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MODIFIED EXPONENTIAL TRIGONOMETRIC CONVEX FUNCTIONS AND SOME INTEGRAL INEQUALITIES

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Abstract. In this paper, a new class of convex functions is defined by utilizing previously defined convex function classes. By using this new class of functions, new integral inequalities are obtained. Hölder inequality, Hölder-İşcan inequality, power mean inequality and generalized power mean inequality are used to put forward new results.

1. Introduction

Some classes of convex functions, which served as an inspiration for proposing a new class of convex functions, have been introduced.

Definition 1.1. A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f(ta + (1-t)b) \le tf(a) + (1-t)f(b) \tag{1.1}$$

is valid for all $a, b \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics.

Let $f : I \to \mathbb{R}$ be a convex function. Then the following double inequality hold

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave.

Definition 1.2. (see [1]) Suppose that I is a subset of $\mathbb{R}$. The mapping $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponentially convex, if

$$f(ta + (1-t)b) \le te^{xa} f(a) + (1-t)e^{xb} f(b) \tag{1.3}$$

for all $a, b \in I, t \in [0,1]$ and $x \in \mathbb{R}$.

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Definition 1.3. (see [5]) A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically convex function on interval $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq \left(\frac{\sin \pi t}{2}\right) f(x) + \left(\frac{\cos \pi t}{2}\right) f(y). \tag{1.4}$$

Definition 1.4. (see [6]) Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called exponential trigonometric convex function if for every $a, b \in I$, $t \in [0, 1]$

$$f(ta + (1-t)b) \leq \frac{\sin \frac{\pi t}{2}}{e^{1-t}} f(a) + \frac{\cos \frac{\pi t}{2}}{e^{t}} f(b). \tag{1.5}$$

Definition 1.5. (see [7]) A mapping $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponentially $quasi-$convex, if

$$f(ta + (1-t)b) \leq \max\left\{e^{xa} f(a), e^{xb} f(b)\right\} \tag{1.6}$$

for all $a, b \in I$, $t \in [0, 1]$ and $x \in \mathbb{R}$.

Definition 1.6. (see [8]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called trigonometrically quasi convex functions if for every $a, b \in I$ and $t \in [0, 1]$

$$f(ta + (1-t)b) \leq \left(\sin \frac{\pi t}{2} + \cos \frac{\pi t}{2}\right) \max\{f(a), f(b)\}. \tag{1.7}$$

In [3] İşcan gave a refinement of the Hölder integral inequality as follows:

Theorem 1.1. *(see [3])Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$ then*

$$\int_a^b |f(x) g(x)| \, dx \tag{1.8}$$

$$\leq \frac{1}{b-a} \left\{ \begin{array}{c} \left(\int_a^b (b-x) |f(x)|^p \, dx\right)^{\frac{1}{p}} \left(\int_a^b (b-x) |g(x)|^q \, dx\right)^{\frac{1}{q}} \\ + \left(\int_a^b (x-a) |f(x)|^p \, dx\right)^{\frac{1}{p}} \left(\int_a^b (x-a) |g(x)|^q \, dx\right)^{\frac{1}{q}} \end{array} \right\}. \tag{1.9}$$

Hölder-İşcan integral inequality provides better approaches than Hölder integral inequality. A refinement of power-mean integral inequality as a result of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.2. *(see [4]) Let $q \geq 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|$, $|f| |g|^q$ are integrable functions on $[a, b]$ then*

$$\int_a^b |f(x) g(x)| \, dx \tag{1.10}$$

$$\leq \frac{1}{b-a} \left\{ \begin{array}{c} \left(\int_a^b (b-x) |f(x)| \, dx\right)^{1-\frac{1}{q}} \left(\int_a^b (b-x) |f(x)| |g(x)|^q \, dx\right)^{\frac{1}{q}} \\ + \left(\int_a^b (x-a) |f(x)| \, dx\right)^{1-\frac{1}{q}} \left(\int_a^b (x-a) |f(x)| |g(x)|^q \, dx\right)^{\frac{1}{q}} \end{array} \right\}.$$

Improved power-mean integral inequality provides better approaches than Power-mean integral inequality.

Lemma 1.1. *(see [2])Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{2}\int_0^1 (1-2t)\,f'(ta+(1-t)b)\,dt. \tag{1.11}$$

Throughout the paper we will denote $\mathbb{R}^+ \cup \{0\}$ by $\mathbb{R}_0^+$.

2. Main Results

Firstly we will introduce a new class of function which is a new version of exponential trigonometrically convexity as the following:

Definition 2.1. $f : I \subseteq \mathbb{R} \to \mathbb{R}_0^+$ is called modified exponential trigonometric convex function if for every $a, b \in I$ and $t \in [0,1]$ the following inequality holds

$$f(ta+(1-t)b) \le \sin\frac{\pi t}{2} e^{(1-t)} f(a) + \cos\frac{\pi t}{2} e^{t} f(b). \tag{2.1}$$

Let us demonstrate this class of function with METC(I).

Remark 2.1. Let

$$h(t) = \sin\frac{\pi t}{2} e^{(1-t)} + \cos\frac{\pi t}{2} e^{t}.$$

It is obvious that $h(t) \ge 1$ for all $t \in [0,1]$. By choosing $a = b$ in definiton of modified exponential trigonometric convex function we get

$$f(ta+(1-t)a) = f(a) \le f(a)\left(\sin\frac{\pi t}{2} e^{(1-t)} + \cos\frac{\pi t}{2} e^{t}\right)$$

that shows the function f has to be nonnegative.

Proposition 2.1. *Every nonnegative convex function defined on I is trigonometric convex function and every trigonometric convex function is modified exponential trigonometric convex function since*

$$\begin{aligned} f(ta+(1-t)b) &\le tf(a)+(1-t)f(b) \\ &\le \sin\frac{\pi t}{2} f(a) + \cos\frac{\pi t}{2} f(b) \\ &\le \sin\frac{\pi t}{2} e^{(1-t)} f(a) + \cos\frac{\pi t}{2} e^{t} f(b). \end{aligned} \tag{2.2}$$

Theorem 2.1. *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}_0^+$. If belongs to $f, g \in$METC(I),*

(i) for $c \in \mathbb{R}$ $(c \ge 0)$ cf belongs to METC(I),
(ii) $f + g$ is a modified exponential trigonometric convex function.

Proof. (i) Let $f\in$METC(I) and $c\in\mathbb{R}(c\geq 0)$, then

$$\begin{aligned}(cf)(ta+(1-t)b) &\leq c\left[\sin\frac{\pi t}{2}e^{(1-t)}f(a)+\cos\frac{\pi t}{2}e^{t}f(b)\right] \\ &= \sin\frac{\pi t}{2}e^{(1-t)}(cf)(a)+\cos\frac{\pi t}{2}e^{t}(cf)(b).\end{aligned} \tag{2.3}$$

(ii) Let f,$g\in$METC(I), then

$$\begin{aligned}(f+g)(ta+(1-t)b) &= f(ta+(1-t)b)+g(ta+(1-t)b) \\ &\leq \sin\frac{\pi t}{2}e^{(1-t)}f(a)+\cos\frac{\pi t}{2}e^{t}f(b) \\ &\quad +\sin\frac{\pi t}{2}e^{(1-t)}g(a)+\cos\frac{\pi t}{2}e^{t}g(b) \\ &= \sin\frac{\pi t}{2}e^{(1-t)}(f+g)(a)+\cos\frac{\pi t}{2}e^{t}(f+g)(b).\end{aligned}$$

□

Theorem 2.2. *If $f:I\subset\mathbb{R}\to J$ is a convex and $g:J\to\mathbb{R}_0^+$ is a modified exponential trigonometric convex function and nondecreasing, then $g\circ f:I\to\mathbb{R}_0^+$ is a modified exponential trigonometric convex function.*

Proof. For $a,b\in I$ and $t\in[0,1]$, we get

$$\begin{aligned}(g\circ f)(ta+(1-t)b) &= g(f(ta+(1-t)b)) \\ &\leq g(tf(a)+(1-t)f(b)) \\ &\leq \sin\frac{\pi t}{2}e^{(1-t)}g(f(a))+\cos\frac{\pi t}{2}e^{t}g(f(b)).\end{aligned} \tag{2.4}$$

This completes the proof of theorem. □

Theorem 2.3. *Let $f:I\subset\mathbb{R}\to\mathbb{R}_0^+$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$ and assume that $f'\in L[a,b]$. If $\left|f'\right|\in$METC(I), then the following inequaltiy holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\ & (b-a)\left[\frac{2\pi e\left(\pi^2-12\right)+4\left(-12+8\sqrt{2e}+8\sqrt{2e}\pi+\left(1-2\sqrt{2e}\right)\pi^2\right)}{\pi^4+8\pi^2+16}\right] \\ & \times A\left(\left|f'(a)\right|,\left|f'(b)\right|\right),\end{aligned} \tag{2.5}$$

where $A(.,.)$ is the arithmetic mean.

Proof. By using Lemma 1.1, properties of absolute value and modified exponential trigonometric convexity of $\left|f'\right|$ we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.6}$$

$$\begin{aligned}
&\le \frac{b-a}{2}\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|dt\\
&\le \frac{b-a}{2}\int_0^1 |1-2t|\left(\sin\frac{\pi t}{2}e^{(1-t)}\left|f'(a)\right|+\cos\frac{\pi t}{2}e^{t}\left|f'(b)\right|\right)dt\\
&= \frac{b-a}{2}\left[\left|f'(a)\right|\int_0^1 |1-2t|\left(\sin\frac{\pi t}{2}e^{(1-t)}\right)dt+\left|f'(b)\right|\int_0^1 |1-2t|\cos\frac{\pi t}{2}e^{t}dt\right]\\
&= (b-a)\left[\frac{2\pi^3e+32\sqrt{2}e^{\frac{1}{2}}-24\pi e+4\pi^2+32\sqrt{2}\pi e^{\frac{1}{2}}-8\sqrt{2}\pi^2e^{\frac{1}{2}}-48}{\pi^4+8\pi^2+16}\right]\\
&\quad\times A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
\int_0^1 |1-2t|\sin\frac{\pi t}{2}e^{(1-t)}dt &= \int_0^1 |1-2t|\cos\frac{\pi t}{2}e^{t}dt\\
&= \frac{2\pi^3e+32\sqrt{2}e^{\frac{1}{2}}-24\pi e+4\pi^2+32\sqrt{2}\pi e^{\frac{1}{2}}-8\sqrt{2}\pi^2e^{\frac{1}{2}}-48}{\left(\pi^2+4\right)^2}
\end{aligned}$$

which completes the proof. □

Theorem 2.4. *Let $f: I\subset\mathbb{R}\to\mathbb{R}_0^+$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$ and assume that $f'\in L[a,b]$. If $\left|f'\right|^q\in$METC(I) then the following inequaltiy holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.8}$$

$$\le \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}2^{\frac{1}{q}}\left(\frac{2\pi e-4}{\pi^2+4}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 1.1, properties of absolute value and Hölder inequality we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.9}$$
$$\leq \frac{b-a}{2}\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|dt$$
$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Using modified exponential trigonometric convexity of $|f'|$ we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(\sin\frac{\pi t}{2}e^{(1-t)}\left|f'(a)\right|^q+\cos\frac{\pi t}{2}e^{t}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}2^{\frac{1}{q}}\left(\frac{2\pi e-4}{\pi^2+4}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)$$

where

$$\int_0^1 |1-2t|^p\,dt=\frac{1}{p+1}$$

$$\int_0^1 \sin\frac{\pi t}{2}e^{(1-t)}dt=\int_0^1\cos\frac{\pi t}{2}e^{t}dt=\frac{2\pi e-4}{\pi^2+4}$$

This completes the proof of theorem. □

Theorem 2.5. *Let $f:I\subset\mathbb{R}\to\mathbb{R}_0^+$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$ and assume that $f'\in L[a,b]$. If $\left|f'\right|^q\in$METC(I) for $q\geq 1$, then the following inequaltiy holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.10}$$
$$\leq \frac{b-a}{2^{2-\frac{2}{q}}}\left[\frac{2\pi e\left(\pi^2-12\right)+4\left(-12+8\sqrt{2e}+8\sqrt{2e}\pi+\left(1-2\sqrt{2e}\right)\pi^2\right)}{\pi^4+8\pi^2+16}\right]^{\frac{1}{q}}$$
$$\times A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).$$

Proof. By using Lemma 1.1, properties of absolute value and power mean inequality we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.11}$$
$$\leq \frac{b-a}{2}\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|dt$$
$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Using modified exponential trigonometric convexity of $|f'|$ we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left(\sin\frac{\pi t}{2}e^{(1-t)}\left|f'(a)\right|^q+\cos\frac{\pi t}{2}e^t\left|f'(b)\right|^q\right)^{\frac{1}{q}}dt\right)^{\frac{1}{q}}$$
$$=\frac{b-a}{2^{2-\frac{2}{q}}}\left[\frac{2\pi^3e+32\sqrt{2}e^{\frac{1}{2}}-24\pi e+4\pi^2+32\sqrt{2}\pi e^{\frac{1}{2}}-8\sqrt{2}\pi^2e^{\frac{1}{2}}-48}{\pi^4+8\pi^2+16}\right]^{\frac{1}{q}}$$
$$\times A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).$$

□

Theorem 2.6. *Let the function $f: I\subset\mathbb{R}\to\mathbb{R}_0^+$ be differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, $q>1$ and assume that $f'\in L[a,b]$. If $\left|f'\right|^q \in$METC(I), then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.12}$$
$$\leq \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}2^{\frac{1}{q}}$$
$$\times\left[\left(\frac{\pi^3e-4\pi e-2\pi^2+8}{\pi^4+8\pi^2+16}\left|f'(a)\right|^q+\frac{8\pi e-16}{\pi^4+8\pi^2+16}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{8e\left(\pi-2e^{-1}\right)}{\pi^4+8\pi^2+16}\left|f'(a)\right|^q+\frac{\pi^3e-4\pi e-2\pi^2+8}{\pi^4+8\pi^2+16}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 1.1, properties of absolute value, Hölder-İşcan inequality and the inequality

$$\left|f'(ta+(1-t)b)\right|^{q} \leq \sin\frac{\pi t}{2}e^{(1-t)}\left|f'(a)\right|^{q}+\cos\frac{\pi t}{2}e^{t}\left|f'(b)\right|^{q} \tag{2.13}$$

which is the definition of the modified exponential trigonometric convex function of $|f'|^{q}$. By using (2.13) we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \qquad (2.14)\\
\leq\ &\frac{b-a}{2}\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|dt\\
\leq\ &\frac{b-a}{2}\left(\int_0^1 (1-t)|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{b-a}{2}\left(\int_0^1 t|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\
=\ &\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times\left[\left(\int_0^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 t\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|f'| \in$METC(I) we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \qquad (2.15)\\
\leq\ &\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times\left[\left(\int_0^1 (1-t)\left(\sin\frac{\pi t}{2}e^{(1-t)}\left|f'(a)\right|^q+\cos\frac{\pi t}{2}e^{t}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 t\left(\sin\frac{\pi t}{2}e^{(1-t)}\left|f'(a)\right|^q+\cos\frac{\pi t}{2}e^{t}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]\\
=\ &\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times\left[\left(2(\pi e-2)\frac{\pi^2-4}{(\pi^2+4)^2}\left|f'(a)\right|^q+16\frac{\pi e-2}{(\pi^2+4)^2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(16\frac{\pi e-2}{\left(\pi^2+4\right)^2}\left|f'(a)\right|^q+2\left(\pi e-2\right)\frac{\pi^2-4}{\left(\pi^2+4\right)^2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Bigg]$$

where

$$\begin{aligned}\int_0^1(1-t)\sin\frac{\pi t}{2}e^{(1-t)}dt &= \int_0^1 t\cos\frac{\pi t}{2}e^t dt=2\left(\pi e-2\right)\frac{\pi^2-4}{\left(\pi^2+4\right)^2}\\ \int_0^1 t\sin\frac{\pi t}{2}e^{(1-t)}dt &= \int_0^1(1-t)\cos\frac{\pi t}{2}e^t dt=16\frac{\pi e-2}{\left(\pi^2+4\right)^2}\end{aligned}$$

which completes the proof. □

Theorem 2.7. *Let the function $f: I\subset\mathbb{R}\to\mathbb{R}_0^+$ be differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, $q\geq 1$ and assume that $f'\in L[a,b]$. If $\left|f'\right|^q\in$METC(I), then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.16}$$
$$\leq \frac{b-a}{2^{3-\frac{3}{q}}}\left(\frac{e}{\left(\pi^2+4\right)^3}\right)^{\frac{1}{q}}$$
$$\times\left[\left(\begin{array}{c}\left(\begin{array}{c}160e^{-1}-96\pi^2e^{-1}+112\pi-\pi^4e^{-1}-96\sqrt{2}e^{-\frac{1}{2}}-\\ \pi^5-32\pi^3-160\sqrt{2}\pi e^{-\frac{1}{2}}+96\sqrt{2}\pi^2e^{-\frac{1}{2}}-24\sqrt{2}\pi^3e^{-\frac{1}{2}}-2\sqrt{2}\pi^4e^{-\frac{1}{2}}\end{array}\right)\left|f'(a)\right|^q\\ +2\left(\begin{array}{c}40\pi^2e^{-1}-128e^{-1}-80\pi+2\pi^4e^{-1}+80\sqrt{2}e^{-\frac{1}{2}}+\\ 12\pi^3+112\sqrt{2}\pi e^{-\frac{1}{2}}-48\sqrt{2}\pi^2e^{-\frac{1}{2}}-4\sqrt{2}\pi^3e^{-\frac{1}{2}}-\sqrt{2}\pi^4e^{-\frac{1}{2}}\end{array}\right)\left|f'(b)\right|^q\end{array}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\begin{array}{c}2\left(\begin{array}{c}40\pi^2e^{-1}-128e^{-1}-80\pi+2\pi^4e^{-1}+80\sqrt{2}e^{-\frac{1}{2}}+\\ 12\pi^3+112\sqrt{2}\pi e^{-\frac{1}{2}}-48\sqrt{2}\pi^2e^{-\frac{1}{2}}-4\sqrt{2}\pi^3e^{-\frac{1}{2}}-\sqrt{2}\pi^4e^{-\frac{1}{2}}\end{array}\right)\left|f'(a)\right|^q\\ +\left(\begin{array}{c}160e^{-1}-96\pi^2e^{-1}+112\pi-\pi^4e^{-1}-96\sqrt{2}e^{-\frac{1}{2}}-\\ \pi^5-32\pi^3-160\sqrt{2}\pi e^{-\frac{1}{2}}+96\sqrt{2}\pi^2e^{-\frac{1}{2}}-24\sqrt{2}\pi^3e^{-\frac{1}{2}}-2\sqrt{2}\pi^4e^{-\frac{1}{2}}\end{array}\right)\left|f'(b)\right|^q\end{array}\right)^{\frac{1}{q}}\right]$$

Proof. By using Lemma 1.1, properties of absolute value and improved power-mean inequality we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.17}$$
$$\leq \frac{b-a}{2}\int_0^1\left|1-2t\right|\left|f'(ta+(1-t)b)\right|dt$$
$$\leq \frac{b-a}{2}\left\{\left(\int_0^1(1-t)\left|1-2t\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left|1-2t\right|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 t\left|1-2t\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|1-2t\right|\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}$$

Since $|f'| \in$METC(I) we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{b-a}{2}\left\{\left(\int_0^1 (1-t)\left|1-2t\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^1 (1-t)\left|1-2t\right|\left(\sin\frac{\pi t}{2}e^{(1-t)}\left|f'(a)\right|^q+\cos\frac{\pi t}{2}e^{t}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1 t\left|1-2t\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|1-2t\right|\left(\sin\frac{\pi t}{2}e^{(1-t)}\left|f'(a)\right|^q+\cos\frac{\pi t}{2}e^{t}\left|f'(b)\right|^q\right)\right)^{\frac{1}{q}}\right\}$$

By simple calculation

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{2e}{(\pi^2+4)^3}\right)^{\frac{1}{q}}$$

$$\times\left[\begin{array}{c}\left(\begin{array}{c}160e^{-1}-96\pi^2e^{-1}+112\pi-\pi^4e^{-1}-96\sqrt{2}e^{-\frac{1}{2}}-\\ \pi^5-32\pi^3-160\sqrt{2}\pi e^{-\frac{1}{2}}+96\sqrt{2}\pi^2e^{-\frac{1}{2}}-24\sqrt{2}\pi^3e^{-\frac{1}{2}}-2\sqrt{2}\pi^4e^{-\frac{1}{2}}\end{array}\right)\left|f'(a)\right|^q\\ +2\left(\begin{array}{c}40\pi^2e^{-1}-128e^{-1}-80\pi+2\pi^4e^{-1}+80\sqrt{2}e^{-\frac{1}{2}}+\\ 12\pi^3+112\sqrt{2}\pi e^{-\frac{1}{2}}-48\sqrt{2}\pi^2e^{-\frac{1}{2}}-4\sqrt{2}\pi^3e^{-\frac{1}{2}}-\sqrt{2}\pi^4e^{-\frac{1}{2}}\end{array}\right)\left|f'(b)\right|^q\end{array}\right]^{\frac{1}{q}}$$

$$+\frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{2e}{(\pi^2+4)^3}\right)^{\frac{1}{q}}$$

$$\times\left[\begin{array}{c}2\left(\begin{array}{c}40\pi^2e^{-1}-128e^{-1}-80\pi+2\pi^4e^{-1}+80\sqrt{2}e^{-\frac{1}{2}}+\\ 12\pi^3+112\sqrt{2}\pi e^{-\frac{1}{2}}-48\sqrt{2}\pi^2e^{-\frac{1}{2}}-4\sqrt{2}\pi^3e^{-\frac{1}{2}}-\sqrt{2}\pi^4e^{-\frac{1}{2}}\end{array}\right)\left|f'(a)\right|^q\\ +\left(\begin{array}{c}160e^{-1}-96\pi^2e^{-1}+112\pi-\pi^4e^{-1}-96\sqrt{2}e^{-\frac{1}{2}}-\\ \pi^5-32\pi^3-160\sqrt{2}\pi e^{-\frac{1}{2}}+96\sqrt{2}\pi^2e^{-\frac{1}{2}}-24\sqrt{2}\pi^3e^{-\frac{1}{2}}-2\sqrt{2}\pi^4e^{-\frac{1}{2}}\end{array}\right)\left|f'(b)\right|^q\end{array}\right]^{\frac{1}{q}}$$

where

$$\int_0^1 (1-t)\left|1-2t\right|dt=\int_0^1 t\left|1-2t\right|dt=\frac{1}{4}$$

$$
\begin{aligned}
& \int_0^1 (1-t)\,|1-2t| \sin\frac{\pi t}{2} e^{(1-t)} dt \\
= & \int_0^1 t\,|1-2t| \cos\frac{\pi t}{2} e^{t} dt \\
= & -2\frac{e}{(\pi^2+4)^3} \\
& \times \begin{pmatrix} 96\pi^2 e^{-1} - 160 e^{-1} - 112\pi + \pi^4 e^{-1} + 96\sqrt{2} e^{-\frac{1}{2}} + \\ 32\pi^3 - \pi^5 + 160\sqrt{2}\pi e^{-\frac{1}{2}} - 96\sqrt{2}\pi^2 e^{-\frac{1}{2}} \\ -24\sqrt{2}\pi^3 e^{-\frac{1}{2}} + 2\sqrt{2}\pi^4 e^{-\frac{1}{2}} \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^1 t\,|1-2t| \sin\frac{\pi t}{2} e^{(1-t)} dt \\
= & \int_0^1 (1-t)\,|1-2t| \cos\frac{\pi t}{2} e^{t} dt \\
= & 4\frac{e}{(\pi^2+4)^3} \\
& \times \begin{pmatrix} 40\pi^2 e^{-1} - 128 e^{-1} - 80\pi + 2\pi^4 e^{-1} + 80\sqrt{2} e^{-\frac{1}{2}} + \\ 12\pi^3 + 112\sqrt{2}\pi e^{-\frac{1}{2}} - 48\sqrt{2}\pi^2 e^{-\frac{1}{2}} \\ -4\sqrt{2}\pi^3 e^{-\frac{1}{2}} - \sqrt{2}\pi^4 e^{-\frac{1}{2}} \end{pmatrix}
\end{aligned}
$$

which is the desired result. □

Remark 2.2. Since $h : [0,\infty) \to \mathbb{R}$, $h(x) = x^s$, $0 < s \leq 1$, is a concave function, for all $u, v \geq 0$ we have

$$
h\left(\frac{u+v}{2}\right) = \left(\frac{u+v}{2}\right)^s \geq \frac{h(u)+h(v)}{2} = \frac{u^s+v^s}{2}.
$$

From here, we get

$$
\begin{aligned}
& \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} 2^{\frac{1}{q}} \qquad (2.18) \\
& \times \left[\left(\frac{\pi^3 e - 4\pi e - 2\pi^2 + 8}{\pi^4 + 8\pi^2 + 16}\left|f'(a)\right|^q + \frac{8\pi e - 16}{\pi^4 + 8\pi^2 + 16}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
& \left. + \left(\frac{8e\left(\pi - 2e^{-1}\right)}{\pi^4 + 8\pi^2 + 16}\left|f'(a)\right|^q + \frac{\pi^3 e - 4\pi e - 2\pi^2 + 8}{\pi^4 + 8\pi^2 + 16}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right] \\
\leq & \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} 2\left[2\frac{\pi e - 2}{\pi^2+4}\left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right)\right]^{\frac{1}{q}}
\end{aligned}
$$

$$= \frac{b-a}{2}2^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{\pi e-2}{\pi^2+4}\right]^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).$$

This show that the inequality (2.12) gives better approximation than the inequality (2.8)

Remark 2.3. The inequality (2.16) gives better approximation than the inequality (2.10.)

Proof. We will do the proof numerically. Since

$$\int_0^1 (1-t)\left|1-2t\right|\sin\frac{\pi t}{2}e^{(1-t)}dt = \int_0^1 t\left|1-2t\right|\cos\frac{\pi t}{2}e^{t}dt \cong 0,152,$$
$$\int_0^1 t\left|1-2t\right|\sin\frac{\pi t}{2}e^{(1-t)}dt = \int_0^1 (1-t)\left|1-2t\right|\cos\frac{\pi t}{2}e^{t}dt \cong 0,263,$$

we can write the inequality (2.16) as

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.19}$$
$$\leq \frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\left(0,152\left|f'(a)\right|^q+0,263\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(0,263\left|f'(a)\right|^q+0,152\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].$$

Similary, since

$$\int_0^1 \left|1-2t\right|\sin\frac{\pi t}{2}e^{(1-t)}dt = \int_0^1 \left|1-2t\right|\cos\frac{\pi t}{2}e^{t}dt \cong 0,415$$

we can write the inequality (2.10) as

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2^{2-\frac{2}{q}}}(0,415)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right). \tag{2.20}$$

If we compare the inequalities (2.19) and (2.20) similar to the method in (2.18) we get

$$\left[\left(0,152\left|f'(a)\right|^q+0,263\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(0,263\left|f'(a)\right|^q+0,152\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]$$
$$\leq 2(0,415)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)$$

This show that the inequality (2.16) gives better approximation than the inequality (2.10). □

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