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GEOMETRIC CONSTRAINTS FOR REGULARITY OF 3D FRACTIONAL MAGNETO-MICROPOLAR SYSTEM

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Abstract. We set up two new geometric constraints: one on the vorticity for the Beale-Kato-Majda type result and the other on the gradient velocity. These are important for keeping the weak solutions to the 3D incompressible fractional magneto-micropolar system from blowing up over a finite time interval. Due to fractional operators and the blow-up of weak solutions, it is worthwhile to study such systems to obtain their finite-time regularity. We use the energy methods to obtain energy bounds for the vorticity and gradient velocity, ensuring smoothness in the interval $[0, T]$.

1. Introduction of the problem

The Cauchy problem of a generalized incompressible magneto-micropolar system is considered in the whole 3D domain, which is $\mathbb{R}^3$, as well as in the finite time domain, which is $[0, T]$:

$$\begin{cases} \mathcal{U}_t + \mathcal{U}\cdot\nabla\mathcal{U} + (-\Delta)^{\alpha}\mathcal{U} + \nabla\psi - \mathcal{V}\cdot\nabla\mathcal{V} - 2(\nabla\times\mathcal{W}) = 0, \\ \mathcal{W}_t + \mathcal{U}\cdot\nabla\mathcal{W} + (-\Delta)^{\beta}\mathcal{W} - \nabla\mathrm{div}\mathcal{W} + 4\mathcal{W} - 2(\nabla\times\mathcal{U}) = 0, \\ \mathcal{V}_t + \mathcal{U}\cdot\nabla\mathcal{V} - \mathcal{V}\cdot\nabla\mathcal{U} + (-\Delta)^{\gamma}\mathcal{V} = 0, \\ \nabla\cdot\mathcal{U} = 0, \quad \nabla\cdot\mathcal{V} = 0, \\ (\mathcal{U}, \mathcal{W}, \mathcal{V})|_{t=0} = (\mathcal{U}_0, \mathcal{W}_0, \mathcal{V}_0), \end{cases} \tag{1.1}$$

where $\mathcal{U}$, $\mathcal{W}$, and $\mathcal{V}$ are the velocity field, micro-rotational velocity, and magnetic field, respectively, while $\psi(x, t)$ is the scalar pressure. Throughout the paper, $\Omega = \nabla\times\mathcal{U}$ denotes the vorticity, generic constant C could change from line to line, and parameters $\alpha, \beta, \gamma \geq 1$. The divergence-free conditions are satisfied by velocity and magnetic field, with initial conditions $\mathcal{U}_0, \mathcal{W}_0$, and $\mathcal{V}_0$ given for velocity, micro-rotation and magnetic field. The Zygmund operator $\Pi = (-\Delta)^{\frac{1}{2}}$ is defined in terms of the Fourier transform:

$$\widehat{\Pi^i f(\xi)} = |\xi|^i \widehat{f(\xi)}, \quad \forall \ \ i \geq 0.$$

Key words and phrases. 3D fractional magneto-micropolar system, Geometric constraints, Weak solutions, Beale-Kato-Majda type regularity criteria, Besov spaces.

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Due to the vast applications of fractional partial differential equations (PDEs) [9] in modeling more complex problems of diffusion in geological and biological systems, signal processing, and long-term memory, it is pertinent to study the fractional magneto-micropolar system (1.1). Fractional PDEs also model the spread of disease and the transport of pollutants in biomedical engineering and environmental sciences. Similarly, system (1.1) is used to study damping and optimization problems in magnetic fluids and industrial processes. One more vital application of the fractional magneto-micropolar system is to model microfluid particles suspended in fluids.

In the earlier nineteenth century, with the introduction of Sobolev spaces, weak solutions, and other harmonic analysis tools, the mathematical theory of partial differential equations advanced effectively. Although the understanding of the differentiability properties of solutions has improved, the problem of the regularity of weak solutions, whose existence is only known, remains open and is considered one of the millennium problems [6]. In terms of velocity, Serrin was the first person to show that the Leray-Hopf weak solutions to the Navier-Stokes equations (NSE) are regular. Later on, results on the other geometric constraints, i.e., pressure, vorticity, and gradient velocity, were also presented (see [4, 5, 16]). System (1.1) is connected to the Navier-Stokes equations with fractional micro-rotation and magnetic diffusion. This enables us to address complex problems such as the movement of liquid crystals, ferromagnetic particles, animal blood flow, and diluted water. There has been a lot of research about the well-posedness problem for the mathematical models (NSE, micropolar systems, magnetohydrodynamic systems, and magneto-micropolar systems) that are part of the system (1.1), and many new results have been shown for them. Recently, Deng and Shang [2] showed the global existence of systems (1.1) using Lebesgue and Sobolev spaces. Fan and Zhong [3] showed the local existence and uniqueness of the system (1.1) and also proved the following geometric constraints for the velocity and gradient velocity that keep the solutions of the given system from blowing up in the given time interval:

$$\begin{aligned} \mathcal{U} &\in L^{\frac{2s}{s-r}}(0,T;\dot{X}_{r,s-1}),\\ \nabla\mathcal{U} &\in L^{\frac{2s}{s-r}}(0,T;\dot{X}_{r}),\\ \nabla\mathcal{U} &\in L^{\frac{2s}{s-r}}(0,T;\dot{X}_{r,s}),\\ \nabla\mathcal{U} &\in L^{1}(0,T;\dot{B}^{0}_{\infty,\infty}), \end{aligned} \tag{1.2}$$

for $0 < r < s \leq \frac{5}{4}$.

Remark 1.1. When $\mathcal{V} = 0$, system (1.1) becomes fractional micropolar system; when $\mathcal{W} = 0$ and $\mathcal{V} = 0$, system (1.1) reduces to fractional NSE.

Recently, Qiu et al. [19] showed the existence of system (1.1) for higher dimensions in the Besov spaces while Yan and Chen [18] proved the following two new results in the Triebel Lizorkin space

$$\psi \in L^{r}(0,T;\dot{F}^{0}_{s,\frac{10s}{5s+6}},$$

for $\frac{2}{r} + \frac{3}{s} = 1 + \frac{9}{5s}$, $\frac{12}{5} < s < 4$,
and

$$\nabla\psi \in L^r(0, T; \dot{F}^0_{s, \frac{8s}{12-3s}},$$

for $\frac{2}{r} + \frac{3}{s} = \frac{11}{4}$, $\frac{12}{11} < s < 4$.

Although numerous well-posedness and regularity results are proven for other systems, the detailed study of the local and global regularity of the system (1.1) clearly lacks. In this article, we aim to address this gap by establishing more stringent geometric constraints for vorticity and gradient velocity, which are crucial for managing the blow-up of any three-dimensional system within a finite time interval. These results are novel and proved in more general and regular Besov function spaces.

Remark 1.2. In this research work, we will focus on improving the condition (1.2) and presenting the new vorticity constraint.

2. Main Results and Their Proofs

First, we present some definitions, inequalities, and propositions useful to prove our results.

Definition 2.1. A function $f \in BMO$, if $\|f\|_{BMO} < \infty$, where

$$\|f\|_{BMO} = \sup_B \frac{1}{m(B)} \int_B |f(x) - f_B| dx,$$

and

$$f_B = \frac{1}{m(B)} \int_B f(x) dx \text{ with } B \text{ a ball over } \mathbb{R}^n.$$

Definition 2.2. [1] Suppose $\forall\, l, \tau \in \mathbb{R}$ and $p, q \in]0, \infty]$, then the homogeneous Besov spaces are the set of all $f \in S'_\infty$. Such that

$$\|f\|_{\dot{B}^{l,\tau}_{p,q}} := \sup_{k\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} 2^{k\sigma\tau} \Big(\sum_{j\geq k} \Big(2^{lj} \|Q_j f\|_{L_p(P_k,\nu)} \Big)^q \Big)^{\frac{1}{q}} < \infty.$$

Proposition 2.1. [2] *Let* $(\mathcal{U}, \mathcal{W}, \mathcal{V})$ *be weak solution of the system (1.1). Then* $\forall\, t \in [0, T]$

$$\|\Pi^s \mathcal{W}\|^2_{L^2} + \|\Pi^s \mathcal{W}\|^2_{L^2} + \|\Pi^s \mathcal{W}\|^2_{L^2} + \int_0^T \left(\|\Pi^{s+\alpha} \mathcal{U}\|^2_{L^2} + \|\Pi^{s+\beta} \mathcal{W}\|^2_{L^2} + \|\Pi^{s+\gamma} \mathcal{V}\|^2_{L^2} \right) dt \leq C. \tag{2.1}$$

In order to prove our results, we will use the following inequalities given in [7, 10, 11, 13], *for* $1 \leq r < \infty$ *and* $s > \frac{5}{2}$ *we have that*

$$\|f\|^2_{L^{2r}} \leq C \|f\|_{L^r} \|f\|_{BMO} \tag{2.2}$$

$$\|f\|_{BMO} \leq C\big(1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \log^{\frac{1}{2}}(1 + \|f\|_{H^{s-1}})\big) \tag{2.3}$$

$$\|(\mathcal{V}\cdot\nabla)\mathcal{V}\|_{L^r} \leq C \|\mathcal{V}\|_{L^r} \|\nabla\mathcal{V}\|_{BMO}$$

$$\|\nabla\mathcal{U}\|^2_{L^4} \leq C \|\Delta\mathcal{U}\|_{L^2} \|\nabla\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}. \tag{2.4}$$

For a detailed discussion of the weak solutions and function spaces, see [15].

2.1. **Beale-Kato-Majda type regularity result.** Next, we establish geometric constraints of the Beale-Kato-Majda type by using intricate energy methods.

Theorem 2.1. *Let* $(\mathcal{U}_0, \mathcal{W}_0, \mathcal{V}_0) \in H^n(\mathbb{R}^3)$ *with* $n > \frac{5}{2}$ *and* $\nabla \cdot \mathcal{U}_0 = 0$, $\nabla \cdot \mathcal{V}_0 = 0$ *in distributional sense. If a weak solution* $(\mathcal{U}, \mathcal{W}, \mathcal{V})$ *of system* (1.1) *satisfies the following constraint for vorticity*

$$\int_0^T \frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{(1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}} dt < \infty, \tag{2.5}$$

then it retains its smoothness in $[0,T]$, *until it blows-up at* $T = T^*$.

Proof. Firstly, testing $(1)_1$ with $\Delta\mathcal{U}$, and using the divergence-free properties of velocity and magnetic field, we have that

$$\begin{aligned} &\int_{\mathbb{R}^3} \mathcal{U}_t \cdot \Delta\mathcal{U}\, dx + \int_{\mathbb{R}^3} \mathcal{U}\cdot\nabla\mathcal{U}\cdot\Delta\mathcal{U}\, dx + \int_{\mathbb{R}^3} (-\Delta)^\alpha \mathcal{U}\cdot\Delta\mathcal{U}\, dx + \int_{\mathbb{R}^3} \nabla\psi\cdot\Delta\mathcal{U}\, dx \\ &- \int_{\mathbb{R}^3} \mathcal{V}\cdot\nabla\mathcal{V}\cdot\Delta\mathcal{U}\, dx - \int_{\mathbb{R}^3} 2(\nabla\times\mathcal{W})\cdot\Delta\mathcal{U}\, dx = 0, \\ &\frac{1}{2}\frac{d}{dt}\|\nabla\mathcal{U}\|^2_{L^2} + \|\Pi^{1+\alpha}\mathcal{U}\|^2_{L^2} = \int_{\mathbb{R}^3} \mathcal{V}\cdot\nabla\mathcal{V}\cdot\Delta\mathcal{U}\, dx - \int_{\mathbb{R}^3} \mathcal{U}\cdot\nabla\mathcal{U}\cdot\Delta\mathcal{U}\, dx. \\ &= X_1 + X_2. \end{aligned} \tag{2.6}$$

Secondly, testing $(1)_2$ with $\Delta\mathcal{W}$, and using the divergence-free properties, we get that

$$\begin{aligned} &\int_{\mathbb{R}^3} \mathcal{W}_t\cdot\Delta\mathcal{W}\, dx + \int_{\mathbb{R}^3} \mathcal{U}\cdot\nabla\mathcal{W}\cdot\Delta\mathcal{W}\, dx + \int_{\mathbb{R}^3} (-\Delta)^\beta\mathcal{W}\cdot\Delta\mathcal{W}\, dx - \int_{\mathbb{R}^3} \nabla\mathrm{div}\mathcal{W}\cdot\Delta\mathcal{W}\, dx \\ &+ \int_{\mathbb{R}^3} 4\mathcal{W}\cdot\Delta\mathcal{W}\, dx - \int_{\mathbb{R}^3} 2(\nabla\times\mathcal{U})\cdot\Delta\mathcal{W}\, dx = 0 \\ &\frac{1}{2}\frac{d}{dt}\|\nabla\mathcal{W}\|^2_{L^2} + \|\Pi^{1+\beta}\mathcal{W}\|^2_{L^2} + 4\|\nabla\mathcal{W}\|^2_{L^2} + \|\nabla\mathrm{div}\mathcal{W}\|^2_{L^2} = -\int_{\mathbb{R}^3} \mathcal{U}\cdot\nabla\mathcal{W}\cdot\Delta\mathcal{W}\, dx \\ &+ \int_{\mathbb{R}^3} 2(\nabla\times\mathcal{U})\cdot\Delta\mathcal{W}\, dx \\ &= X_3 + X_4. \end{aligned} \tag{2.7}$$

Thirdly, testing $(1)_3$ with $\Delta\mathcal{V}$, and employing the divergence-free conditions, results in

$$\begin{aligned} &\int_{\mathbb{R}^3} \mathcal{V}_t\cdot\Delta\mathcal{V}\, dx + \int_{\mathbb{R}^3} \mathcal{U}\cdot\nabla\mathcal{V}\cdot\Delta\mathcal{V}\, dx - \int_{\mathbb{R}^3} \mathcal{V}\cdot\nabla\mathcal{U}\cdot\Delta\mathcal{V}\, dx + \int_{\mathbb{R}^3} (-\Delta)^\gamma\mathcal{V}\cdot\Delta\mathcal{V}\, dx = 0 \\ &\frac{1}{2}\frac{d}{dt}\|\nabla\mathcal{V}\|^2_{L^2} + \|\Pi^{1+\gamma}\mathcal{V}\|^2_{L^2} = -\int_{\mathbb{R}^3} \mathcal{U}\cdot\nabla\mathcal{V}\cdot\Delta\mathcal{V}\, dx + \int_{\mathbb{R}^3} \mathcal{V}\cdot\nabla\mathcal{U}\cdot\Delta\mathcal{V}\, dx \\ &= X_5 + X_6. \end{aligned} \tag{2.8}$$

Now, adding (2.6), (2.7), and (2.8), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla\mathcal{U}\|^2_{L^2} + \|\nabla\mathcal{W}\|^2_{L^2} + \|\nabla\mathcal{V}\|^2_{L^2}) + (\|\Pi^{1+\alpha}\mathcal{U}\|^2_{L^2} + \|\Pi^{1+\beta}\mathcal{W}\|^2_{L^2} + \|\Pi^{1+\gamma}\mathcal{V}\|^2_{L^2}) \\ &+4\|\nabla\mathcal{W}\|^2_{L^2} + \|\nabla\mathrm{div}\mathcal{W}\|^2_{L^2} = \sum_{i=1}^{6} X_i. \end{aligned} \tag{2.9}$$

Next, we need an estimate for each of X_i appearing in (2.9). For X_2, we derive

$$|X_2| \leq \int_{\mathbb{R}^3} |\nabla \mathcal{U}|^3 \leq C\|\nabla \mathcal{U}\|_{L^2}\|\nabla \mathcal{U}\|_{L^4}^2.$$

Here, employing (2.2), we get that

$$|X_2| \leq C\|\nabla \mathcal{U}\|_{BMO}\|\nabla \mathcal{U}\|_{L^2}^2. \tag{2.10}$$

Using (2.3) and the following inequality:

$$\|\nabla \mathcal{U}\|_{\dot{B}^0_{\infty,\infty}} \leq C\|\Omega\|_{\dot{B}^0_{\infty,\infty}},$$

the above inequality (2.10) yields

$$\begin{aligned} |X_2| &\leq C\|\nabla \mathcal{U}\|_{L^2}^2\Big(1 + \frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}}\Big)\log(e+\|\mathcal{U}\|_{H^s}) \\ &\leq C\|\nabla \mathcal{U}\|_{L^2}^2\Big(1 + \frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}}\Big)\log(e+\lambda(t)), \end{aligned} \tag{2.11}$$

where $\lambda(t) := \sup_{T_* \leq s \leq t}\|\Pi^s \mathcal{U}\|_{L^2}^2 + \|\Pi^s \mathcal{W}\|_{L^2}^2 + \|\Pi^s V\|_{L^2}^2$. For the detailed reasoning and proven bounds on $s = 3$, see [14].

Next, we get bounds for X_1

$$|X_1| \leq \int_{\mathbb{R}^3} |\nabla \mathcal{V}|^2 \cdot \nabla \mathcal{U}\, dx.$$

Now, using Hölder's inequality with $\frac{1}{2} + \frac{5}{12} + \frac{1}{12} = 1$,

$$|X_1| \leq C\|\nabla \mathcal{U}\|_{L^2}\|\nabla \mathcal{V}\|_{L^{12}}\|\nabla \mathcal{V}\|_{L^{\frac{12}{5}}},$$

which in view of Sobolev inequalities gives

$$|X_1| \leq C\|\nabla \mathcal{U}\|_{L^2}^2 + C\|\Pi^\gamma \mathcal{V}\|_{L^2}^2 + C\|\Pi^\gamma \mathcal{V}\|_{L^2}^2\|\nabla \mathcal{U}\|_{L^2}^2 + C\|\Pi^{1+\gamma}\mathcal{V}\|_{L^2}^2 + C. \tag{2.12}$$

Following the same steps as taken for X_1, the estimates for X_5, X_6, and X_3, are given as

$$|X_5| = |X_6| \leq C\|\nabla \mathcal{U}\|_{L^2}^2 + C\|\Pi^\gamma \mathcal{V}\|_{L^2}^2 + C\|\Pi^\gamma \mathcal{V}\|_{L^2}^2\|\nabla \mathcal{U}\|_{L^2}^2 + C\|\Pi^{1+\gamma}\mathcal{V}\|_{L^2}^2 + C \tag{2.13}$$

$$|X_3| \leq C\|\nabla \mathcal{U}\|_{L^2}^2 + C\|\Pi^\beta \mathcal{W}\|_{L^2}^2 + C\|\Pi^\beta \mathcal{W}\|_{L^2}^2\|\nabla \mathcal{U}\|_{L^2}^2 + C\|\Pi^{1+\beta}\mathcal{W}\|_{L^2}^2 + C. \tag{2.14}$$

Lastly, for X_4, we obtain

$$\begin{aligned} |X_4| &\leq C\|\nabla \mathcal{U}\|_{L^2}\|\Delta \mathcal{W}\|_{L^2} \\ &\leq C\Big(\|\nabla \mathcal{U}\|_{L^2}^2 + \|\nabla \mathcal{W}\|_{L^2}^2 + \|\Pi^{1+\beta}\mathcal{W}\|_{L^2}^2\Big). \end{aligned} \tag{2.15}$$

Putting (2.11), (2.12), (2.13), (2.14), and (2.15) in (2.9), after arranging and conserving norms we have that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla \mathcal{U}\|_{L^2}^2 + \|\nabla \mathcal{W}\|_{L^2}^2 + \|\nabla \mathcal{V}\|_{L^2}^2) + (\|\Pi^{1+\alpha}\mathcal{U}\|_{L^2}^2 + \|\Pi^{1+\beta}\mathcal{W}\|_{L^2}^2 + \|\Pi^{1+\gamma}\mathcal{V}\|_{L^2}^2) \\ &+4\|\nabla \mathcal{W}\|_{L^2}^2 + \|\nabla \mathrm{div} \mathcal{W}\|_{L^2}^2 \\ &\leq C\|\nabla \mathcal{U}\|_{L^2}^2\Big(1 + \frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}}\Big)\log(e+\lambda(t)) + C(\|\nabla \mathcal{U}\|_{L^2}^2 + \|\nabla \mathcal{W}\|_{L^2}^2) \\ &\quad +C\|\Pi^\gamma \mathcal{V}\|_{L^2}^2\|\nabla \mathcal{U}\|_{L^2}^2 + C\|\Pi^\gamma \mathcal{V}\|_{L^2}^2 + C + C\|\Pi^\beta \mathcal{W}\|_{L^2}^2 + C\|\Pi^\beta \mathcal{W}\|_{L^2}^2\|\nabla \mathcal{U}\|_{L^2}^2 + C. \end{aligned}$$

Continuing the same methodology as above, we apply the Gronwall's inequality to obtain

$$\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{W}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2+\int_0^T\|\Pi^{\alpha}\mathcal{U}\|_{L^2}^2+\|\Pi^{\beta}\mathcal{W}\|_{L^2}^2+\|\Pi^{\gamma}\mathcal{V}\|_{L^2}^2+\|\Pi^{1+\alpha}\mathcal{U}\|_{L^2}^2$$
$$+\|\Pi^{1+\beta}\mathcal{W}\|_{L^2}^2+\|\Pi^{1+\gamma}\mathcal{V}\|_{L^2}^2\,dt$$
$$\leq C_0\exp\Big(C\int_{T_*}^T\Big(1+\frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}}\Big)dt(1+\ln(e+\lambda(t))\Big)$$
$$\leq C_0(e+\lambda(t))^{C\epsilon}, \tag{2.16}$$

Where $C_0=\Big(\|\nabla\mathcal{U}(\cdot,T_*)\|_{L^2}^2+\|\nabla\mathcal{W}(\cdot,T_*)\|_{L^2}^2+\|\nabla\mathcal{V}(\cdot,T_*)\|_{L^2}^2+1\Big)$ and for infinitesimally small constant $\epsilon>0$, $\exists\, T_*<T$, such that

$$\int_{T_*}^T\Big(1+\frac{\|\Omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log(e+\|\Omega\|_{\dot{B}^0_{\infty,\infty}})}}\Big)dt<\epsilon.$$

The bounds for $\lambda(t)$, and for $\Big(\|\Pi^{\alpha}\mathcal{U}\|_{L^2}^2+\|\Pi^{\beta}\mathcal{W}\|_{L^2}^2+\|\Pi^{\gamma}\mathcal{V}\|_{L^2}^2+\|\Pi^{1+\alpha}\mathcal{U}\|_{L^2}^2+\|\Pi^{1+\beta}\mathcal{W}\|_{L^2}^2+\|\Pi^{1+\gamma}\mathcal{V}\|_{L^2}^2\Big)$ are elaborately proved in [3].

For the L^2 bounds, taking inner product of $\mathcal{U}$ with $(1)_1$, $\mathcal{W}$ with $(1)_2$, and $\mathcal{V}$ with $(1)_3$, estimating all the terms and adding, it is easy to see we get that

$$\|\mathcal{U}\|_{L^2}^2+\|\mathcal{W}\|_{L^2}^2+\|\mathcal{V}\|_{L^2}^2+\int_0^T(\|\Pi^{\alpha}\mathcal{U}\|_{L^2}^2+\|\Pi^{\beta}\mathcal{W}\|_{L^2}^2+\|\Pi^{\gamma}\mathcal{V}\|_{L^2}^2)dt\leq C. \tag{2.17}$$

Hence, (2.16) together with (2.1) and (2.17) results in the following inequality:

$$\|\nabla(\mathcal{U},\mathcal{W},\mathcal{V})\|_{L^2}^2\leq C.$$

By controlling one of the geometric constraints of vorticity in regular homogeneous Besov spaces we proved our solutions will keep their blow-up until reaching singular time. □

2.2. **Gradient velocity regularity result.** This section is concerned with demonstrating gradient velocity regularity in the critical Besov spaces.

Theorem 2.2. *Suppose* $(\mathcal{U}_0,\mathcal{W}_0,\mathcal{V}_0)\in H^n(\mathbb{R}^3)$ *with* $n>\frac{5}{2}$ *and* $\nabla\cdot\mathcal{U}_0=0$, $\nabla\cdot\mathcal{V}_0=0$ *in distributional sense. If a weak solution* $(\mathcal{U},\mathcal{W},\mathcal{V})$ *of system (1.1) satisfies the following condition for gradient velocity*

$$\int_0^T\frac{\|\nabla\mathcal{U}\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\{}1+\log(e+\|\nabla\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}})\}dt<\infty, \tag{2.18}$$

then it retains its regularity in the interval $\mathbb{R}^3\times[0,T]$.

Proof. To prove result (2.18), we continue from equation (2.9). Using (2.4), we get new estimates for X_2

$$|X_2|\leq C\|\nabla\mathcal{U}\|_{L^2}\|\nabla\mathcal{U}\|_{L^4}^2$$
$$\leq C\|\nabla\mathcal{U}\|_{L^2}^2\|\nabla\mathcal{U}\|^2_{\dot{B}^{-1}_{\infty,\infty}}+\|\Delta\mathcal{U}\|_{L^2}^2.$$

All the other estimates X_i's are evaluated in a way similar to the previous result. Putting all the new estimates in (2.9) and following on the same steps as for (2.16), we have that

$$\begin{aligned}\|\nabla\mathcal{U}\|_{L^2}^2 + \|\nabla\mathcal{W}\|_{L^2}^2 + \|\nabla\mathcal{V}\|_{L^2}^2 + \int_0^T \|\Pi^\alpha\mathcal{U}\|_{L^2}^2 + \|\Pi^\beta\mathcal{W}\|_{L^2}^2 + \|\Pi^\gamma\mathcal{V}\|_{L^2}^2 + \|\Pi^{1+\alpha}\mathcal{U}\|_{L^2}^2 \\ +\|\Pi^{1+\beta}\mathcal{W}\|_{L^2}^2 + \|\Pi^{1+\gamma}\mathcal{V}\|_{L^2}^2 \, dt \\ \leq C_0 \exp\Big(C\int_{T_*}^T \Big(1 + \frac{\|\nabla\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{1+\log(e+\|\nabla\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}})}\Big)dt(1+\ln(e+\lambda(t))\Big) \\ \leq C\exp(C_0\epsilon\ln(e+\lambda(t))) \leq C_0(e+\lambda(t))^{C\epsilon}. \qquad (2.19)\end{aligned}$$

Whereas, for $\epsilon > 0$, $\exists\, T_* < T$, such that

$$\int_{T_*}^T \Big(1 + \frac{\|\nabla\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{1+\log(e+\|\nabla\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}})}\Big)dt < \epsilon.$$

Now, (2.19) together with (2.18) shows that for a finite time the solution remains regular. □

Corollary 2.1. *The fact* $\|\nabla\mathcal{U}\|_{\dot{B}^{-1}_{\infty,\infty}} \approx \|\mathcal{U}\|_{\dot{B}^{0}_{\infty,\infty}}$ *implies the new improved result for (1.1) via velocity constraint*

$$\int_0^T \frac{\|\mathcal{U}\|_{\dot{B}^{0}_{\infty,\infty}}^2}{1+log(e+\|\mathcal{U}\|_{\dot{B}^{0}_{\infty,\infty}})}dt < \infty. \qquad (2.20)$$

3. Conclusions

This work employs the functional theoretical approach to the system (1.1) to prove two new geometric constraints. Considering its structural properties, these constraints are vital for analyzing the turbulence over a finite time interval for a possible blow-up. The results (2.5), (2.18), and (2.20) are proved in Besov spaces that are important due to their complexity and scale-invariant properties. Although NSE has been extensively analyzed for finite time regularity or partial regularity (the regularity of the singular sets), the detailed regularity and partial regularity analyses of the well-posedness of systems (1.1) are missing from the literature. Until now, a few regularity results for (1.1) in Lebesgue spaces, BMO, and Besov spaces have been obtained; the better results should be presented on this system. The proposed future problems related to the system (1.1) include the improved results of component reduction regularity in terms of velocity and pressure. We can improve the already proven results by simply presenting the regularity condition on one component of velocity, i.e., $\mathcal{U}_1$, in more regular function spaces. Similarly, we could improve the gradient velocity result in more general function spaces.

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