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A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED IN TERMS OF RUSCHEWEYH DERIVATIVE

MUMTAZ ALI[1] AND MUHAMMAD ARSHAD[2]

Abstract. In this paper, we have used Ruscheweyh differential operator in order to study another subclass of analytic functions $M(\alpha, \eta, \lambda)$ univalent in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| = 1\}$. We study some co-efficient inequalities for functions in the class which generalizes the already known results of coefficient inequalities considered earlier by Owa, Polatoglu and Yavuz for uniformly convex and starlike functions.

1. Introduction and preliminaries

Geometric function theory is a beautiful branch of complex analysis, comprising the study of the properties of normalized univalent functions. The use of differential and integral operators has further enlarged and enriched this field for study and research.

In the present paper, we use the idea of a differential operators introduced by Ruscheweyh [12]. The obtained results are generalizations of the classes of uniformly convex and starlike functions available in the literature.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For two functions f and $g \in A$, the convolution or Hadmard product of f and g is denoted by $(f * g)$ and is defined by

$$(f * g)(z) = z + \sum_{n=1}^{\infty} a_1 a_2.. \tag{1.2}$$

Definition 1.1. A function $f(z) \in A$ is said to be starlike in $\triangle$ if and only if $f'(0) \neq 0$ and $\Re e\left(\frac{zf'(z)}{f(z)}\right) > 0$. The class of starlike functions is denoted by S^* and is defined by

$$S^* = \left\{ f(z) \in S : \Re e\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \triangle \right\}.$$

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Definition 1.2. A function $f(z) \in A$ is said to be convex in $\triangle$ if and only if $f'(0) \neq 0$ and satisfy the inequality $\Re e\left(1+\frac{zf''(z)}{f'(z)}\right) > 0$.The class of convex functions is denoted by κ and is defined by

$$\kappa = \left\{f(z) \in A : \Re e\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, z \in \triangle\right\}.$$

Definition 1.3. A function $f(z) \in A$ is said to be starlike function of order α $(0 \leq \alpha \leq 1)$, denoted by $S^*(\alpha)$ if and only if $\Re e\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \triangle$. The class of starlike univalent functions of order α is defined by

$$S^*(\alpha) = \left\{f(z) \in A : \Re e\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \triangle\right\}.$$

Similarly, the class of convex functions of order α $(0 \leq \alpha \leq 1)$ satisfy the condition

$$\Re e\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha,$$

and is given by

$$\kappa(\alpha) = \left\{f(z) \in A : \Re e\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \triangle\right\}.$$

Definition 1.4. A function $f(z) \in A$ is said to be in UCV, the class of uniformly convex functions, if it satisfies the inequality

$$\Re e\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \left|\frac{zf''(z)}{f'(z)}\right|, z \in \triangle.$$

Likewise, a function $f(z) \in A$ is said to be in UST, the class of uniformly starlike functions, if it satisfies the inequality

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \triangle.$$

Definition 1.5. ([4], [5]) A function $f(z)$ is said to be in the class of uniformly starlike function of order $\alpha, UST(\alpha)$, if it satisfies the inequality

$$\Re e\left(\frac{zf'(z)}{f(z)} - \alpha\right) \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \triangle, -1 \leq \alpha \leq 1.$$

Similarly, a function $f(z)$ is said to be in the class $UCV(\alpha)$, if and only if $zf'(z) \in UST(\alpha)$ and satisfy

$$\Re e\left(1+\frac{zf''(z)}{f'(z)} - \alpha\right) \geq \left|\frac{zf''(z)}{f'(z)}\right|, z \in \triangle, -1 \leq \alpha \leq 1.$$

Over the last few years, many authors have generalized and extended the already known classes of uniformly convex and starlike functions and have succeeded in developing new subclasses of univalent functions. For example, the class $v_k^\lambda(\beta, b, \delta)$ was introduced by Latha and Nanjunda Rao [8]. Similarly, Shams, Kulkarni and Jehangiri [9] collectively introduced the classes $SD(\alpha, \beta)$ and $KD(\alpha, \beta)$ as the subclass of A consisting of functions satisfying the inequalities

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) > \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta, z \in \triangle,$$

and

$$\Re e\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha\left|\frac{zf''(z)}{f'(z)}\right| + \beta, z \in \triangle,$$

for $\alpha \geq 0$ and $\beta\,(0 \leq \beta < 1)$.

For $f \in A$ and $\lambda \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$, Ruscheweyh in [12] introduced the following nth order derivative operator

$$D^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) = \left(z + \sum_{n=2}^{\infty} \frac{(n+\lambda-1)!}{\lambda!(n-1)!} z^n \right) * \left(z + \sum_{n=2}^{\infty} a_n z^n \right),$$

or

$$\begin{aligned} D^{\lambda} f(z) &= z + \sum_{n=2}^{\infty} \frac{(n+\lambda-1)!}{\lambda!(n-1)!} a_n z^n, \\ &= z + \sum_{n=2}^{\infty} c_n(\lambda) a_n z^n, \text{ where } c_n(\lambda) = \frac{(n+\lambda-1)!}{\lambda!(n-1)!}, n \geq 2, \lambda \geq 0. \end{aligned} \tag{1.3}$$

For $\lambda = 0, 1, 2, ...$ we can write

$$\begin{aligned} &D^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n \\ &D^1 f(z) = z f'(z) = z + \sum_{n=2}^{\infty} n a_n z^n \\ &... \\ &(\lambda+1) D^{\lambda+1} f(z) = z \left(D^{\lambda} f(z) \right)' + \lambda D^{\lambda} f(z). \end{aligned}$$

Then the operator $D^{\lambda} f(z)$ is called Ruscheweyh derivative operator. The Ruscheweyh derivative provides an important tool for the generalization of various classes of univalent functions. Using Ruscheweyh derivative operator $D^{\lambda} f(z)$ we can generalize the well known classes of uniformly convex and starlike functions.

Definition 1.6. We say that a function $f \in A$ is in the class $M(\alpha, \eta, \lambda)$ if it satisfies the inequality

$$\Re \left\{ \frac{D^{\lambda+1} f(z)}{D^{\lambda} f(z)} \right\} > \alpha \left| \frac{D^{\lambda+1} f(z)}{D^{\lambda} f(z)} - 1 \right| + \eta,$$

for $\alpha \geq 0, 0 \leq \eta < 1$ and $\lambda \in \mathbb{N}_0$.

It is important to note that for particular values of the parameters α, η and λ, the class $M(\alpha, \eta, \lambda)$,includes several subclasses of univalent functions studied earlier:

1. $M(1, 0, 0) \equiv S_p$ ([5])
2.$M(1, \eta, 0) \equiv S_p(\eta)$ ([5])
3. $M(\alpha, \eta, 0) \equiv SD(\alpha, \eta)$ ([9])
4. $M(\alpha, \eta, 1) \equiv KD(\alpha, \eta)$ ([9])
5. $M(\alpha, \eta, 0) \equiv Sp(\alpha, \eta)$ ([4], [5])
6.$M(\alpha, \eta, 1) \equiv UCV(\alpha, \eta)$ ([4], [5])
7. $M(0, \eta, 0) \equiv S^*(\eta)$ ([11])
8. $M(0, \eta, 1) \equiv C(\eta)$ ([11])
9. $M\left(0, \frac{1}{2}, \lambda\right) \equiv K_n$ ([12])

2. Main results

In this section we obtain some coefficient inequalities for functions in the class $M(\alpha, \eta, \lambda)$.

Theorem 2.1. *If $f \in M(\alpha, \eta, \lambda)$ with $0 \leq \alpha \leq \eta$, then $f \in M\left(\frac{\eta-\alpha}{1-\alpha}, \lambda\right)$.*

Proof. Let $f \in M(\alpha, \eta, \lambda)$, then we can write

$$\Re \left\{ \frac{D^{\lambda+1} f(z)}{D^{\lambda} f(z)} \right\} > \alpha \left| \frac{D^{\lambda+1} f(z)}{D^{\lambda} f(z)} - 1 \right| + \eta, \ z \in D,$$

which in equivalent form can also be written as

$$\Re\left\{\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right\} > \frac{\eta-\alpha}{1-\alpha},$$

If $0 \leq \alpha \leq \eta$, then we have

$$0 \leq \frac{\eta-\alpha}{1-\alpha} < 1.$$

□

Corollary 2.1. *For $\lambda = 0$, we get Theorem 2.1 in [10] reads as: If $f(z) \in SD(\alpha, \eta)$ with $0 \leq \alpha \leq \eta$ or $\alpha > \frac{1+\eta}{2}$, then $f(z) \in S^{*}\left(\frac{\eta-\alpha}{1-\alpha}\right)$.*

Corollary 2.2. *For the parametric value $\lambda = 1$, we get Theorem 2.1 in [11] reads as: If $f(z) \in KD(\alpha, \eta)$ with $0 \leq \alpha \leq \eta$, then $f(z) \in K\left(\frac{\eta-\alpha}{1-\alpha}\right)$.*

The following lemma is important for the proof of our main result given below.

Lemma 2.1. [3]*If $p \in P$, then $|p_k| \leq 2$ for each $k \in \mathbb{N}$, where P is the family of all functions $p(z)$ analytic in D for which $\Re(p(z)) > 0$ and*

$$p(z) = 1 + p_1(z) + p_2(z) + ...$$

Theorem 2.2. *If $f \in SM(\alpha, \eta, \lambda)$ with $0 \leq \alpha \leq \eta$, then*

$$|a_2| \leq \frac{(1+\lambda) \times (1-\eta)}{(2+\lambda)|1-\alpha| B_n(\lambda)}, \tag{2.1}$$

and

$$|a_n| \leq \frac{(1+\lambda)\times(1-\eta)}{[2+(n-1)\lambda]|1-\alpha|B_n(\lambda)} \prod_{j=1}^{k-2}\left[1 + \frac{(1-\eta)(1+\lambda)}{(2+j\lambda)|1-\alpha|}\right], n \geq 3. \tag{2.2}$$

Proof. In view of $f(z) \in SM(\alpha, \eta, \lambda)$ and $0 \leq \alpha \leq \lambda$,we have

$$\Re\left\{\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right\} > \frac{\eta-\alpha}{1-\alpha}, z \in D.$$

We can define the function $p(z)$ by

$$p(z) = \frac{(1-\alpha)\left[\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right]-(\eta-\alpha)}{(1-\eta)}, z \in D. \tag{2.3}$$

Hence, $p(z)$ is analytic in Δ with $p(0) = 1$ and $\Re\{p(z)\} > 0$, $(z \in \Delta)$. If we let

$$p(z) = 1 + p_1 z + p_2 z^2 + \ ...$$

then, we can write

$$\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)} = 1 + \left(\frac{1-\eta}{1-\alpha}\right)\sum_{n=1}^{\infty} p_n z^n,$$

or

$$D^{\lambda+1}f(z) = D^{\lambda}f(z)\left(1 + \frac{1-\eta}{1-\alpha}\sum_{n=1}^{\infty} p_n z^n\right). \tag{2.4}$$

In equivalent form we have

$$D^{\lambda+1}f(z) = D^{\lambda}f(z)\,(1 + \frac{1-\eta}{1-\alpha}(p_1 z + p_2 z^2 + ...)$$

or

$$\left[\frac{z\left(D^{\lambda}f(z)\right)'}{(\lambda+1)} + \frac{\lambda}{(\lambda+1)}\left(D^{\lambda}f(z)\right)\right] = D^{\lambda}f(z)\,(1 + \frac{1-\eta}{1-\alpha}(p_1 z + p_2 z^2 + ...)\,. \tag{2.5}$$

From (2.4), we obtain

$$\begin{aligned} & z+\tfrac{2+\lambda}{1+\lambda}B_2\left(\lambda\right)a_2z^2+\tfrac{3+\lambda}{1+\lambda}B_3\left(\lambda\right)a_3z^3+... \\ & =z+\left(\tfrac{1-\eta}{1-\alpha}\right)\left[p_1z^2+\left(p_2+B_2\left(\lambda\right)p_1a_2\right)z^3+...\right], \end{aligned}$$

in equivalent form the above equality can be written as

$$z+\textstyle\sum_{n=2}^{\infty}\tfrac{2+(n-1)\lambda}{1+\lambda}B_n\left(\lambda\right)a_nz^n=z+\tfrac{1-\eta}{1-\alpha}\textstyle\sum_{n=2}^{\infty}\left(\sum_{\lambda=1}^{n-1}B_n\left(\lambda\right)p_{n-\lambda}a_nz^n\right). \tag{2.6}$$

Now, the equality of coefficients on both sides of z^n in (2.5) gives us

$$\tfrac{2+(n-1)\lambda}{1+\lambda}B_n\left(\lambda\right)a_n=\tfrac{1-\eta}{1-\alpha}\left(\textstyle\sum_{\lambda=1}^{n-1}B_n\left(\lambda\right)p_{n-\lambda}a_\lambda\right),$$

or

$$a_n=\tfrac{(1+\lambda)\times(1-\eta)}{[2+(n-1)\lambda](1-\alpha)B_n(\lambda)}\textstyle\sum_{\lambda=1}^{n-1}B_n\left(\lambda\right)p_{n-\lambda}a_\lambda.$$

and by Lemma 2.1 for $|p_n|\leq 2$, $(n\geq 1)$ we have

$$\begin{aligned} |a_n| & \leq\tfrac{(1+\lambda)\times(1-\eta)}{[2+(n-1)\lambda]|1-\alpha|B_n(\lambda)}\left[B_n\left(\lambda\right)|p_{n-\lambda}|\,|a_\lambda|\right] \\ & \leq\tfrac{2(1+\lambda)\times(1-\eta)}{[2+(n-1)\lambda]|1-\alpha|B_n(\lambda)}\textstyle\sum_{\lambda=1}^{n-1}B_n\left(\lambda\right)|a_\lambda|. \end{aligned} \tag{2.7}$$

For $n=2$, we have

$$|a_2|\leq\tfrac{2(1+\lambda)\times(1-\eta)}{(2+\lambda)|1-\alpha|B_2(\lambda)}.$$

Similarly, for $n=3$,

$$|a_3|\leq\tfrac{2(1-\eta)}{2|1-\alpha|B_3(\lambda)}\left[1+\tfrac{2(1-\eta)(1+\lambda)^2}{|1-\alpha|(2+\lambda)}\right].$$

Therefore, the coefficient inequality (2.2) holds for $n=2$ and $n=3$.

Suppose now, that the inequality (2.2) holds for $n=k$, that is

$$|a_k|\leq\tfrac{2(1+\lambda)\times(1-\eta)}{[2+(k-1)\lambda]|1-\alpha|B_k(\lambda)}\textstyle\prod_{j=1}^{k-2}\left[1+\tfrac{2(1-\eta)(1+\lambda)}{(2+j\lambda)|1-\alpha|}\right].$$

To prove for $n=k+1$, consider

$$\begin{aligned} & |a_{k+1}|\leq\tfrac{2(1+\lambda)\times(1-\eta)}{[(2+k)\lambda]|1-\alpha|B_{k+1}(\lambda)}\left(1+\tfrac{2(1+\lambda)\times(1-\eta)}{(2+\lambda)|1-\alpha|B_2(\lambda)}\right)+ \\ & \tfrac{2(1-\eta)}{2|1-\alpha|B_3(\lambda)}\left(1+\tfrac{2(1-\eta)(1+\lambda)^2}{|1-\alpha|(2+\lambda)}\right)+...+\tfrac{2(1+\lambda)\times(1-\eta)}{(k-1)|1-\alpha|}\textstyle\prod_{j=1}^{k-2}\left(1+\tfrac{2(1-\eta)(1+\lambda)}{(2+j\lambda)|1-\alpha|}\right), \\ & =\tfrac{2(1+\lambda)\times(1-\eta)}{[(2+k)\lambda]|1-\alpha|B_{k+1}(\lambda)}\textstyle\prod_{j=1}^{k-1}\left(1+\tfrac{2(1-\eta)(1+\lambda)}{(2+j\lambda)|1-\alpha|}\right), \end{aligned}$$

which means that the inequality (2.2) holds for $n=k+1$ and by mathematical induction it holds true for all $n\geq 3$. □

Corollary 2.3. *If we put $\alpha=0$ in Theorem 2.2, we get the inequality*

$$|a_k|\leq 2\left(1+\lambda\right)\left(1-\eta\right)\textstyle\prod_{j=1}^{k-2}\left(1+\tfrac{2(1+\lambda)(1-\eta)}{(2+j\lambda)}\right),\ \left(k\geq 2\right).$$

Corollary 2.4. *If we put $\alpha=\lambda=0$ in Theorem 2.2, then we get the inequality*

$$|a_k|\leq 2\left(1-\eta\right)\textstyle\prod_{j=1}^{k-2}\left(2-\eta\right)^j,\left(k\geq 2\right).$$

Theorem 2.3. *If $f \in KD(\alpha, \eta, \lambda)$ with $0 \leq \alpha \leq \eta$, then*

$$|a_n| \leq \frac{2(1-\beta)}{n(n-1)|1-\alpha|} \prod_{j=2}^{k-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|}\right), (n \geq 4, 5, 6, ...),$$

and

$$|a_2| \leq \frac{(1-\beta)}{|1-\alpha|}.$$

Similarly,

$$|a_3| \leq \frac{2(1-\beta)}{3|1-\alpha|} (1 + 2|a_2|).$$

3. Conclusion

A new class $M(\alpha, \eta, \lambda)$ of uniformly convex and starlike functions using Ruscheweyh derivative operator is introduced. Characterization of coefficient inequalities for functions in the class are estimated.

References

[1] K. Al-Shaqsi, M. Darrus, *On certain subclass of analytic univalent functions with negative coefficients*, Appl. Math. Sci., **1**(23) (2007), 1121–1128.
[2] C. Carathéodory, *Über den variabilitätsbereich der Fourier'schen konstanten von possitiven harmonischen funktionen,* Rend. Circ. Palermo, **32** (1911), 193–217.
[3] P. L. Duren, *Univalent Functions,* Sringer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
[4] F. Ronning, *On starlike functions associated with parabolic region*, Ann. Univ. Mariae-Curie Sklodowska Sect., **45** (1991), 117–122.
[5] F. Ronning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118** (1993), 180–196.
[6] A. W. Goodman, *On uniformly convex functions,* Ann. Polon. Math., **56** (1991), 87–92.
[7] S. Latha, *Coefficient inequalities for certain classes of Ruscheweyh type analytic functions,* Journal of Inequalities in Pure and Applied Mathematics, **9**(2) (2008), Article No 52.
[8] S. Latha, S. Najunda Rao. *Convex combinations of n analytic functions in generalized Ruscheweyh class*, Int. J. Math. Sci. Technology, **25** (1994), 791–795.
[9] S. Shams, S. R. Kulkarn, J. M. Jehangiri, *Classes of uniformly starlike functions,* Internet. J. Math. and Math. Sci., **55** (2004), 2959–2961.
[10] S. Owa, Y. Polatoglu, E. Yavuz, *Coefficient inequalities for classes of uniformly starlike and convex functions,* J. Ineq. in Pure and Applied Math., **7**(5) (2006), Article 160.
[11] M. S. Robertson, *On the theory of univalent functions,* Ann. Math., **37**(2) (1936), 374–408.
[12] S. Ruscheweyh, *New criteria for univalent function,* Proc. Amer. Math. Soc., **49** (1975), 109–115.
[13] H. Silverman, *Univalent functions with negative coefficients,* Proc. Amer. Math. Soc., **51**(1) (1975), 109–116.

[1]Department of Mathematics,
Federal Government College,
Mardan (KPK), Pakistan.
Email address: mumtaz16102@gmail.com

[2]Department of Mathematics and Statistics,
International Islamic University,
H-10 Islamabad Pakistan
Email address: marshadzia@iiu.edu.pk