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ON GRAND CESÁRO SEQUENCE SPACES

OĞUZ OĞUR[1]

Abstract. In this paper, we introduce the grand Cesàro sequence space, inspired by [15], and characterize its fundamental properties. Furthermore, we establish inclusion relations using newly derived inequalities.

1. Introduction

Let $1 \leq t < \infty$. the space ces_t is defined as follows;

$$ces_t = \left\{ z \in w : \sum_{r=1}^{\infty} \left\{ \frac{1}{r} \sum_{s=1}^{r} |z(s)| \right\}^t < \infty \right\}, \tag{1.1}$$

where w is the space of all sequences, equipped with the norm

$$\|z\|_{ces_t} = \left[\sum_{r=1}^{\infty} \left\{ \frac{1}{r} \sum_{s=1}^{r} |z(s)| \right\}^t \right]^{\frac{1}{t}}. \tag{1.2}$$

Cesáro sequence spaces were first introduced by the Dutch Mathematical Society at the end of 1968 as the question of finding the duals of these spaces ([2]). Shiue gave a solution to this problem and also examined some properties of Cesaro sequence spaces ([18]). (For more details see [4,10,11,13,17,19,20]).

The grand spaces were firstly defined by Iwaniec and Sbordone . They gave the grand Lebesgue spaces $L^{t)}$ to benefit the solution of partial differential equations ([7]). Samko and Umarkhadzhiev worked on these spaces on sets whose measure is infinite ([16]). Later, Rafeiro et al. introduced the grand Lebesgue sequence spaces $\ell^{t),\nu} = \ell^{t),\nu}(Y)$ and examined properties of several operators ([15]). The grand Lebesgue sequence space $\ell^{t),\nu} = \ell^{t),\nu}(Y)$ is defined as follows;

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$$\|z\|_{\ell^{t),\nu}(Y)} = \sup_{\gamma>0}\left(\gamma^{\theta}\sum_{s\in Y}|z(s)|^{t(1+\gamma)}\right)^{\frac{1}{t(1+\gamma)}} = \sup_{\gamma>0}\gamma^{\frac{\nu}{t(1+\gamma)}}\|z\|_{\ell^{t(1+\gamma)}(Y)} \tag{1.3}$$

where Y is a set from the collection $\mathbb{Z}^n, \mathbb{N}_0, \mathbb{N}, \mathbb{Z}$ for $1 \leq t < \infty$, $\nu > 0$. Finally, Ogur defined grand Lorentz sequence spaces as a generalization of grand Lebesgue sequence spaces and characterized the multiplication operator defined on these spaces ([14]).

In this work, we define the grand Cesáro sequence space and study some of basic properties. Let $1 < t < \infty$ and $\nu > 0$. Then, the set $ces_{t),\nu}$ consists of all $z \in w$ such that

$$sup_{\gamma>0}\left[\gamma^{\nu}\sum_{r=1}^{\infty}\left\{\frac{1}{r}\sum_{s=1}^{r}|z(s)|\right\}^{t(1+\gamma)}\right]^{\frac{1}{t(1+\gamma)}} < \infty. \tag{1.4}$$

2. Main Results

Here, for $1 < t < \infty$ and $\nu > 0$ the space $ces_{t),\nu}$ is examined.

Theorem 2.1. *The set $ces_{t),\nu}$ is a real linear space for $1 < t < \infty$ and $\nu > 0$*

Proof. Let $z, u \in ces_{t),\nu}$ and $\lambda, \mu \in \mathbb{R}$. Then, we have

$$sup_{\gamma>0}\left[\gamma^{\nu}\sum_{r=1}^{\infty}\left\{\frac{1}{r}\sum_{s=1}^{r}|\lambda z(s)+\mu u(s)|\right\}^{t(1+\gamma)}\right]^{\frac{1}{t(1+\gamma)}}$$

$$= sup_{\gamma>0}\gamma^{\frac{\nu}{t(1+\gamma)}}\left[\sum_{r=1}^{\infty}\left\{\frac{1}{r}\sum_{s=1}^{r}|\lambda z(s)+\mu u(s)|\right\}^{t(1+\gamma)}\right]^{\frac{1}{t(1+\gamma)}}$$

$$\leq sup_{\gamma>0}\gamma^{\frac{\nu}{t(1+\gamma)}}\left[\sum_{r=1}^{\infty}\left\{\frac{1}{r}\sum_{s=1}^{r}\lambda|z(s)|+\mu|u(s)|\right\}^{t(1+\gamma)}\right]^{\frac{1}{t(1+\varepsilon)}}$$

$$\leq sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{t(1+\varepsilon)}}\left[2^{t(1+\varepsilon)-1}\sum_{r=1}^{\infty}\left\{\frac{1}{r}\sum_{s=1}^{r}\lambda|z(s)|\right\}^{t(1+\varepsilon)}+\left\{\frac{1}{r}\sum_{s=1}^{r}\mu|u(s)|\right\}^{t(1+\varepsilon)}\right]^{\frac{1}{t(1+\varepsilon)}}$$

$$= sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{t(1+\varepsilon)}}2^{\frac{t(1+\varepsilon)-1}{t(1+\varepsilon)}}\left[\sum_{r=1}^{\infty}\left\{\frac{1}{r}\sum_{s=1}^{r}\lambda|z(s)|\right\}^{t(1+\varepsilon)}+\left\{\frac{1}{r}\sum_{s=1}^{r}\mu|u(s)|\right\}^{t(1+\varepsilon)}\right]^{\frac{1}{t(1+\varepsilon)}}$$

$$\leq 2\lambda sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{t(1+\varepsilon)}}\left[\sum_{r=1}^{\infty}\left\{\frac{1}{r}\sum_{s=1}^{r}|z(s)|\right\}^{t(1+\varepsilon)}\right]^{\frac{1}{t(1+\varepsilon)}}$$

$$+ 2\mu sup_{\varepsilon>0}\left[\varepsilon^{\theta}\sum_{r=1}^{\infty}\left\{\frac{1}{r}\sum_{s=1}^{r}|u(s)|\right\}^{t(1+\varepsilon)}\right]^{\frac{1}{t(1+\varepsilon)}}$$

$$< \infty.$$

This shows that $ces_{t),\theta}$ is a real linear space. $\square$

Theorem 2.2. *Let $1 < t < \infty$ and $\theta > 0$. Then, the space $ces_{t),\theta}$ is a normed linear space with the function*

$$\|x\|_{t),\theta} = sup_{\varepsilon>0} \left[\varepsilon^{\theta} \sum_{r=1}^{\infty} \left\{ \frac{1}{r} \sum_{s=1}^{r} |z(s)| \right\}^{t(1+\varepsilon)} \right]^{\frac{1}{t(1+\varepsilon)}} = sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{t(1+\varepsilon)}} \|z\|_{t(1+\varepsilon)} . \tag{2.1}$$

Here, $\|z\|_{t(1+\varepsilon)}$ is the norm of Lebesgue sequence space.

Proof. It is enough to show the triangle inequality. Let $z, u \in ces_{t),\theta}$. Thus, we get

$$\begin{aligned} \|z+u\|_{t),\theta} &= \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{t(1+\varepsilon)}} \|z+u\|_{t(1+\varepsilon)} \\ &\leq \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{t(1+\varepsilon)}} \left(\|z\|_{t(1+\varepsilon)} + \|u\|_{t(1+\varepsilon)} \right) \\ &\leq \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{t(1+\varepsilon)}} \|z\|_{t(1+\varepsilon)} + \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{t(1+\varepsilon)}} \|u\|_{t(1+\varepsilon)} \\ &= \|z\|_{t),\theta} + \|u\|_{t),\theta} . \end{aligned}$$

$\square$

Theorem 2.3. *Let $1 < t < \infty$ and $\theta > 0$. The space $ces_{t),\theta}$ is a Banach space with its norm.*

Proof. Let $\{z_n\}_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in the space $ces_{t),\theta}$. Then, for $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$\|z_n - z_m\|_{t),\theta} = sup_{\varepsilon>0} \left[\varepsilon^{\theta} \sum_{r=1}^{\infty} \left\{ \frac{1}{r} \sum_{s=1}^{r} |z_n(s) - z_m(s)| \right\}^{t(1+\varepsilon)} \right]^{\frac{1}{t(1+\varepsilon)}} < \delta$$

whenever $n, m > N$. Thus, we have that the sequence $\{z_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $\ell^{t),\theta}$. So, we have $z \in \ell^{t),\theta}$ such that $\{z_n\}_{n\in\mathbb{N}}$ converges to z in $\ell^{t),\theta}$. Also, by the inequality

$$\|z\|_{t),\theta} \leq \|z_n - z\|_{t),\theta} + \|z_n\|_{t),\theta}$$

we get that $x \in ces_{t),\theta}$.

$\square$

Theorem 2.4. *Let $1 < t < \infty$ and $\theta > 0$. Then, ces_t is included by $ces_{t),\theta}$.*

Proof. Let $z \in ces_t$. Then, there exists $M > 0$ such that

$$\|x\|_{ces_t} = \left[\sum_{r=1}^{\infty} \left\{ \frac{1}{r} \sum_{s=1}^{r} |z(s)| \right\}^{t} \right]^{\frac{1}{t}} \leq M.$$

Since the function $\|z\|_{t(1+\varepsilon)}$ is a decreasing function for ε, we have

$$\|z\|_{t),\theta} = sup_{\varepsilon>0}\left[\varepsilon^{\theta}\sum_{r=1}^{\infty}\left\{\frac{1}{r}\sum_{s=1}^{r}|z(s)|\right\}^{t(1+\varepsilon)}\right]^{\frac{1}{t(1+\varepsilon)}}$$
$$= sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{t(1+\varepsilon)}}\|Cz\|_{t(1+\varepsilon)}$$
$$\leq sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{t(1+\varepsilon)}}\|Cz\|_{p}$$
$$\leq \varepsilon_0^{\frac{\theta}{t(1+\varepsilon_0)}}\|Cz\|_{t}$$
$$\leq M\varepsilon_0^{\frac{\theta}{t(1+\varepsilon_0)}}.$$

Here, $\varepsilon_0 = \frac{1}{W(1/e)} \simeq 3.59$ and $W:\mathbb{R}_+\to\mathbb{R}_+, W(a) = ae^a$ is Lambert function (for more details see [1]). □

Theorem 2.5. *Let* $1 < t < \infty$ *and* $\theta > 0$*. Then,* $l^{t),\theta}$ *is contained by* $ces_{t),\theta}$*.*

Proof. Let $z \in \ell^{t),\theta}$. Then, there exists $M > 0$ such that

$$\|x\|_{\ell^{t),\theta}} = sup_{\varepsilon>0}\left[\varepsilon^{\theta}\sum_{r=1}^{\infty}|z(r)|^{t(1+\varepsilon)}\right]^{\frac{1}{t(1+\varepsilon)}} = sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{t(1+\varepsilon)}}\|z\|_{t(1+\varepsilon)} \leq M.$$

Thus, by the Hardy inequality we get

$$\|z\|_{t),\theta} = sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{t(1+\varepsilon)}}\|Cz\|_{t(1+\varepsilon)} \leq sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{t(1+\varepsilon)}}\frac{t(1+\varepsilon)}{t(1+\varepsilon)-1}\|z\|_{t(1+\varepsilon)}.$$

Let define $f(x) = \frac{t(1+x)}{t(1+x)-1}$. So $f'(x) = \frac{-t}{(t(1+x)-1)^2}$ and since $1 < t < \infty$, we have $f'(x) < 0$. This shows that the function f is a decreasing. Thus, we get

$$\|z\|_{t),\theta} \leq sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{t(1+\varepsilon)}}\frac{t}{t-1}\|z\|_{t(1+\varepsilon)} = \frac{t}{t-1}\|z\|_{\ell^{t),\theta}}$$

which completes the proof. □

Theorem 2.6. *Let* $1 < t < \infty$ *and* $\theta > 0$*. If* $t < q$*, we have the inclusion* $ces_{t),\theta} \subseteq ces_{q),\theta}$

Proof. Let $z \in ces_{t),\theta}$. Thus, there exists $M > 0$ such that $\|z\|_{t),\theta} \leq M$. Since the function $\|z\|_{t(1+\varepsilon)}$ is decreasing, we get

$$\|z\|_{q(1+\varepsilon)} = sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{q(1+\varepsilon)}}\|Cz\|_{q(1+\varepsilon)}$$
$$\leq sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{q(1+\varepsilon)}}\|Cz\|_{t(1+\varepsilon)}$$
$$\leq sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{t(1+\varepsilon)}}\|Cz\|_{t(1+\varepsilon)}$$
$$\leq M$$
$$< \infty.$$

□

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[1]Department of Mathematics,
Giresun University,
Giresun, Turkey
Email address: oguz.ogur@giresun.edu.tr