) (2025), 1-10.

By the Koebe One-Quarter Theorem [10], every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}) \qquad \text{and} \qquad f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r^*; \quad r^* \geq \frac{1}{4}\right).$$

For the inverse function $g := f^{-1}$, we obtain

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &= w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{1.2}$$

If the functions f and f^{-1} are univalent in $\mathbb{U}$, then f is called bi-univalent function. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). The class Σ was first introduced and studied by Lewin [21]. For more details and examples of functions belong to the class Σ, see [26] (see also [9, 16, 25, 26, 30, 31]).

Brannan and Taha [5] and Taha [27] investigated certain subclasses of Σ and found non-sharp estimates on $|a_2|$ and $|a_3|$. But the bounds on the general coefficient $|a_n|$ for $n > 3$ are not much known. In this study, we use the Faber polynomial expansions introduced by Faber [11] to determine general coefficient bound $|a_n|$ of analytic and bi-univalent functions whose coefficients are related to Van der Pol numbers. Some works investigated the general coefficient bounds $|a_n|$ using Faber polynomial expansions can be found in [3, 4, 6, 13, 14, 18, 24, 32], (see also [7, 12, 15]).

Van der Pol [29] studied the sequence $\{\wp_n\}_{n\geq 0}$ by using the function

$$\mathfrak{P}(z) = \frac{z^3}{6\left[e^z(z-2) + (z+2)\right]} = \sum_{n=0}^{\infty} \frac{\wp_n}{n!} z^n \qquad (z \in \mathbb{U}), \tag{1.3}$$

where the numbers $\wp_n$ were later named Van der Pol numbers and utilized in the process of unsmoothing a smoothed function of three variables.

The first four values of Van der Pol numbers were calculated as (see [17])

$$\wp_0 = 1, \quad \wp_1 = -\frac{1}{2}, \quad \wp_2 = \frac{1}{5}, \quad \wp_3 = -\frac{1}{20},$$

thus, we have

$$\mathfrak{P}(z) = 1 - \frac{1}{2}z + \frac{1}{10}z^2 - \frac{1}{120}z^3 + \cdots.$$

The function $\mathfrak{P}$ given by (1.3) is analytic in $\mathbb{U}$ and it maps $\mathbb{U}$ onto a convex domain included in the right half plane (for more details, see [23], and also see [22]).

The analogous of these numbers for functions of one variable are the Bernoulli numbers. The connection between Van der Pol numbers and Rayleigh function is shown by Howard [17]. The Rayleigh function is defined in terms of the zeros of the Bessel function (for more details, see [8, 19, 20]).

Throughout this paper, " $\prec$ " denotes the subordination, and we assume that $0 \leq \lambda \leq 1$. By considering the function $\mathfrak{P}$,

$$\mathfrak{P}(z) = \frac{z^3}{6\left[e^z(z-2) + (z+2)\right]}$$

given by (1.3), we introduce a new subclass of analytic bi-univalent functions as follows.

Definition 1.1. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{V}_{\Sigma}^{\lambda}(\mathfrak{P})$ if the following conditions are satisfied:

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} \prec \mathfrak{P}(z) \tag{1.4}$$

and

$$\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda) g(w) + \lambda w g'(w)} \prec \mathfrak{P}(w) \tag{1.5}$$

where $z, w \in \mathbb{U}$ and $g = f^{-1}$ is defined by (1.2).

Remark 1.1. We obtain the following bi-univalent function classes which consists of functions $f \in \Sigma$ for the special values of λ.
(**i**) For $\lambda = 0$,

$$\mathcal{S}_{\Sigma}(\mathfrak{P}) = \left\{ f \in \Sigma : \frac{zf'(z)}{f(z)} \prec \mathfrak{P}(z) \qquad \text{and} \qquad \frac{wg'(w)}{g(w)} \prec \mathfrak{P}(w) \right\}.$$

(**ii**) For $\lambda = 1$,

$$\mathcal{K}_{\Sigma}(\mathfrak{P}) = \left\{ f \in \Sigma : 1 + \frac{zf''(z)}{f'(z)} \prec \mathfrak{P}(z) \qquad \text{and} \qquad 1 + \frac{wg''(w)}{g'(w)} \prec \mathfrak{P}(w) \right\}.$$

2. Main results

Using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse function $g = f^{-1}$ can be expressed as, [1]:

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n, \tag{2.1}$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!\,(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!\,(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!\,(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!\,(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2 \right] \\ &+ \frac{(-n)!}{(-2n+5)!\,(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4 \right] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \tag{2.2}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$, [2]. For $n = 2, 3, 4$, we obtain

$$K_1^{-2} = -2a_2, \qquad K_2^{-3} = 3\left(2a_2^2 - a_3\right), \qquad K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right), \tag{2.3}$$

respectively. In general case, for any $p \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, an expansion of K_{n-1}^{p} is given by (see [1])

$$K_{n-1}^{p} = pa_n + \frac{p(p-1)}{2} D_{n-1}^2 + \frac{p!}{(p-3)!\,3!} D_{n-1}^3 + \cdots + \frac{p!}{(p-n+1)!\,(n-1)!} D_{n-1}^{n-1}, \tag{2.4}$$

where

$$D_{n-1}^{p}=D_{n-1}^{p}\left(a_2,a_3,\ldots,a_n\right),$$

and by [28],

$$D_{n-1}^{m}\left(a_2,\ldots,a_n\right)=\sum\frac{m!}{\upsilon_1!\ldots\upsilon_{n-1}!}a_2^{\upsilon_1}\ldots a_n^{\upsilon_{n-1}}$$

and the sum is taken over all non-negative integers $i_1,\ldots,i_n$ satisfying

$$\begin{cases}\upsilon_1+\upsilon_2+\cdots+\upsilon_{n-1}=m\\ \upsilon_1+2\upsilon_2+\cdots+(n-1)\,\upsilon_{n-1}=n-1\end{cases}.$$

It is clear that

$$D_{n-1}^{n-1}\left(a_2,\ldots,a_n\right)=a_2^{n-1}.$$

Lemma 2.1. [10] *Let* $p(z)=1+c_1z+c_2z^2+\cdots\in\mathcal{P}$, *then* $|c_k|\leq 2$ *for* $k\in\mathbb{N}$.

Lemma 2.2. [33] *Let* $k,l\in\mathbb{R}$ *and* $z_1,z_2\in\mathbb{C}$. *If* $|z_1|<R$ *and* $|z_2|<R$, *then*

$$\left|(k+l)\,z_1+(k-l)\,z_2\right|\leq\begin{cases}2R\,|k| & , \quad |k|\geq|l|\\ \\ 2R\,|l| & \quad |k|\leq|l|\end{cases}.$$

Theorem 2.1. *Let the function* $f\in\mathcal{V}_{\Sigma}^{\lambda}(\mathfrak{P})$ *be given by* (1.1). *If* $a_k=0\ \ (2\leq k\leq n-1)$, *then*

$$|a_n|\leq\frac{1}{2\,(n-1)\,\left[1+(n-1)\,\lambda\right]}\qquad(n\geq 3).$$

Proof. Let $f\in\mathcal{V}_{\Sigma}^{\lambda}(\mathfrak{P})$ be of the form (1.1). Then, we have

$$\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)\,f(z)+\lambda zf'(z)}=1+\sum_{n=2}^{\infty}\left\{(n-1)\,\left[1+(n-1)\,\lambda\right]\,a_n+\sum_{j=1}^{n-2}F_j\,a_{n-j}\right\}z^{n-1},\tag{2.5}$$

where

$$F_j=(n-j-1)\,\left[1+(n-j-1)\,\lambda\right]\,K_j^{-1}\left((1+\lambda)\,a_2,\ldots,(1+j\lambda)\,a_{j+1}\right).$$

Similarly, for the inverse map $g=f^{-1}$ given by (1.2), we get

$$\frac{wg'(w)+\lambda w^2g''(w)}{(1-\lambda)\,g(w)+\lambda wg'(w)}=1+\sum_{n=2}^{\infty}\left\{(n-1)\,\left[1+(n-1)\,\lambda\right]\,A_n+\sum_{j=1}^{n-2}G_j\,A_{n-j}\right\}w^{n-1},\tag{2.6}$$

where

$$G_j=(n-j-1)\,\left[1+(n-j-1)\,\lambda\right]\,K_j^{-1}\left((1+\lambda)\,A_2,\ldots,(1+j\lambda)\,A_{j+1}\right).$$

On the other hand, since $f\in\mathcal{V}_{\Sigma}^{\lambda}(\mathfrak{P})$, by the subordination principle, there exist the Schwarz's function $\varkappa(z)$,

$$\varkappa\in\mathcal{H}:\ \varkappa(0)=0\quad\text{and}\quad|\varkappa(z)|<1\ \ (z\in\mathbb{U})$$

such that

$$\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)\,f(z)+\lambda zf'(z)}=\mathfrak{P}(\varkappa(z))\qquad(z\in\mathbb{U}).\tag{2.7}$$

Now for $p \in \mathcal{P}$ of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

we can write

$$\varkappa(z) = \frac{p(z) - 1}{p(z) + 1}. \tag{2.8}$$

Solving $\varkappa(z)$ in terms of $p(z)$ in (2.8), we obtain

$$\begin{aligned} \varkappa(z) &= \frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 \\ &\quad + \frac{1}{2}\left(c_4 - c_1 c_3 + \frac{3}{4}c_1^2 c_2 - \frac{1}{2}c_2^2 - \frac{1}{8}c_1^4\right) z^4 + \cdots. \end{aligned} \tag{2.9}$$

Using (2.9) in (1.3), we find

$$\mathfrak{P}(\varkappa(z)) = 1 - \frac{1}{4}c_1 z + \left(-\frac{1}{4}c_2 + \frac{3}{20}c_1^2\right) z^2 + \left(-\frac{1}{4}c_3 + \frac{3}{10}c_1 c_2 - \frac{17}{192}c_1^3\right) z^3 + \cdots. \tag{2.10}$$

Define

$$\varkappa(z) := \sum_{n=1}^{\infty} \varphi_n z^n.$$

Thus, we can write

$$\mathfrak{P}(\varkappa(z)) = 1 + \sum_{n=1}^{\infty} \sum_{r=1}^{n} \frac{\wp_r}{r!} D_n^r(\varphi_1, \varphi_2, \ldots, \varphi_n) z^n. \tag{2.11}$$

Similarly, since $g \in \mathcal{V}_\Sigma^\lambda(\mathfrak{P})$, there exist the Schwarz's function $\tau(w)$,

$$\tau \in \mathcal{H} : \ \tau(0) = 0 \quad \text{and} \quad |\tau(w)| < 1 \ (w \in \mathbb{U})$$

such that

$$\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda) g(w) + \lambda w g'(w)} = \mathfrak{P}(\tau(w)) \qquad (w \in \mathbb{U}). \tag{2.12}$$

Now for $q \in \mathcal{P}$ of the form

$$q(w) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \cdots,$$

we can write

$$\tau(w) = \frac{q(w) - 1}{q(w) + 1} = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \cdots. \tag{2.13}$$

Solving $\tau(w)$ in terms of $q(w)$ in (2.13), we obtain

$$\begin{aligned} \tau(w) &= \frac{1}{2}d_1 w + \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right) w^2 + \frac{1}{2}\left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right) w^3 \\ &\quad + \frac{1}{2}\left(d_4 - d_1 d_3 + \frac{3}{4}d_1^2 d_2 - \frac{1}{2}d_2^2 - \frac{1}{8}d_1^4\right) w^4 + \cdots. \end{aligned} \tag{2.14}$$

Using (2.14) in (1.3), we find

$$\mathfrak{P}(\tau(w)) = 1 - \frac{1}{4}d_1 w + \left(-\frac{1}{4}d_2 + \frac{3}{20}d_1^2\right) w^2 + \left(-\frac{1}{4}d_3 + \frac{3}{10}d_1 d_2 - \frac{17}{192}d_1^3\right) w^3 + \cdots. \tag{2.15}$$

Define

$$\tau(w) := \sum_{n=1}^{\infty} \psi_n w^n.$$

Thus, we can write

$$\mathfrak{P}(\tau(w)) = 1 + \sum_{n=1}^{\infty} \sum_{r=1}^{n} \frac{\wp_r}{r!} D_n^r(\psi_1, \psi_2, \ldots, \psi_n) w^n. \tag{2.16}$$

Considering (2.5) and (2.11) in equality (2.7), for any $n \geq 2$, we get

$$(n-1)\ [1+(n-1)\lambda]\ a_n + \sum_{j=1}^{n-2} F_j\, a_{n-j} = \sum_{r=1}^{n-1} \frac{\wp_r}{r!} D_{n-1}^r(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}), \tag{2.17}$$

and similarly, (2.6) and (2.16) in equality (2.12), we obtain

$$(n-1)\ [1+(n-1)\lambda]\ A_n + \sum_{j=1}^{n-2} G_j\, A_{n-j} = \sum_{r=1}^{n-1} \frac{\wp_r}{r!} D_{n-1}^r(\psi_1, \psi_2, \ldots, \psi_{n-1}). \tag{2.18}$$

By hypothesis, since $a_k = 0\ \ (2 \leq k \leq n-1)$, we have

$$A_n = -a_n,$$

$$\varphi_1 = \cdots = \varphi_{n-2} = 0, \qquad \varphi_{n-1} = \frac{1}{2} c_{n-1}$$

and

$$\psi_1 = \cdots = \psi_{n-2} = 0, \qquad \psi_{n-1} = \frac{1}{2} d_{n-1}.$$

So (2.17) and (2.18) imply that

$$(n-1)\ [1+(n-1)\lambda]\ a_n = \wp_1 \varphi_{n-1} \qquad \text{and} \qquad -(n-1)\ [1+(n-1)\lambda]\ a_n = \wp_1 \psi_{n-1},$$

respectively. Using the fact that $\wp_1 = -\frac{1}{2}$ and Lemma 2.1, we obtain

$$|a_n| = \frac{|\wp_1|\ |\varphi_{n-1}|}{(n-1)\ [1+(n-1)\lambda]} = \frac{|\wp_1|\ |\psi_{n-1}|}{(n-1)\ [1+(n-1)\lambda]} \leq \frac{1}{2(n-1)\ [1+(n-1)\lambda]},$$

which completes the proof of the Theorem 2.1. □

By setting $\lambda = 0$ and $\lambda = 1$ in Theorem 2.1, we obtain the following consequences, respectively.

Corollary 2.1. *Let the function $f \in \mathcal{S}_\Sigma(\mathfrak{P})$ be given by* (1.1)*. If $a_k = 0\ \ (2 \leq k \leq n-1)$, then*

$$|a_n| \leq \frac{1}{2(n-1)} \qquad (n \geq 3).$$

Corollary 2.2. *Let the function $f \in \mathcal{K}_\Sigma(\mathfrak{P})$ be given by* (1.1)*. If $a_k = 0\ \ (2 \leq k \leq n-1)$, then*

$$|a_n| \leq \frac{1}{2n(n-1)} \qquad (n \geq 3).$$

Theorem 2.2. *Let the function* $f \in \mathcal{V}_{\Sigma}^{\lambda}(\mathfrak{P})$ *be given by* (1.1). *Then*

$$|a_2| \leq \frac{1}{2(1+\lambda)} \tag{2.19}$$

and

$$|a_3| \leq \begin{cases} \frac{5}{2(7+14\lambda+17\lambda^2)} & , \quad 0 \leq \lambda \leq \frac{3+\sqrt{60}}{17} \\ \\ \frac{1}{4(1+2\lambda)} & , \quad \frac{3+\sqrt{60}}{17} \leq \lambda \leq 1 \end{cases}. \tag{2.20}$$

Proof. If we set $n=2$ and $n=3$ in (2.17) and (2.18), respectively, we get

$$(1+\lambda)\; a_2 = \wp_1\varphi_1, \tag{2.21}$$

$$2(1+2\lambda)\; a_3 - (1+\lambda)^2\; a_2^2 = \wp_1\varphi_2 + \frac{\wp_2}{2}\varphi_1^2, \tag{2.22}$$

$$-(1+\lambda)\; a_2 = \wp_1\psi_1, \tag{2.23}$$

$$\left(3+6\lambda-\lambda^2\right)\; a_2^2 - 2(1+2\lambda)\, a_3 = \wp_1\psi_2 + \frac{\wp_2}{2}\psi_1^2, \tag{2.24}$$

where

$$\begin{aligned} \varphi_1 &= \frac{1}{2}c_1, \quad \varphi_2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right), \\ \psi_1 &= \frac{1}{2}d_1, \quad \psi_2 = \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right). \end{aligned}$$

Note that

$$\varphi_1 = -\psi_1 \qquad \text{and} \qquad \varphi_1^2 + \psi_1^2 = 8(1+\lambda)^2 a_2^2$$

or equivalently

$$c_1 = -d_1 \qquad \text{and} \qquad c_1^2 + d_1^2 = 32(1+\lambda)^2 a_2^2. \tag{2.25}$$

From (2.21) and (2.23), we find

$$|a_2| = \frac{|\wp_1|\;|\varphi_1|}{1+\lambda} = \frac{|\wp_1|\;|\psi_1|}{1+\lambda} \leq \frac{1}{2(1+\lambda)}. \tag{2.26}$$

Also from (2.22) and (2.24), we obtain

$$2\left(1+2\lambda-\lambda^2\right) a_2^2 = \wp_1(\varphi_2+\psi_2) + \frac{\wp_2}{2}\left(\varphi_1^2+\psi_1^2\right)$$

or equivalently

$$2\left(1+2\lambda-\lambda^2\right) a_2^2 = -\frac{1}{4}(c_2+d_2) + \frac{3}{20}\left(c_1^2+d_1^2\right). \tag{2.27}$$

Substituting the value of $c_1^2+d_1^2$ from (2.25) in the right hand side of the above equality, we deduce that

$$a_2^2 = \frac{5(c_2+d_2)}{8(7+14\lambda+17\lambda^2)}. \tag{2.28}$$

Using the Lemma 2.1, we get

$$|a_2| \leq \sqrt{\frac{5}{2(7+14\lambda+17\lambda^2)}}, \tag{2.29}$$

and combining this with the inequality (2.26), we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (2.19).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.24) from (2.22). We thus get

$$4(1+2\lambda)\left(a_3-a_2^2\right)=\frac{\wp_1}{2}(c_2-d_2)$$

or

$$a_3=a_2^2-\frac{c_2-d_2}{16(1+2\lambda)}. \tag{2.30}$$

Upon substituting the value of a_2^2 from (2.28) into (2.30), it follows that

$$a_3=\left(\Lambda(\lambda)-\frac{1}{16(1+2\lambda)}\right)c_2+\left(\Lambda(\lambda)+\frac{1}{16(1+2\lambda)}\right)d_2,$$

where

$$\Lambda(\lambda)=\frac{5}{8(7+14\lambda+17\lambda^2)}.$$

By Lemma 2.2, we get the desired estimate on the coefficient $|a_3|$ as asserted in (2.20). $\square$

By setting $\lambda=0$ and $\lambda=1$ in Theorem 2.2, we obtain the following consequences, respectively.

Corollary 2.3. *Let the function $f\in\mathcal{S}_\Sigma(\mathfrak{P})$ be given by (1.1). Then*

$$|a_2|\leq\frac{1}{2}\qquad\textit{and}\qquad|a_3|\leq\frac{5}{14}.$$

Corollary 2.4. *Let the function $f\in\mathcal{K}_\Sigma(\mathfrak{P})$ be given by (1.1). Then*

$$|a_2|\leq\frac{1}{4}\qquad\textit{and}\qquad|a_3|\leq\frac{1}{12}.$$

Theorem 2.3. *Let the function $f\in\mathcal{V}_\Sigma^\lambda(\mathfrak{P})$ be given by (1.1). Then, for any $\delta\in\mathbb{R}$, we have*

$$\left|a_3-\delta a_2^2\right|\leq\begin{cases}\frac{1}{4(1+2\lambda)} & ,\quad \delta\in[\rho,\eta]\\[2ex] \frac{5|1-\delta|}{2(7+14\lambda+17\lambda^2)} & ,\quad \delta\in(-\infty,\rho]\cup[\eta,\infty)\end{cases},$$

where

$$\rho=\frac{3+6\lambda-17\lambda^2}{10(1+2\lambda)}\qquad\textit{and}\qquad\eta=\frac{17(1+\lambda)^2}{10(1+2\lambda)}.$$

Proof. For the function $f\in\mathcal{V}_\Sigma^\lambda(\mathfrak{P})$ of the form (1.1), from (2.28) and (2.30) we have

$$a_3-\delta a_2^2=\left(\mathfrak{h}(\delta)-\frac{1}{16(1+2\lambda)}\right)c_2+\left(\mathfrak{h}(\delta)+\frac{1}{16(1+2\lambda)}\right)d_2,$$

where

$$\mathfrak{h}(\delta)=\frac{5(1-\delta)}{8(7+14\lambda+17\lambda^2)}.$$

Then by Lemma 2.1 and Lemma 2.2, we conclude that

$$\left|a_3-\delta a_2^2\right|\leq\begin{cases}\frac{1}{4(1+2\lambda)} & ,\quad |\mathfrak{h}(\delta)|\leq\frac{1}{16(1+2\lambda)}\\[2ex] 4|\mathfrak{h}(\delta)| & ,\quad |\mathfrak{h}(\delta)|\geq\frac{1}{16(1+2\lambda)}\end{cases}.$$

$\square$

By setting $\lambda = 0$ and $\lambda = 1$ in Theorem 2.3, we obtain the following consequences, respectively.

Corollary 2.5. *Let the function $f \in \mathcal{S}_{\Sigma}(\mathfrak{P})$ be given by (1.1). Then, for any $\delta \in \mathbb{R}$, we have*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{1}{4} & , \quad \delta \in \left[\frac{3}{10}, \frac{17}{10}\right] \\ \frac{5|1-\delta|}{14} & , \quad \delta \in \left(-\infty, \frac{3}{10}\right] \cup \left[\frac{17}{10}, \infty\right) \end{cases}.$$

Corollary 2.6. *Let the function $f \in \mathcal{K}_{\Sigma}(\mathfrak{P})$ be given by (1.1). Then, for any $\delta \in \mathbb{R}$, we have*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{1}{12} & , \quad \delta \in \left[-\frac{4}{15}, \frac{34}{15}\right] \\ \frac{5|1-\delta|}{76} & , \quad \delta \in \left(-\infty, -\frac{4}{15}\right] \cup \left[\frac{34}{15}, \infty\right) \end{cases}.$$

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