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ON THE BERNOULLI INEQUALITY WITH A REAL EXPONENT: AN INFINITE PRODUCT APPROACH

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ABSTRACT. The Bernoulli inequality with a real exponent is a key result in mathematics, with applications in various fields such as analysis, number theory and probability. Although numerous proofs of this inequality already exist, it is always interesting to offer innovative points of view for a better understanding and to search for original perspectives. In this article, we contribute to the field by developing a new and elegant approach. Specifically, we determine an infinite product expansion based on a reasonable logarithmic decomposition and the Seidel formula. Some applications are presented, including one related to the notion of q-number. Most importantly, we show how the desired Bernoulli inequality can be derived from this result. Then, based on this approach, some improvements are discussed and illustrated.

1. INTRODUCTION

The Bernoulli inequality is a fundamental result in mathematics. Its basic version states that, for any $x > -1$ and $n \in \mathbb{N}$, we have $(1 + x)^n \geq 1 + nx$. Thanks to its simplicity and versatility, this inequality has various applications in different branches of mathematics, including analysis, combinatorics, and probability theory.

More specifically, in analysis, it is often used in the study of sequences and series, providing information about their convergence behavior. In combinatorics, it is applied to some enumerative problems, where it helps to derive inequalities related to counting and ordering objects. It is also used in probability theory to establish various bounds on the probabilities and expectations of random variables.

Overall, the basic Bernoulli inequality serves as a basis for proving inequalities and advanced theorems. A more general version of this inequality, with a real exponent instead of the integer n, is recalled in the result below.

Key words and phrases. Bernoulli's inequality, Seidel's formula, Infinite series expansions, q-numbers, General inequalities.

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Theorem 1.1. *(Bernoulli's inequality) For any $a \in \mathbb{R} \setminus (0,1)$ and $x > -1$, we have*

$$(1+x)^a \geq 1 + ax.$$

This inequality is reversed for $a \in (0,1)$, i.e.,

$$(1+x)^a \leq 1 + ax.$$

There are various proofs of this theorem, using methods such as induction, arithmetic and geometric means, geometric series, the binomial theorem or convexity. The references to the Bernoulli inequality with a real exponent and its extensions are numerous. To name a few, we mention [1, 2, 4, 5, 9, 10] and [12]. We also refer to [7], where a comprehensive review is given, including some equivalent relations between the Bernoulli inequality and other known inequalities. The references therein should also be consulted.

This article presents a new perspective on this inequality, as well as variants of potential interest. We first recall the Seidel formula, which can be presented as a special infinite series formula centered on the logarithmic function. We highlight the fact that it is applicable to the whole range of values of the logarithmic function, which makes it interesting for inequalities. We then use it to derive an infinite series expansion for the following ratio-type function: $ax/[(1+x)^a - 1]$. In a sense, our result has the merit of "perfectly filling the gap" between the functions ax and $(1+x)^a - 1$, which are the main ingredients of the Bernoulli inequality under consideration. To the best of our knowledge, this is an unexplored approach in the literature. A direct application to the notion of q-number is discussed. Some visual illustrations are also provided. We then use this new expansion to establish the Bernoulli inequality with a real exponent. As a further contribution, we offer a new general variant with some potential for improvement.

The structure of this article is outlined as follows: Our alternative approach is presented in Section 2. An extension of the Bernoulli inequality with a real exponent is given in Section 3. The article concludes with a summary in Section 4.

2. Results

In this section, we present our infinite product expansion approach and show how to derive the Bernoulli inequality with a real exponent.

2.1. An original infinite decomposition.

First, we recall the Seidel formula in the theorem below.

Theorem 2.1. *(Seidel's formula) For any $t > 0$, we have*

$$\log(t) = (t-1) \prod_{k=1}^{+\infty} \frac{2}{1 + t^{2^{-k}}}.$$

As far as we know, this formula is best known for the special case $t = 2$, giving an infinite product expansion for $\log(2)$. Surprisingly, in its general form, it has few references and seems not to have been really studied in depth. However, it has the merit of being defined for any $t > 0$, unlike most existing series decompositions for $\log(t)$. From our point of view, this is a real advantage for using it to establish inequalities, among other things. For more details, see [8, page 354] and [3]. For the sake of completeness, a comprehensive proof of this formula based on an iterative technique is given in Appendix.

Based on the Seidel formula, we derive an original infinite expansion for $ax/[(1+x)^a-1]$ described in the proposition below.

Proposition 2.1. *For any $a \in \mathbb{R}\setminus\{0\}$ and $x>-1$, we have*

$$\frac{ax}{(1+x)^a-1}=\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}}.$$

Proof. The proof is based on a judicious use of the logarithmic function and the Seidel formula. For any $a \in \mathbb{R}\setminus\{0\}$ and $x>-1$, we first decompose a in the following (unexpected) way:

$$a=a\times\frac{\log(1+x)}{\log(1+x)}=\frac{\log[(1+x)^a]}{\log(1+x)}.$$

Then, by applying the Seidel formula for both logarithmic terms, with $t=(1+x)^a$ and $t=1+x$, respectively, we get

$$a=\frac{[(1+x)^a-1]\prod_{k=1}^{+\infty}\frac{2}{1+(1+x)^{a2^{-k}}}}{[(x+1)-1]\prod_{k=1}^{+\infty}\frac{2}{1+(1+x)^{2^{-k}}}}=\frac{(1+x)^a-1}{x}\prod_{k=1}^{+\infty}\left[\frac{\frac{2}{1+(1+x)^{a2^{-k}}}}{\frac{2}{1+(1+x)^{2^{-k}}}}\right]$$

$$=\frac{(1+x)^a-1}{x}\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}}.$$

Hence, by rearranging the expressions on both sides of the equality, we have

$$\frac{ax}{(1+x)^a-1}=\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}},$$

which is the desired result. ☐

In addition to its elegance, two major interests in this decomposition are that it is valid with minimal restrictions on the possible values of x and a, and that the coefficients involved in the product are all positive and can be greater or less than 1, depending on the values of x and a. These aspects will be crucial in the main proof.

2.2. **Some complements.**

Some additions and discussion centered on Proposition 2.1 are given in this subsection.

2.2.1. *Explicit formulas.*

We now exhibit the formula in Proposition 2.1 applied to some selected negative and positive integer values on a. They are presented in Table 1, where the column "Main function" gives $ax/[(1+x)^a-1]$.

TABLE 1. Specific formulas based on Proposition 2.1 for selected integer values of a.

a	Main function	Infinite product
-6	$\frac{6x^6+36x^5+90x^4+120x^3+90x^2+36x+6}{x^5+6x^4+15x^3+20x^2+15x+6}$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{-3\times 2^{1-k}}}$
-5	$\frac{5x^5+25x^4+50x^3+50x^2+25x+5}{x^4+5x^3+10x^2+10x+5}$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{-5\times 2^{-k}}}$
-4	$\frac{4x^4+16x^3+24x^2+16x+4}{x^3+4x^2+6x+4}$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{-2^{2-k}}}$
-3	$\frac{3x^3+9x^2+9x+3}{x^2+3x+3}$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{3\times 2^{-k}}}$
-2	$\frac{2x^2+4x+2}{x+2}$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{-2^{1-k}}}$
-1	$1+x$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{-2^{-k}}}$
1	1	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{2^{-k}}}=1$
2	$\frac{2}{x+2}$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{2^{1-k}}}$
3	$\frac{3}{x^2+3x+3}$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{3\times 2^{-k}}}$
4	$\frac{4}{x^3+4x^2+6x+4}$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{2^{2-k}}}$
5	$\frac{5}{x^4+5x^3+10x^2+10x+5}$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{5\times 2^{-k}}}$
6	$\frac{6}{x^5+6x^4+15x^3+20x^2+15x+6}$	$\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{3\times 2^{1-k}}}$

2.2.2. *Secondary results.* The result below presents an immediate consequence of Proposition 2.1.

Proposition 2.2. *For any $a \in \mathbb{R}\setminus\{0\}$ and $x \in (-1,+\infty)\setminus\{0\}$, we have*

$$\log\left[\frac{ax}{(1+x)^a-1}\right]=\sum_{k=1}^{+\infty}\log\left[\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}}\right]$$

and

$$\frac{ax}{(1+x)^a-1}=\exp\left\{\sum_{k=1}^{+\infty}\log\left[\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}}\right]\right\}.$$

Proof. It is sufficient to note that all the ratio terms in Proposition 2.1 are positive, and we conclude by taking the logarithmic function of both sides, then the exponential function for the second decomposition. □

In addition to its novelty, an interesting aspect of this result is that there are many different bounds for the logarithmic and exponential functions. Thus, they can be used to bound terms in the sum and then to derive original bounds for the main functions, i.e., $\log\{ax/[(1+x)^a-1]\}$ or $ax/[(1+x)^a-1]$. To illustrate this claim, using the famous exponential inequality $\exp(t)\geq 1+t$ for any $t\in\mathbb{R}$, we get for any $a\leq 1$ and $x\geq 0$,

$$\frac{ax}{(1+x)^a-1}\geq 1+\sum_{k=1}^{+\infty}\log\left[\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}}\right]\geq 1+\log\left[\frac{1+\sqrt{1+x}}{1+(1+x)^{a/2}}\right].$$

This is only an illustrative example. We will not explore this direction further in this article, focusing on the formula in Proposition 2.1 without using intermediate results.

Another direct application of Proposition 2.1 is connected with the notion of "q-number". For any $q\in(0,1)$ and $a\in\mathbb{R}$, the q-number of a is defined as

$$[a]_q=\frac{q^a-1}{q-1}.$$

It is an essential component of the q-binomial coefficient involved in the q-Gauss binomial formula. For further information, we may refer to [6]. Based on Proposition 2.1, the result below gives a new infinite product expansion for this particular number.

Proposition 2.3. *For any $q\in(0,1)$ and $a\in\mathbb{R}$, we have*

$$[a]_q=a\prod_{k=1}^{+\infty}\frac{1+q^{a2^{-k}}}{1+q^{2^{-k}}}.$$

Proof. For $a=0$, it is immediate that

$$[a]_q=0=0\times\prod_{k=1}^{+\infty}\frac{1+q^{0\times 2^{-k}}}{1+q^{2^{-k}}}.$$

For any $a\in\mathbb{R}\setminus\{0\}$, by applying Proposition 2.1 with $x=q-1\in(-1,0)$, we get

$$\frac{a(q-1)}{q^a-1}=\prod_{k=1}^{+\infty}\frac{1+q^{2^{-k}}}{1+q^{a2^{-k}}}.$$

By identifying $a/[a]_q$ as the exact left term and isolating it by the reciprocal function, we get

$$[a]_q=a\prod_{k=1}^{+\infty}\frac{1+q^{a2^{-k}}}{1+q^{2^{-k}}}.$$

This ends the proof. □

This original expansion of the q-number is new in the literature to the best of our knowledge.

2.3. **A graphical analysis.** Before making the theoretical connection between Proposition 2.1 and the Bernoulli inequality, a graphical analysis is suggested. Let us consider the following truncated-difference function:

$$f_m(x;a) = \frac{ax}{(1+x)^a - 1} - \prod_{k=1}^{m} \frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}},$$

where $m \in \mathbb{N} \setminus \{0\}$. From Proposition 2.1, it is clear that $\lim_{m\to+\infty} f_m(x;a) = 0$. With this in mind, Figure 1 displays the plots of $f_m(x;a)$ for $m = 1, 2, \ldots, 12$, and arbitrary values for a and x. We want to see how fast the convergence is with respect to m under different parameter configurations.

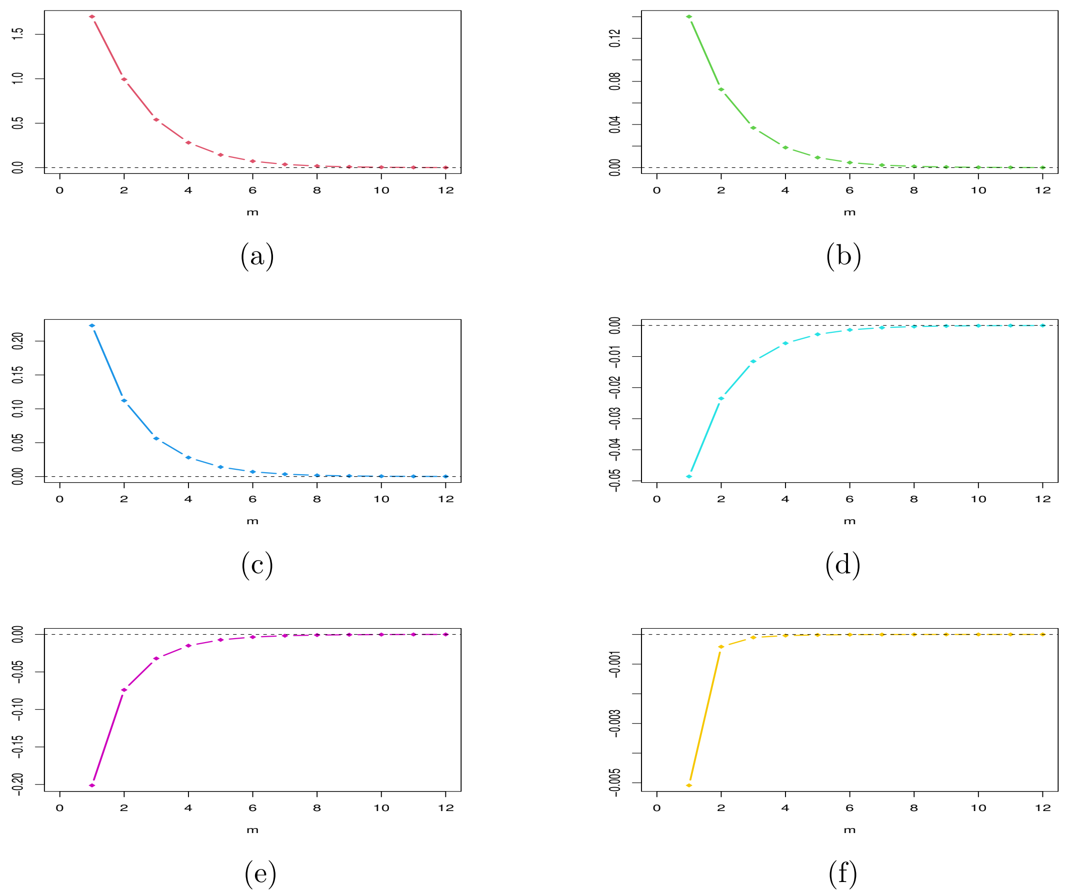


FIGURE 1. Plots of $f_m(x;a)$ for $m = 1, 2, \ldots, 12$ and (a) $a = -1.4$ and $x = 2$, (b) $a = -4$ and $x = 0.1$, (c) $a = 0.5$ and $x = 2.5$, (d) $a = 1.5$ and $x = 0.5$, (e) $a = 3$ and $x = 2.5$, and (f) $a = 8$ and $x = 4$.

It is clear from this figure that the convergence of $f_m(x; a)$ to 0 is fast; it is very close to the axis associated with $y = 0$, starting at $m = 8$. So we have graphical support for Proposition 2.1.

2.4. **Connection with the Bernoulli inequality.** The proposition below formulates the connection between the Bernoulli inequality with a real exponent and Proposition 2.1.

Proposition 2.4. *The infinite product expansion in Proposition 2.1 implies the Bernoulli inequality with a real exponent, as described in Theorem 1.1.*

Proof. Let us consider the "standard" Bernoulli inequality with a real exponent; the reverse version will be shown as a second part. Let us distinguish between two different cases covering $x > -1$ and $a \in \mathbb{R} \setminus (0, 1)$: Case 1: $x \in (-1, 0)$ and $a \geq 1$, or $x \geq 0$ and $a \leq 0$, and Case 2: $x \in (-1, 0)$ and $a \leq 0$, or $x \geq 0$ and $a \geq 1$.

Case 1:: For any $x \in (-1, 0)$ and $a \geq 1$, or $x \geq 0$ and $a \leq 0$, we have $(1 + x)^{a-1} \leq 1$ and $(1 + x)^a \leq 1$ at the same time. The inequality $(1 + x)^{a-1} \leq 1$ implies that $(1 + x)^a \leq 1 + x$, and, for any integer $k \geq 1$, $1 + (1 + x)^{2^{-k}} \geq 1 + (1 + x)^{a2^{-k}} > 0$, so that

$$\frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \geq 1.$$

Applying Proposition 2.1, we get

$$\frac{ax}{(1 + x)^a - 1} = \prod_{k=1}^{+\infty} \frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \geq 1.$$

On the other hand, the inequality $(1 + x)^a \leq 1$ yields $(1 + x)^a - 1 \leq 0$. Therefore, we obtain $ax \leq (1 + x)^a - 1$, which is equivalent to the desired inequality, i.e.,

$$(1 + x)^a \geq 1 + ax.$$

Case 2:: For any $x \in (-1, 0)$ and $a \leq 0$, or $x \geq 0$ and $a \geq 1$, we have $(1 + x)^{a-1} \geq 1$ and $(1 + x)^a \geq 1$ at the same time. The inequality $(1 + x)^{a-1} \geq 1$ implies that $(1 + x)^a \geq 1 + x$, and, for any integer $k \geq 1$, $0 < 1 + (1 + x)^{2^{-k}} \leq 1 + (1 + x)^{a2^{-k}}$, so that

$$\frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \leq 1.$$

Thanks to Proposition 2.1, we get

$$\frac{ax}{(1 + x)^a - 1} = \prod_{k=1}^{+\infty} \frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \leq 1.$$

In addition, the inequality $(1 + x)^a \geq 1$ gives $(1 + x)^a - 1 \geq 0$. As a result, we have $ax \leq (1 + x)^a - 1$, which yields

$$(1 + x)^a \geq 1 + ax.$$

The standard Bernoulli inequality with a real exponent is demonstrated.

The reverse version can be shown in a similar way, but with different cases. More precisely, we now distinguish between two different cases covering $x > -1$ and $a \in (0, 1)$: Case 1: $x \in (-1, 0)$ and $a \in (0, 1)$, and Case 2: $x \geq 0$ and $a \in (0, 1)$.

Case 1:: For any $x \in (-1, 0)$ and $a \in (0, 1)$, we have $(1 + x)^{a-1} \geq 1$ and $(1 + x)^a \leq 1$ at the same time. The inequality $(1 + x)^{a-1} \geq 1$ implies that $(1 + x)^a \geq 1 + x$, and, for any integer $k \geq 1$, $0 < 1 + (1 + x)^{2^{-k}} \leq 1 + (1 + x)^{a2^{-k}}$, so that

$$\frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \leq 1.$$

Proposition 2.1 yields

$$\frac{ax}{(1 + x)^a - 1} = \prod_{k=1}^{+\infty} \frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \leq 1.$$

On the other hand, the inequality $(1 + x)^a \leq 1$ gives $(1 + x)^a - 1 \leq 0$. Therefore, we obtain $ax \geq (1 + x)^a - 1$. This is equivalent to the desired reverse inequality, i.e.,

$$(1 + x)^a \leq 1 + ax.$$

Case 2:: For any $x \geq 0$ and $a \in (0, 1)$, we have $(1 + x)^{a-1} \leq 1$ and $(1 + x)^a \geq 1$ at the same time. The inequality $(1 + x)^{a-1} \leq 1$ implies that $(1 + x)^a \leq 1 + x$, and, for any integer $k \geq 1$, $1 + (1 + x)^{2^{-k}} \geq 1 + (1 + x)^{a2^{-k}} > 0$, so that

$$\frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \geq 1.$$

Owing to Proposition 2.1, we get

$$\frac{ax}{(1 + x)^a - 1} = \prod_{k=1}^{+\infty} \frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \geq 1.$$

In addition, the inequality $(1 + x)^a \geq 1$ yields $(1 + x)^a - 1 \geq 0$. As a result, we have $ax \geq (1 + x)^a - 1$, which gives

$$(1 + x)^a \leq 1 + ax.$$

The reverse Bernoulli inequality with a real exponent is demonstrated. This concludes the proof. □

This proof is interesting because of the original use of the infinite product and the fact that the Bernoulli inequality comes quite naturally with basic intermediate mathematical tools.

When this proof is examined more closely, it becomes clear that it is a bit crude, sacrificing substantial information to fit the desired Bernoulli inequality. With this in mind, a refined inequality is proposed in the next subsection, making more thorough use of Proposition 2.1.

3. Extension

The theorem below provides an elegant extension of the Bernoulli inequality with a real exponent and its reverse version. It is still based on Proposition 2.1.

Theorem 3.1. *For any $a \in \mathbb{R}$, $x > -1$ and $K \subseteq \mathbb{N} \setminus \{0\}$, let us set*

$$g(x; a, K) = \prod_{k \in K} \frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}}$$

and we complete it with $g(x; a, \{0\}) = 1$. It thus may be a finite product containing one or several functions of x. Then, for any $a \in \mathbb{R} \setminus (0, 1)$, $x > -1$, $K \subseteq \mathbb{N} \setminus \{0\}$ and $L \in \{\{0\}, K\}$, we have

$$[(1 + x)^a - 1]g(x; a, L) + 1 \geq 1 + ax.$$

Let us notice that there are two "1" on both sides that can be simplified, but we chose this formulation to align with that of the former Bernoulli inequality with a real exponent.

This inequality is reversed for $a \in (0, 1)$, i.e.,

$$[(1 + x)^a - 1]g(x; a, L) + 1 \leq 1 + ax.$$

Proof. The proof follows the structure of that of Proposition 2.4. It suffices to be more precise about the bound on the product term by exploiting $g(x; a, L)$. Let us first consider the standard Bernoulli inequality with a real exponent; the reverse version will be shown as a second part. Let us distinguish between two different cases covering $x > -1$ and $a \in \mathbb{R} \setminus (0, 1)$: Case 1: $x \in (-1, 0)$ and $a \geq 1$, or $x \geq 0$ and $a \leq 0$, and Case 2: $x \in (-1, 0)$ and $a \leq 0$, or $x \geq 0$ and $a \geq 1$.

Case 1:: For any $x \in (-1, 0)$ and $a \geq 1$, or $x \geq 0$ and $a \leq 0$, we have $(1 + x)^{a-1} \leq 1$ and $(1 + x)^a \leq 1$ at the same time. The inequality $(1 + x)^{a-1} \leq 1$ implies that $(1 + x)^a \leq 1 + x$, and, for any integer $k \geq 1$, $1 + (1 + x)^{2^{-k}} \geq 1 + (1 + x)^{a2^{-k}} > 0$, so that

$$\frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \geq 1.$$

It follows from Proposition 2.1 and the definition of $g(x; a, L)$ that

$$\frac{ax}{(1 + x)^a - 1} = \prod_{k=1}^{+\infty} \frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \geq \prod_{k \in K} \frac{1 + (1 + x)^{2^{-k}}}{1 + (1 + x)^{a2^{-k}}} \geq g(x; a, L).$$

Let us notice that, with the choice $L = \{0\}$, we have $g(x; a, L) = 1$, refinding the lower bound in the proof of Proposition 2.4.

On the other hand, we have $(1+x)^a \leq 1$, implying that $(1+x)^a - 1 \leq 0$. Therefore, we obtain

$$ax \leq g(x; a, L)[(1 + x)^a - 1],$$

which is equivalent to the desired inequality, i.e.,

$$1 + ax \leq 1 + g(x; a, L)[(1 + x)^a - 1].$$

Case 2:: For any $x \in (-1, 0)$ and $a \leq 0$, or $x \geq 0$ and $a \geq 1$, we have $(1+x)^{a-1} \geq 1$ and $(1+x)^a \geq 1$ at the same time. The inequality $(1+x)^{a-1} \geq 1$ implies that $(1+x)^a \geq 1+x$, and, for any integer $k \geq 1$, $0 < 1+(1+x)^{2^{-k}} \leq 1+(1+x)^{a2^{-k}}$, so that

$$\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}} \leq 1.$$

Owing to Proposition 2.1 and the definition of $g(x; a, L)$, we get

$$\frac{ax}{(1+x)^a - 1} = \prod_{k=1}^{+\infty} \frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}} \leq \prod_{k \in K} \frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}} \leq g(x; a, L).$$

In addition, the inequality $(1+x)^a \geq 1$ gives $(1+x)^a - 1 \geq 0$. As a result, we have

$$ax \leq g(x; a, L)[(1+x)^a - 1],$$

which yields

$$1 + ax \leq 1 + g(x; a, L)[(1+x)^a - 1].$$

The proposed generalization is demonstrated.

The reverse version can be shown in a similar way. Let us distinguish between two different cases covering $x > -1$ and $a \in (0, 1)$: Case 1: $x \in (-1, 0)$ and $a \in (0, 1)$, and Case 2: $x \geq 0$ and $a \in (0, 1)$.

Case 1:: For any $x \in (-1, 0)$ and $a \in (0, 1)$, we have $(1+x)^{a-1} \geq 1$ and $(1+x)^a \leq 1$ at the same time. The inequality $(1+x)^{a-1} \geq 1$ implies that $(1+x)^a \geq 1+x$, and, for any integer $k \geq 1$, $0 < 1+(1+x)^{2^{-k}} \leq 1+(1+x)^{a2^{-k}}$, so that

$$\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}} \leq 1.$$

Proposition 2.1 and the definition of $g(x; a, L)$ yield

$$\frac{ax}{(1+x)^a - 1} = \prod_{k=1}^{+\infty} \frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}} \leq \prod_{k \in K} \frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}} \leq g(x; a, L).$$

On the other hand, the inequality $(1+x)^a \leq 1$ gives $(1+x)^a - 1 \leq 0$. Therefore, we obtain

$$ax \geq g(x; a, L)[(1+x)^a - 1].$$

This is equivalent to the desired reverse inequality, i.e.,

$$1 + ax \geq 1 + g(x; a, L)[(1+x)^a - 1].$$

Case 2:: For any $x \geq 0$ and $a \in (0, 1)$, we have $(1+x)^{a-1} \leq 1$ and $(1+x)^a \geq 1$ at the same time. The inequality $(1+x)^{a-1} \leq 1$ implies that $(1+x)^a \leq 1+x$, and, for any integer $k \geq 1$, $1+(1+x)^{2^{-k}} \geq 1+(1+x)^{a2^{-k}} > 0$, so that

$$\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}} \geq 1.$$

Owing to Proposition 2.1 and the use of $g(x;a,L)$, we get

$$\frac{ax}{(1+x)^a-1}=\prod_{k=1}^{+\infty}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}}\geq\prod_{k\in K}\frac{1+(1+x)^{2^{-k}}}{1+(1+x)^{a2^{-k}}}\geq g(x;a,L).$$

In addition, the inequality $(1+x)^a\geq 1$ yields $(1+x)^a-1\geq 0$. As a result, we have

$$ax\geq g(x;a,L)[(1+x)^a-1],$$

which gives

$$1+ax\geq 1+g(x;a,L)[(1+x)^a-1],$$

The reverse version of the proposed generalization is demonstrated. The proof is complete. □

As sketched in the proof, by applying Theorem 3.1 with $L=\{0\}$, we have $g(x;a,L)=1$ and we refind Theorem 1.1 (and Proposition 2.4). In order to highlight the originality of our theorem, let us focus on the special case $L=\{1\}$. In this simple case, we have

$$g(x;a,K)=\frac{1+\sqrt{1+x}}{1+(1+x)^{a/2}}$$

By applying Theorem 3.1, for any $a\in\mathbb{R}\setminus(0,1)$ and $x>-1$, we establish that

$$[(1+x)^a-1]g(x;a,L)+1\geq 1+ax,$$

which can be expressed as

$$[(1+x)^a-1]\frac{1+\sqrt{1+x}}{1+(1+x)^{a/2}}+1\geq 1+ax.$$

Upon simplification, it can be rearranged as

$$[(1+x)^{a/2}-1][1+\sqrt{1+x}]\geq ax.$$

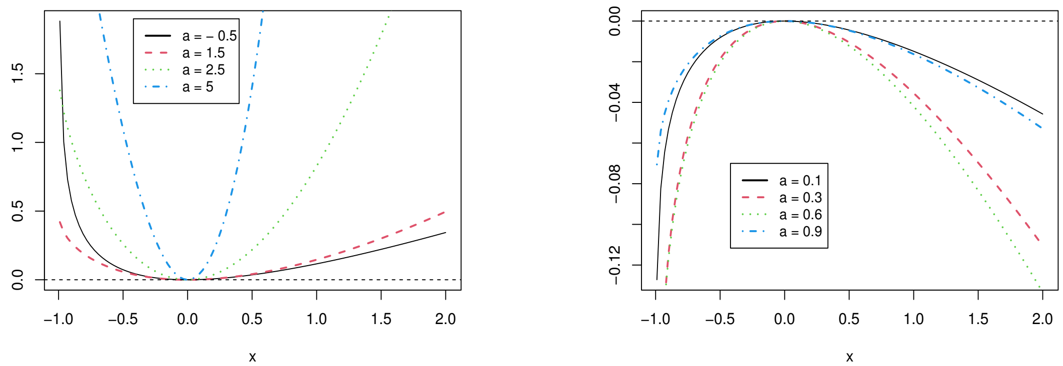
This inequality is reversed for $a\in(0,1)$, i.e.,

$$[(1+x)^{a/2}-1][1+\sqrt{1+x}]\leq ax.$$

To the best of our knowledge, these inequalities are new to the literature. Let us now illustrate them. For any $a\in\mathbb{R}$ and $x>-1$, we set

$$h(x;a)=[(1+x)^{a/2}-1][1+\sqrt{1+x}]-ax$$

Figure 2 displays the plots of $h(x;a)$ for several arbitrary values for a.



(a) (b)

FIGURE 2. Plots of $h(x;a)$ for (a) $a=-0.5$, $a=1.5$, $a=2.5$ and $a=5$, and (b) $a=0.1$, $a=0.3$, $a=0.6$ and $a=0.9$ for $x\in(-1,2]$.

We observe that $h(x;a)\geq 0$ for the values of a satisfying $a\in\mathbb{R}\setminus(0,1)$ and $x>-1$, and $h(x;a)\leq 0$ for the values of a satisfying $a\in(0,1)$ and $x>-1$.

This is just one example of specific inequalities derived from our results; many more can be derived from, among others, Theorem 3.1.

4. Conclusion

In this article, we develop an infinite product expansion approach to prove the Bernoulli inequality with a real exponent, as well as new variants. Specifically, after manipulating the logarithmic function, a new infinite product expansion is derived from the Seidel formula. We then show how this result can be used to establish the desired Bernoulli inequality. We also discuss possible improvements to the proof. A general result in this sense is also established. The originality and generality of the approach can be a source of inspiration to go further in this direction. In particular, it is possible that the product series expansion found for the q-number may be useful in combinatorics and other fields. We thus make a contribution to rehabilitating the use of product series expansions in the derivation of various key properties, including sharp inequalities.

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Appendix: Known proof of Theorem 2.1

We proceed with an iterative approach. First, we have $t-1=(1+t^{2^{-1}})(t^{2^{-1}}-1)$. Now, let us notice that $t^{2^{-1}}-1=(1+t^{2^{-2}})(t^{2^{-2}}-1)$, and a similar product decomposition holds for $t^{2^{-2}}-1$, etc. As a result, for any integer $m \geq 1$, we establish that

$$t-1=\left[\prod_{k=1}^{m}(1+t^{2^{-k}})\right](t^{2^{-m}}-1)=\left[\prod_{k=1}^{m}\frac{1+t^{2^{-k}}}{2}\right]2^m(t^{2^{-m}}-1).$$

Therefore, we have

$$2^m(t^{2^{-m}}-1)=(t-1)\prod_{k=1}^{m}\frac{2}{1+t^{2^{-k}}}.$$

By applying $m\to+\infty$, we obtain

$$\log(t)=\lim_{m\to+\infty}2^m(t^{2^{-m}}-1)=(t-1)\left[\lim_{m\to+\infty}\prod_{k=1}^{m}\frac{2}{1+t^{2^{-k}}}\right]=(t-1)\prod_{k=1}^{+\infty}\frac{2}{1+t^{2^{-k}}}.$$

The Seidel formula is proved. Further details or alternative proofs can be found in [8, page 354] and [3].

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