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RECENT DEVELOPMENTS OF INTEGRAL INEQUALITIES OF THE HARDY-HILBERT TYPE

MEHMET ZEKI SARIKAYA[1] AND MEHMET SABIR BINGÖL[1]

Abstract. Our aim in this study will be to obtain a new Hardy-Hilbert type of inequalities, taking into account the two studies by given Sulaiman and Wei-Lei.

1. Introduction

Hardy-Hilbert's integral inequalities constitute a significant cornerstone in the field of mathematical analysis, offering profound insights into the behaviour of integral operators and their associated functions. Named after the eminent mathematicians G.H. Hardy and D. Hilbert, these inequalities have found wide-ranging applications across various branches of mathematics, including functional analysis, partial differential equations, and harmonic analysis.

The well-known Hilbert's inequality and its equivalent form are presented first [2]:

Theorem 1.1. *If $f,g \in L_2([0,\infty))$, then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leq \pi \left\{ \int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy \right\}^{\frac{1}{2}} \tag{1.1}$$

and

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx \right)^2 dy \leq \pi^2 \int_0^\infty f^2(x)\,dx$$

where π and π^2 are the best possible constants.

The classical Hilbert's integral inequality (1.1) had been generalized by Hardy- Riesz (see [1]) in 1925 as the following result. If f,g are nonnegative functions such that $0 <$

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$\int_0^\infty f^p(x)\,dx < \infty$ and $0 < \int_0^\infty g^q(x)\,dx < \infty$, where $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}} \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible constant. When $p=q=2$, inequality (1.2) is reduced to (1.1). In recent years, a number of mathematicians had given lots of generalizations of these inequalities. We mention here some of these contributions in this direction: Li et al. [3] have proved the following Hardy- Hilbert's type inequality using the hypotheses of (1.1):

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y+\max\{x,y\}}dxdy \le c\left\{\int_0^\infty f^2(x)\,dx\int_0^\infty g^2(y)\,dy\right\}^{\frac{1}{2}}$$

Where the constant factor $c=\sqrt{2}\left(\pi-2-\tan^{-1}\sqrt{2}\right)$ is the best possible. Other mathematicians have presented generalizations or new kinds of the above Hardy-Hilbert inequalities, as follows:

Theorem 1.2. [9] *Let $f,g>0$. If $p>1,\lambda>0,\ \frac{1}{p}+\frac{1}{q}=1$, and $0<\lambda=2-\frac{1}{p}+\frac{1}{q}\le 1$, then one has*

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{(x+y)^\lambda}dxdy \le k\left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}.$$

Here, k depends on p and q; only if $\frac{1}{p}+\frac{1}{q}=1,\ \lambda=2-\frac{1}{p}+\frac{1}{q}=1$, k is the best possible.

Theorem 1.3. [7] *Let $f,g>0$. If $p>1,\ q>1$, and $\frac{1}{p}+\frac{1}{q}=1$, are such that*

$$0<\int_0^\infty x^{p-1-\lambda}f^p(x)\,dx<\infty \text{ and } 0<\int_0^\infty y^{q-1-\lambda}g^q(y)\,dy<\infty,$$

then one has

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy \le \frac{pq}{\lambda}\left(\int_0^\infty x^{p-1-\lambda}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q-1-\lambda}g^q(y)\,dy\right)^{\frac{1}{q}},$$

where the constant factor $\frac{pq}{\lambda}$ is the best possible.

In [8], Sulaiman provided a comprehensive generalization of Hardy-Hilbert type inequalities applicable to homogeneous kernels of order λ. This extension significantly broadened the scope of these inequalities, allowing for their application to a wider range of mathematical contexts. Building upon Sulaiman's groundwork, Wei and Lei further contributed to the advancement of this field in [11] by offering an alternative proof technique for Hardy-Hilbert type inequalities. Despite addressing the same functions and kernels, Wei-Lei's approach introduced novel insights and methodologies, enriching the mathematical discourse surrounding these inequalities. Regarding Hardy-Hilbert integral inequalities regarding different types of functions and approximations see [4–6, 10], where further references are given.

Our objective in this study is to derive novel Hardy-Hilbert type inequalities, building upon the groundwork laid by the aforementioned studies. By incorporating insights from both Sulaiman's generalization of 2010 and Wei-Lei's alternative proof technique of 2011, we aim to contribute to the ongoing development of this field. Our endeavor is to explore new avenues and refine existing methodologies to further enhance the applicability and theoretical understanding of Hardy-Hilbert type inequalities.

2. Main Results

To prove our main results, we require the following theorem:

Theorem 2.1. *Assume that* $f, h_1, h_2, k > 0$, $h_1, h_2, k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, h_1, h_2 *homogeneous of degree* λ, *and* k *is nondecreasing,* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

i) $k(x) \neq 1$, *or in general,* $k(x) \neq c$, *(c is constant)*

$$\int_0^\infty y^{(\lambda-1)(p-1)} \left(\int_0^\infty \frac{f(x)\,dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\} + h_2(x,y)\min\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \right)^p dy \tag{2.1}$$

$$\leq C_1^{p-1} C_2 \int_0^\infty x^{1-\lambda} f^p(x)\,dx$$

where $C_1 = I_1 + I_2$ *and* $C_2 = J_1 + J_2$,

$$I_1 = \int_0^1 \frac{du}{h_1(u,1)\,k(u^{-1}) + h_2(u,1)\,k(u)}, \qquad I_2 = \int_1^\infty \frac{du}{h_1(u,1)\,k(u) + h_2(u,1)\,k(u^{-1})} \tag{2.2}$$

and

$$J_1 = \int_0^1 \frac{du}{h_1(1,u)\,k(u^{-1}) + h_2(1,u)\,k(u)}, \qquad J_2 = \int_1^\infty \frac{du}{h_1(1,u)\,k(u) + h_2(1,u)\,k(u^{-1})}. \tag{2.3}$$

ii) $k(x) = 1$, *(*α *and* β *are both arbitrary constants),*

$$\int_0^\infty y^{\left(\frac{q\beta}{p}+\lambda-1-\alpha\right)\frac{p}{q}} \left(\int_0^\infty \frac{f(x)\,dx}{h_1(x,y) + h_2(x,y)} \right)^p dy \leq K_1^{\frac{p}{q}} K_2 \int_0^\infty x^{1+\beta-\lambda-\alpha\frac{p}{q}} f^p(x)\,dx \tag{2.4}$$

where

$$K_1 = \int_0^\infty \frac{u^\alpha du}{h_1(u,1) + h_2(u,1)}, \qquad K_2 = \int_0^\infty \frac{u^\beta}{h_1(1,u) + h_2(1,u)} du. \tag{2.5}$$

Proof. i) From Hölder's inequality, we get

$$\int_0^\infty \frac{f(x)\,dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\} + h_2(x,y)\min\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}}$$

$$\leq \left(\int_0^\infty \frac{f^p(x)\,dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty \frac{dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \right)^{\frac{1}{q}}$$

which yields

$$\left(\int_0^\infty \frac{f(x)\,dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \right)^{p} \tag{2.6}$$

$$\leq \int_0^\infty \frac{f^p(x)\,dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}$$

$$\times \left(\int_0^\infty \frac{dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \right)^{\frac{p}{q}}.$$

We first consider the following integral:

$$\int_0^\infty \frac{dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \tag{2.7}$$

$$= \int_0^y \frac{dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}$$

$$+ \int_y^\infty \frac{dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}$$

$$= M_1 + M_2.$$

For the case M_1, since $x \leq y$ implies $\frac{x}{y} \leq \frac{y}{x}$, and hence $k\left(\frac{x}{y}\right) \leq k\left(\frac{y}{x}\right)$, then we have

$$\begin{aligned} M_1 &= \int_0^y \frac{dx}{h_1(x,y)\,k\left(\frac{y}{x}\right)+h_2(x,y)\,k\left(\frac{x}{y}\right)} \\ &= \int_0^y \frac{dx}{h_1\left(\frac{x}{y}y,y\right)k\left(\frac{y}{x}\right)+h_2\left(\frac{x}{y}y,y\right)k\left(\frac{x}{y}\right)} \\ &= y^{-\lambda}\int_0^y \frac{dx}{h_1\left(\frac{x}{y},1\right)k\left(\frac{y}{x}\right)+h_2\left(\frac{x}{y},1\right)k\left(\frac{x}{y}\right)}. \end{aligned}$$

Let $\frac{x}{y}=u$, it follows that

$$M_1 = y^{1-\lambda}\int_0^1 \frac{du}{h_1(u,1)\,k(u^{-1})+h_2(u,1)\,k(u)} = y^{1-\lambda}I_1$$

and similarly, since $x\geq y$ implies $\frac{x}{y}\geq\frac{y}{x}$, and hence $k\left(\frac{x}{y}\right)\geq k\left(\frac{y}{x}\right)$, then we have

$$\begin{aligned} M_2 &= \int_y^\infty \frac{dx}{h_1(x,y)\,k\left(\frac{x}{y}\right)+h_2(x,y)\,k\left(\frac{y}{x}\right)} \\ &= y^{1-\lambda}\int_1^\infty \frac{du}{h_1(u,1)\,k(u)+h_2(u,1)\,k(u^{-1})} = y^{1-\lambda}I_2. \end{aligned}$$

By using M_1 and M_2 in (2.7), we have

$$\begin{aligned} &\int_0^\infty \frac{dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \\ &= y^{1-\lambda}(I_1+I_2) = C_1 y^{1-\lambda}. \end{aligned}$$

Hence, from (2.6) and (2.7), we get

$$\begin{aligned} &\left(\int_0^\infty \frac{f(x)\,dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^p \\ &\leq C_1^{\frac{p}{q}} y^{(1-\lambda)\frac{p}{q}} \int_0^\infty \frac{f^p(x)\,dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}. \end{aligned}$$

If we multiply both sides of the above inequality by $y^{(\lambda-1)(p-1)}$ and integrate with respect to the y variable , we get

$$\begin{aligned} &\int_0^\infty y^{(\lambda-1)(p-1)}\left(\int_0^\infty \frac{f(x)\,dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^p dy \\ &\leq C_1^{\frac{p}{q}} \int_0^\infty\int_0^\infty \frac{f^p(x)\,dxdy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \\ &= C_1^{p-1}\int_0^\infty f^p(x)\left(\int_0^\infty \frac{dy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)dx. \end{aligned} \tag{2.8}$$

By calculating the inner integral above, with the same method as above, it follows that

$$\int_0^\infty \frac{dy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \tag{2.9}$$

$$
\begin{aligned}
&= \int_0^x \frac{dy}{h_1(x,y)\,k\left(\frac{x}{y}\right)+h_2(x,y)\,k\left(\frac{y}{x}\right)}+\int_x^\infty \frac{dy}{h_1(x,y)\,k\left(\frac{y}{x}\right)+h_2(x,y)\,k\left(\frac{x}{y}\right)} \\
&= x^{1-\lambda}\int_0^1 \frac{du}{h_1(1,u)\,k(u^{-1})+h_2(1,u)\,k(u)}+x^{1-\lambda}\int_1^\infty \frac{du}{h_1(1,u)\,k(u)+h_2(1,u)\,k(u^{-1})} \\
&= x^{1-\lambda}(J_1+J_2)=C_2x^{1-\lambda}.
\end{aligned}
$$

If (2.9) is substituted into (2.8) which shows that the inequality (2.1) holds.

ii) Similarly, according to Hölder's inequality, we get

$$
\begin{aligned}
&\int_0^\infty \frac{f(x)\,dx}{h_1(x,y)+h_2(x,y)} \\
&= \int_0^\infty \frac{f(x)\,y^{\frac{\beta}{p}}}{[h_1(x,y)+h_2(x,y)]^{\frac{1}{p}}\,x^{\frac{\alpha}{q}}}\frac{x^{\frac{\alpha}{q}}}{[h_1(x,y)+h_2(x,y)]^{\frac{1}{q}}\,y^{\frac{\beta}{p}}}dx \\
&\le \left(\int_0^\infty \frac{f^p(x)\,y^\beta dx}{[h_1(x,y)+h_2(x,y)]\,x^{\frac{p\alpha}{q}}}\right)^{\frac{1}{p}}\times\left(\int_0^\infty \frac{x^\alpha dx}{[h_1(x,y)+h_2(x,y)]\,y^{\frac{q\beta}{p}}}\right)^{\frac{1}{q}}
\end{aligned}
$$

which yields

$$
\left(\int_0^\infty \frac{f(x)\,dx}{h_1(x,y)+h_2(x,y)}\right)^p \le \int_0^\infty \frac{f^p(x)\,y^\beta dx}{[h_1(x,y)+h_2(x,y)]\,x^{\frac{p\alpha}{q}}}\left(\int_0^\infty \frac{x^\alpha dx}{[h_1(x,y)+h_2(x,y)]\,y^{\frac{q\beta}{p}}}\right)^{\frac{p}{q}}. \tag{2.10}
$$

We consider the above last integral

$$
\begin{aligned}
\int_0^\infty \frac{x^\alpha dx}{[h_1(x,y)+h_2(x,y)]\,y^{\frac{q\beta}{p}}} &= y^{-\lambda-\frac{q\beta}{p}}\int_0^\infty \frac{x^\alpha dx}{h_1\left(\frac{x}{y},1\right)+h_2\left(\frac{x}{y},1\right)} \\
&= y^{1-\lambda-\frac{q\beta}{p}+\alpha}\int_0^\infty \frac{u^\alpha du}{h_1(u,1)+h_2(u,1)} \\
&= y^{1-\lambda-\frac{q\beta}{p}+\alpha}K_1,
\end{aligned}
$$

then we can obtain

$$
\left(\int_0^\infty \frac{f(x)\,dx}{h_1(x,y)+h_2(x,y)}\right)^p \le y^{\left(1-\lambda-\frac{q\beta}{p}+\alpha\right)\frac{p}{q}}K_1^{\frac{p}{q}}\int_0^\infty \frac{f^p(x)\,y^\beta dx}{[h_1(x,y)+h_2(x,y)]\,x^{\frac{p\alpha}{q}}}.
$$

If we multiply both sides of the above inequality by $y^{\left(\frac{q\beta}{p}+\lambda-1-\alpha\right)\frac{p}{q}}$ and integrate with respect to the y variable , we get

$$
\int_0^\infty y^{\left(\frac{q\beta}{p}+\lambda-1-\alpha\right)\frac{p}{q}}\left(\int_0^\infty \frac{f(x)\,dx}{h_1(x,y)+h_2(x,y)}\right)^p dy \tag{2.11}
$$

$$\leq \int_0^\infty y^{\left(\frac{q\beta}{p}+\lambda-1-\alpha\right)\frac{p}{q}} y^{\left(1-\lambda-\frac{q\beta}{p}+\alpha\right)\frac{p}{q}} K_1^{\frac{p}{q}} \int_0^\infty \frac{f^p(x)\, y^\beta dx}{\left[h_1(x,y)+h_2(x,y)\right] x^{\frac{p\alpha}{q}}} dy$$

$$= K_1^{\frac{p}{q}} \int_0^\infty f^p(x) \left(\int_0^\infty \frac{y^\beta}{\left[h_1(x,y)+h_2(x,y)\right] x^{\frac{p\alpha}{q}}} dy \right) dx.$$

Now, by computing the above integrals

$$\begin{aligned}
\int_0^\infty \frac{y^\beta}{\left[h_1(x,y)+h_2(x,y)\right] x^{\frac{p\alpha}{q}}} dy &= x^{-\alpha\frac{p}{q}-\lambda} \int_0^\infty \frac{y^\beta}{\left[h_1\left(1,\frac{y}{x}\right)+h_2\left(1,\frac{y}{x}\right)\right]} dy \\
&= x^{1+\beta-\lambda-\alpha\frac{p}{q}} \int_0^\infty \frac{u^\beta}{\left[h_1(1,u)+h_2(1,u)\right]} du \\
&= x^{1+\beta-\lambda-\alpha\frac{p}{q}} K_2.
\end{aligned}$$

This final result is written instead of (2.11), it follows that

$$\int_0^\infty y^{\left(\frac{q\beta}{p}+\lambda-1-\alpha\right)\frac{p}{q}} \left(\int_0^\infty \frac{f(x)\, dx}{h_1(x,y)+h_2(x,y)} \right)^p dy \leq K_1^{\frac{p}{q}} K_2 \int_0^\infty x^{1+\beta-\lambda-\alpha\frac{p}{q}} f^p(x)\, dx$$

which shows that the inequality (2.4) holds. □

Theorem 2.2. *Assume that $f, g, h_1, h_2, k > 0$, $h_1, h_2, k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, h_1, h_2 homogeneous of degree λ, and k is nondecreasing, $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$. Then*

i) $k(x) \neq 1$, or in general, $k(x) \neq c$, (c is constant)

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)\, dx dy}{h_1(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\} + h_2(x,y) \min\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \tag{2.12}$$

$$\leq C_1^{\frac{1}{p}} C_2^{\frac{1}{q}} \left(\int_0^\infty x^{1-\lambda} f^p(x)\, dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{1-\lambda} g^q(y)\, dy \right)^{\frac{1}{1}}$$

where $C_1 = I_1 + I_2$ and $C_2 = J_1 + J_2$ are defined by (2.2) and (2.3).

ii) $k(x) = 1$, (α and β are both arbitrary constants),

$$\begin{aligned}
\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)\, dx dy}{h_1(x,y)+h_2(x,y)} &\leq K_1^{\frac{1}{q}} K_2^{\frac{1}{p}} \left(\int_0^\infty x^{1+\beta-\lambda-\alpha(p-1)} f^p(x)\, dx \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty y^{1+\alpha-\lambda-\beta(p-1)} g^q(y)\, dy \right)^{\frac{1}{q}}
\end{aligned} \tag{2.13}$$

where K_1 and K_2 are defined by (2.5).

Proof. i) By using Hölder's inequality, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)\,dxdy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \tag{2.14}$$

$$\begin{aligned}
&\leq \left(\int_0^\infty \int_0^\infty \frac{f^p(x)\,dxdy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^\infty \int_0^\infty \frac{g^q(y)\,dxdy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^{\frac{1}{q}} \\
&= M^{\frac{1}{p}}N^{\frac{1}{q}}.
\end{aligned}$$

We first consider the following integral:

$$M=\int_0^\infty \int_0^\infty \frac{f^p(x)\,dxdy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \tag{2.15}$$

$$=\int_0^\infty f^p(x)\left(\int_0^\infty \frac{dy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)dx$$

and writing the inner integral as follows

$$\begin{aligned}
&\int_0^\infty \frac{dy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \\
&= \int_0^x \frac{dy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \\
&\quad+\int_x^\infty \frac{dy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}+h_2(x,y)\min\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}} \\
&= M'+M''.
\end{aligned}$$

For the case M'', since $x\leq y$ implies $\frac{x}{y}\leq\frac{y}{x}$, and hence $k\left(\frac{x}{y}\right)\leq k\left(\frac{y}{x}\right)$, then we have

$$M''=\int_0^x \frac{dy}{h_1(x,y)\,k\left(\frac{y}{x}\right)+h_2(x,y)\,k\left(\frac{x}{y}\right)}=x^{-\lambda}\int_0^x \frac{dy}{h_1\left(\frac{x}{y},1\right)k\left(\frac{y}{x}\right)+h_2\left(\frac{x}{y},1\right)k\left(\frac{x}{y}\right)}.$$

Let $\frac{x}{y}=u$, it follows that

$$M''=x^{1-\lambda}\int_0^1 \frac{du}{h_1(u,1)\,k\left(u^{-1}\right)+h_2(u,1)\,k(u)}=x^{1-\lambda}I_1.$$

and similarly, since $x \geq y$ implies $\frac{x}{y} \geq \frac{y}{x}$, and hence $k\left(\frac{x}{y}\right) \geq k\left(\frac{y}{x}\right)$, then we have

$$\begin{aligned} M' &= \int_x^\infty \frac{dy}{h_1(x,y)\,k\left(\frac{x}{y}\right) + h_2(x,y)\,k\left(\frac{y}{x}\right)} \\ &= x^{1-\lambda} \int_1^\infty \frac{du}{h_1(u,1)\,k(u) + h_2(u,1)\,k(u^{-1})} = x^{1-\lambda} I_2. \end{aligned}$$

From M' and M''

$$\int_0^\infty \frac{dy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\} + h_2(x,y)\min\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} = x^{1-\lambda}(I_1 + I_2) = C_1 x^{1-\lambda}.$$

If this last result is substituted into (2.15), we have

$$\begin{aligned} M &= \int_0^\infty \int_0^\infty \frac{f^p(x)\,dxdy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\} + h_2(x,y)\min\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \\ &= C_1 \int_0^\infty x^{1-\lambda} f^p(x)\,dx. \end{aligned}$$

Similarly,

$$N = \int_0^\infty g^q(y) \left(\int_0^\infty \frac{dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\} + h_2(x,y)\min\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \right) dy.$$

By calculating the inner integral above, with the same method as above, it follows that

$$\begin{aligned} &\int_0^\infty \frac{dx}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\} + h_2(x,y)\min\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \\ &= \int_0^y \frac{dx}{h_1(x,y)\,k\left(\frac{x}{y}\right) + h_2(x,y)\,k\left(\frac{y}{x}\right)} + \int_y^\infty \frac{dx}{h_1(x,y)\,k\left(\frac{y}{x}\right) + h_2(x,y)\,k\left(\frac{x}{y}\right)} \\ &= y^{1-\lambda} \int_0^1 \frac{du}{h_1(1,u)\,k(u^{-1}) + h_2(1,u)\,k(u)} + y^{1-\lambda} \int_1^\infty \frac{du}{h_1(1,u)\,k(u) + h_2(1,u)\,k(u^{-1})} \\ &= y^{1-\lambda}(J_1 + J_2) = C_2 y^{1-\lambda} \end{aligned}$$

and so

$$N = C_2 \int_0^\infty y^{1-\lambda} g^q(y)\,dy.$$

If M and N are written in (2.14), we have

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)\,dxdy}{h_1(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\} + h_2(x,y)\min\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}}$$

$$\leq C_1^{\frac{1}{p}} C_2^{\frac{1}{q}} \left(\int_0^\infty x^{1-\lambda} f^p(x)\, dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{1-\lambda} g^q(y)\, dy \right)^{\frac{1}{1}}$$

which shows that the inequality (2.12) holds.

ii) Similarly, according to Hölder's inequality, we get

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)\, dxdy}{h_1(x,y) + h_2(x,y)} \tag{2.16}$$

$$= \int_0^\infty \int_0^\infty \frac{f(x)\, y^{\frac{\beta}{p}}}{[h_1(x,y) + h_2(x,y)]^{\frac{1}{p}}\, x^{\frac{\alpha}{q}}} \frac{g(y)\, x^{\frac{\alpha}{q}}}{[h_1(x,y) + h_2(x,y)]^{\frac{1}{q}}\, y^{\frac{\beta}{p}}} dxdy$$

$$\leq \left(\int_0^\infty \int_0^\infty \frac{f^p(x)\, y^\beta dxdy}{[h_1(x,y) + h_2(x,y)]\, x^{\frac{p\alpha}{q}}} \right)^{\frac{1}{p}} \times \left(\int_0^\infty \int_0^\infty \frac{g^q(y)\, x^\alpha dxdy}{[h_1(x,y) + h_2(x,y)]\, y^{\frac{q\beta}{p}}} \right)^{\frac{1}{q}}$$

$$= \left(\int_0^\infty x^{-\alpha(p-1)} f^p(x) \left(\int_0^\infty \frac{y^\beta dy}{[h_1(x,y) + h_2(x,y)]} \right) dx \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty y^{-\beta(p-1)} g^q(y) \left(\int_0^\infty \frac{x^\alpha dx}{[h_1(x,y) + h_2(x,y)]} \right) dy \right)^{\frac{1}{q}}.$$

We consider the above last integrals with change of variables, we have

$$\begin{aligned} \int_0^\infty \frac{y^\beta dy}{h_1(x,y) + h_2(x,y)} &= x^{-\lambda} \int_0^\infty \frac{y^\beta dy}{h_1(1, \frac{y}{x}) + h_2(1, \frac{y}{x})} \\ &= x^{1+\beta-\lambda} \int_0^\infty \frac{u^\beta du}{h_1(1,u) + h_2(1,u)} = x^{1+\beta-\lambda} K_2 \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \int_0^\infty \frac{x^\alpha dx}{h_1(x,y) + h_2(x,y)} &= y^{-\lambda} \int_0^\infty \frac{x^\alpha dx}{h_1\left(\frac{x}{y} y, y\right) + h_2\left(\frac{x}{y} y, y\right)} \\ &= y^{1+\alpha-\lambda} \int_0^\infty \frac{u^\alpha du}{h_1(u,1) + h_2(u,1)} = y^{1+\alpha-\lambda} K_1. \end{aligned} \tag{2.18}$$

Therefore, (2.17) and (2.18) are written into (2.16), it follows that

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)\, dxdy}{h_1(x,y) + h_2(x,y)} \leq K_1^{\frac{1}{q}} K_2^{\frac{1}{p}} \left(\int_0^\infty x^{1+\beta-\lambda-\alpha(p-1)} f^p(x)\, dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{1+\alpha-\lambda-\beta(p-1)} g^q(y)\, dy \right)^{\frac{1}{q}}$$

which shows that the inequality (2.13) holds. □

3. Applications

Corollary 3.1. *Assume that* $f \geq 0,\ \lambda > \frac{1}{2},\ p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_0^\infty y^{(\lambda-1)(p-1)} \left(\int_0^\infty \frac{f(x)\,dx}{(xy)^\lambda \max\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\} + (xy)^\lambda \min\left\{\left(\frac{x}{y}\right)^\lambda, \left(\frac{y}{x}\right)^\lambda\right\}} \right)^p dy$$

$$\leq \left[\frac{2\lambda+1}{2(2\lambda-1)}\right]^p \int_0^\infty x^{1-\lambda} f^p(x)\,dx$$

Proof. The result is obtained from result (i) in Theorem 2.1 by putting

$$h_1(x,y) = (xy)^\lambda,\ h_2(x,y) = (xy)^{-\lambda},\ k(x) = x^{-\lambda}$$

thus

$$I_1 = J_1 = \int_0^1 \frac{du}{u^\lambda u^{-\lambda} + u^\lambda u^{-\lambda}} = \frac{1}{2},$$

$$I_2 = J_2 = \int_1^\infty \frac{du}{u^{2\lambda} + u^{-2\lambda}} = \int_1^\infty \frac{u^{2\lambda}du}{u^{4\lambda}+1} \leq \int_1^\infty \frac{u^{2\lambda}du}{u^{4\lambda}} = \frac{1}{2\lambda-1}.$$

The desired result can now be obtained. □

Corollary 3.2. *Assume that* $f \geq 0,\ \lambda > \frac{1}{2},\ p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_0^\infty y^{(\lambda-1)(p-1)} \left(\int_0^\infty \frac{f(x)\,dx}{(x+y)^\lambda + (x+y)^{-\lambda}} \right)^p dy$$

$$\leq B^{p-1}\left(\frac{1}{q}, \lambda - \frac{1}{q}\right) B\left(\frac{1}{p}, \lambda - \frac{1}{p}\right) \int_0^\infty x^{1-\lambda} f^p(x)\,dx$$

Proof. The result is obtained from result (ii) in Theorem 2.1 by putting

$$h_1(x,y) = (x+y)^\lambda,\ h_2(x,y) = (x+y)^{-\lambda},\ \alpha = \frac{1}{q} - 1,\ \beta = \frac{1}{p} - 1,$$

thus

$$K_1 = \int_0^\infty \frac{u^{\frac{1}{q}-1}(u+1)^\lambda du}{(u+1)^{2\lambda}+1} \leq \int_0^\infty \frac{u^{\frac{1}{q}-1}du}{(u+1)^\lambda} = B\left(\frac{1}{q}, \lambda - \frac{1}{q}\right),$$

$$K_2 = \int_0^\infty \frac{u^{\frac{1}{p}-1}(u+1)^\lambda du}{(u+1)^{2\lambda}+1} \leq B\left(\frac{1}{p}, \lambda - \frac{1}{p}\right).$$

The desired result can now be obtained. □

4. Conclusion

In conclusion, the works of Sulaiman in 2010 and Wei-Lei in 2011 represent significant advancements in the theory of Hardy-Hilbert type inequalities. Sulaiman's generalization widened the applicability of these inequalities by accommodating homogeneous kernels of order λ, thereby extending their utility across various mathematical domains. Wei-Lei's alternative proof technique not only provided a fresh perspective but also introduced new methodologies, further enriching the understanding and discourse surrounding Hardy-Hilbert type inequalities. Collectively, these contributions lay a strong foundation for future research endeavours, encouraging continued exploration and innovation in this field.

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[1]Düzce University,
Department of Mathematics,
Faculty of Science and Arts, Düzce, Turkey
Email address: sarikayamz@gmail.com
Email address: mehmetsabirbingol@gmail.com