Turkish J. Ineq., 8 (2) (2024), Pages 1 –15 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

ERROR BOUNDS OF NEWTON-COTES QUADRATURE RULES INVOLVING AT MOST FIVE POINTS VIA EXTENDED s-CONVEXITY

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Abstract. In this paper, we propose a new parametrized integral identity that spans several quadrature formulas. Based on this equality, we establish bounds on the error estimates of several quadrature formulas under the assumption that the absolute value of the first derivatives belongs to the class of extended s-convexity. Some special cases are discussed. Applications to inequalities involving special means are provided.

1. Introduction

Let I be a real interval.

Definition 1.1 ([50]). A function $\mathcal{S}: I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$\mathcal{S}(h\vartheta_1 + (1-h)\vartheta_2) \leq h\mathcal{S}(\vartheta_1) + (1-h)\mathcal{S}(\vartheta_2)$$

holds for all $\vartheta_1, \vartheta_2 \in I$ and $h \in [0,1]$.

Convexity plays a central role in contemporary analysis. Due to its many applications, it has a considerable influence on both pure and applied sciences. Due to its wide scope, this class of functions has become the most important class in the field of analysis. Many authors have given it a keen interest. As a consequence, several scientists have extended the concept of classical convexity. Among these generalizations, we recall the extended s-convexity.

Definition 1.2 ([59]). A nonnegative function $\mathcal{S}: I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in [-1,1]$, if

$$\mathcal{S}(h\vartheta_1 + (1-h)\vartheta_2) \leq h^s\mathcal{S}(\vartheta_1) + (1-h)^s\mathcal{S}(\vartheta_2)$$

holds for all $\vartheta_1, \vartheta_2 \in I$ and $h \in [0,1]$.

Key words and phrases. Newton-Cotes inequalities, s-convex functions, parametrized inequalities.

2010 *Mathematics Subject Classification.* Primary: 65D32. Secondary: 26D10, 26D15.

Received: 28/03/2024 *Accepted*: 08/10/2024.

Cite this article as: B. Meftah, C. Menai, Error bounds of Newton-Cotes quadrature rules involving at most five points via extended s-convexity, Turkish Journal of Inequalities, 8(2) (2024), 1-15.

This class of functions encompasses the class of convex [50], s-convex [10], P-function [17], Godunova-Levin functions [22] and the s-Godunova-Levin functions [19].

One of the powerful tools used to solve a variety of mathematical problems and closely related to the principle of convexity is the theory of inequalities. This allows the study of the properties of the solutions of differential and integral equations. Also very much in demand in numerical analysis which provide the error boundaries to divers quadrature rules as trapezium [18,36,37,54], midpoint [7,40,48], Simpson [27,28,47,53,55], Dual Simpson [13,14,42,52], Corrected Simpson [26], Corrected dual Simpson [24,43], Bullen [20,56,58], Maclaurin [41,46,57], Milne [1,2,16,25,44,61], Bullen-Simpson [45], Pachpatte [38] and Boole [3,23].

For papers using other type inequalities as well as s- and extended s-convex one can see [4,5,9,11–13,21,29–32,34,35,39,49,54,60].

Inequalities are not limited to only one-variable expressions but can also involve several variables or complex functions [6,8,33,38,51].

In this investigation we propose a new parametrized integral identity that covers several quadrature formulas. Based on this equality, we establish the bounds of the error estimates of several quadratures under the assumption that the absolute value of the first derivatives belong to the class of extended s-convexity. Some particularly cases are discussed. Applications to inequalities involving special means are provided.

2. Main results

To prove our main results, we need the following integral identity.

Lemma 2.1. *Let $\mathcal{S} : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\vartheta_1, \vartheta_2 \in I^\circ$ with $\vartheta_1 < \vartheta_2$, and $\mathcal{S}' \in L^1[\vartheta_1, \vartheta_2]$, then the following equality*

$$
\begin{aligned}
&Q(\vartheta_1, x, \vartheta_2; \mathcal{S}) - \tfrac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} \mathcal{S}(u)\,du \\
=&\tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\int_0^1 \left(h - \tfrac{\lambda}{1+\gamma}\right)\mathcal{S}'((1-h)\vartheta_1 + hx)\,dh \\
&+ \tfrac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\int_0^1 \left(h - \tfrac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right)\mathcal{S}'\left((1-h)x + h\tfrac{\vartheta_1+\vartheta_2}{2}\right)dh \\
&+ \tfrac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\int_0^1 \left(h - \tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right)\mathcal{S}'\left((1-h)\tfrac{\vartheta_1+\vartheta_2}{2} + h(\vartheta_1+\vartheta_2-x)\right)dh \\
&+ \tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\int_0^1 \left(h - \tfrac{1+\gamma-\lambda}{1+\gamma}\right)\mathcal{S}'(((1-h)(\vartheta_1+\vartheta_2-x) + h\vartheta_2))\,dh,
\end{aligned}
$$

holds for all real number $\lambda,\gamma\in[0,1]$ *and* $x\in\left(\vartheta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)$, *where*

$$\begin{aligned}Q(\vartheta_1,x,\vartheta_2;\mathcal{S})=&\tfrac{\lambda(x-\vartheta_1)}{(1+\gamma)(\vartheta_2-\vartheta_1)}\mathcal{S}(\vartheta_1)+\tfrac{(\vartheta_2-\vartheta_1)-2\lambda(x-\vartheta_1)}{2(1+\gamma)(\vartheta_2-\vartheta_1)}\mathcal{S}(x)+\tfrac{\gamma}{1+\gamma}\mathcal{S}\left(\tfrac{\vartheta_1+\vartheta_2}{2}\right)\\&+\tfrac{(\vartheta_2-\vartheta_1)-2\lambda(x-\vartheta_1)}{2(1+\gamma)(\vartheta_2-\vartheta_1)}\mathcal{S}(\vartheta_1+\vartheta_2-x)+\tfrac{\lambda(x-\vartheta_1)}{(1+\gamma)(\vartheta_2-\vartheta_1)}\mathcal{S}(\vartheta_2).\end{aligned}$$

Proof. Let

$$I=\tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}I_1+\tfrac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}I_2+\tfrac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}I_3+\tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}I_4, \tag{2.1}$$

where

$$I_1=\int_0^1\left(h-\tfrac{\lambda}{1+\gamma}\right)\mathcal{S}'((1-h)\vartheta_1+hx)\,dh,$$

$$I_2=\int_0^1\left(h-\tfrac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right)\mathcal{S}'\left((1-h)x+h\tfrac{\vartheta_1+\vartheta_2}{2}\right)dh,$$

$$I_3=\int_0^1\left(h-\tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right)\mathcal{S}'\left((1-h)\tfrac{\vartheta_1+\vartheta_2}{2}+h(\vartheta_1+\vartheta_2-x)\right)dh,$$

and

$$I_4=\int_0^1\left(h-\tfrac{1+\gamma-\lambda}{1+\gamma}\right)\mathcal{S}'(((1-h)(\vartheta_1+\vartheta_2-x)+h\vartheta_2))\,dh.$$

Integrating by parts I_1, then using a change of variable, we obtain

$$\begin{aligned}I_1=&\tfrac{1}{x-\vartheta_1}\left(h-\tfrac{\lambda}{1+\gamma}\right)\mathcal{S}((1-h)\vartheta_1+hx)\Big|_0^1-\tfrac{1}{x-\vartheta_1}\int_0^1\mathcal{S}((1-h)\vartheta_1+hx)\,dh\\=&\tfrac{1+\gamma-\lambda}{(1+\gamma)(x-\vartheta_1)}\mathcal{S}(x)+\tfrac{\lambda}{(1+\gamma)(x-\vartheta_1)}\mathcal{S}(\vartheta_1)-\tfrac{1}{x-\vartheta_1}\int_0^1\mathcal{S}((1-h)\vartheta_1+hx)\,dh\\=&\tfrac{\lambda}{(1+\gamma)(x-\vartheta_1)}\mathcal{S}(\vartheta_1)+\tfrac{1+\gamma-\lambda}{(1+\gamma)(x-\vartheta_1)}\mathcal{S}(x)-\tfrac{1}{(x-\vartheta_1)^2}\int_{\vartheta_1}^x\mathcal{S}(u)\,du.\end{aligned}\tag{2.2}$$

Similarly, we have

$$\begin{aligned}I_2=&\tfrac{2}{\vartheta_1+\vartheta_2-2x}\left(h-\tfrac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right)\mathcal{S}\left((1-h)x+h\tfrac{\vartheta_1+\vartheta_2}{2}\right)\Big|_0^1\\&-\tfrac{2}{\vartheta_1+\vartheta_2-2x}\int_0^1\mathcal{S}\left((1-h)x+h\tfrac{\vartheta_1+\vartheta_2}{2}\right)dh\\=&\tfrac{2\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)^2}\mathcal{S}\left(\tfrac{\vartheta_1+\vartheta_2}{2}\right)+\tfrac{2(\vartheta_2-x)-2(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)^2}\mathcal{S}(x)\\&-\tfrac{4}{(\vartheta_1+\vartheta_2-2x)^2}\int_x^{\frac{\vartheta_1+\vartheta_2}{2}}\mathcal{S}(u)\,du,\end{aligned}\tag{2.3}$$

$$I_3=\tfrac{2}{\vartheta_1+\vartheta_2-2x}\left(h-\tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right)\mathcal{S}\left((1-h)\tfrac{\vartheta_1+\vartheta_2}{2}+h(\vartheta_1+\vartheta_2-x)\right)\Big|_0^1$$

$$
\begin{aligned}
&-\tfrac{2}{\vartheta_1+\vartheta_2-2x}\int_0^1 \mathcal{S}\left((1-h)\tfrac{\vartheta_1+\vartheta_2}{2}+h(\vartheta_1+\vartheta_2-x)\right)dt\\
=&\tfrac{2(\vartheta_2-x)-2(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)^2}\mathcal{S}(\vartheta_1+\vartheta_2-x)+\tfrac{2\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)^2}\mathcal{S}\left(\tfrac{\vartheta_1+\vartheta_2}{2}\right)\\
&-\tfrac{4}{(\vartheta_1+\vartheta_2-2x)^2}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_1+\vartheta_2-x}\mathcal{S}(u)\,du \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 =& \tfrac{1}{x-\vartheta_1}\left(h-\tfrac{1+\gamma-\lambda}{1+\gamma}\right)\mathcal{S}\left(((1-h)(\vartheta_1+\vartheta_2-x)+h\vartheta_2)\right)\Big|_0^1\\
&-\tfrac{1}{x-\vartheta_1}\int_0^1 \mathcal{S}\left(((1-h)(\vartheta_1+\vartheta_2-x)+h\vartheta_2)\right)dh\\
=&\tfrac{\lambda}{(1+\gamma)(x-\vartheta_1)}\mathcal{S}(\vartheta_2)+\tfrac{1+\gamma-\lambda}{(1+\gamma)(x-\vartheta_1)}\mathcal{S}(\vartheta_1+\vartheta_2-x)\\
&-\tfrac{1}{(x-\vartheta_1)^2}\int_{\vartheta_1+\vartheta_2-x}^{\vartheta_2}\mathcal{S}(u)\,du. \qquad (2.5)
\end{aligned}
$$

Substituting (2.2)-(2.5) in (2.1), we get the desired result. □

We are now able to establish our results for this, we admit throughout the paper that $0 \leq \vartheta_1 < \vartheta_2, x \in \left(\vartheta_1, \frac{\vartheta_1+\vartheta_2}{2}\right), \lambda, \gamma \in [0,1]$ and $s \in (-1,1]$.

Theorem 2.1. *Let $\mathcal{S}$ be as in Lemma 2.1. If $|\mathcal{S}'|$ is extended s-convex on $[\vartheta_1, \vartheta_2]$, then we have*

$$
\begin{aligned}
&\left|Q(\vartheta_1,x,\vartheta_2;\mathcal{S})-\tfrac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{S}(u)\,du\right|\\
\leq&\tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(L_1(s,\lambda,\gamma)\left(|\mathcal{S}'(\vartheta_1)|+|\mathcal{S}'(\vartheta_2)|\right)\right.\\
&\left.+L_2(s,\lambda,\gamma)\left(|\mathcal{S}'(x)|+|\mathcal{S}'(\vartheta_1+\vartheta_2-x)|\right)\right)\\
&+\tfrac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(N_{s,\gamma}(x,\vartheta_1,\vartheta_2)\left(|\mathcal{S}'(x)|+|\mathcal{S}'(\vartheta_1+\vartheta_2-x)|\right)\right.\\
&\left.+2M_{s,\gamma}(x,\vartheta_1,\vartheta_2)\left|\mathcal{S}'\left(\tfrac{\vartheta_1+\vartheta_2}{2}\right)\right|\right),
\end{aligned}
$$

where $L_1(s,\lambda,\gamma)$, $L_2(s,\lambda,\gamma)$, $N_{s,\gamma}(x,\vartheta_1,\vartheta_2)$ and $M_{s,\gamma}(x,\vartheta_1,\vartheta_2)$ are given by (2.6), (2.7), (2.8) and (2.9), respectively.

Proof. Using Lemma 2.1, the absolute value and the extended s-convexity of $|\mathcal{S}'|$, we get

$$
\left|Q(\vartheta_1,x,\vartheta_2;\mathcal{S})-\tfrac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{S}(u)\,du\right|
$$

$$\leq \frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\int_0^1 \left|h-\frac{\lambda}{1+\gamma}\right| \left|\mathcal{S}'\left((1-h)\vartheta_1+hx\right)\right| dh$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\int_0^1 \left|h-\frac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right| \left|\mathcal{S}'\left((1-h)x+h\frac{\vartheta_1+\vartheta_2}{2}\right)\right| dh$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\int_0^1 \left|h-\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right| \left|\mathcal{S}'\left((1-h)\frac{\vartheta_1+\vartheta_2}{2}+h(\vartheta_1+\vartheta_2-x)\right)\right| dh$$

$$+\frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\int_0^1 \left|h-\frac{1+\gamma-\lambda}{1+\gamma}\right| \left|\mathcal{S}'\left(((1-h)(\vartheta_1+\vartheta_2-x)+h\vartheta_2)\right)\right| dh$$

$$\leq \frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\int_0^1 \left|h-\frac{\lambda}{1+\gamma}\right| \left((1-h)^s\left|\mathcal{S}'(\vartheta_1)\right|+h^s\left|\mathcal{S}'(x)\right|\right) dh$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\int_0^1 \left|h-\frac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right| \left((1-h)^s\left|\mathcal{S}'(x)\right|+h^s\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|\right) dh$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\int_0^1 \left|h-\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right| \left((1-h)^s\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|+h^s\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|\right) dh$$

$$+\frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\int_0^1 \left|h-\frac{1+\gamma-\lambda}{1+\gamma}\right| \left((1-h)^s\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|+h^s\left|\mathcal{S}'(\vartheta_2)\right|\right) dh$$

$$=\frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\left|\mathcal{S}'(\vartheta_1)\right|\int_0^1 \left|h-\frac{\lambda}{1+\gamma}\right|(1-h)^s dh+\left|\mathcal{S}'(x)\right|\int_0^1 \left|h-\frac{\lambda}{1+\gamma}\right| h^s dh\right)$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\left|\mathcal{S}'(x)\right|\int_0^1 \left|h-\frac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|(1-h)^s dh\right.$$

$$\left.+\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|\int_0^1 \left|h-\frac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right| h^s dh\right)$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|\int_0^1 \left|h-\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|(1-h)^s dh\right.$$

$$\left.+\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|\int_0^1 \left|h-\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right| h^s dh\right)$$

$$+\frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|\int_0^1 \left|h-\frac{1+\gamma-\lambda}{1+\gamma}\right|(1-h)^s dh+\left|\mathcal{S}'(\vartheta_2)\right|\int_0^1 \left|h-\frac{1+\gamma-\lambda}{1+\gamma}\right| h^s dh\right)$$

$$
\begin{aligned}
&= \tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1} \left(L_1(s,\lambda,\gamma) \left(\left|\mathbb{S}'(\vartheta_1)\right| + \left|\mathbb{S}'(\vartheta_2)\right| \right) \right. \\
&\quad \left. + L_2(s,\lambda,\gamma) \left(\left|\mathbb{S}'(x)\right| + \left|\mathbb{S}'(\vartheta_1+\vartheta_2-x)\right| \right) \right) \\
&\quad + \tfrac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)} \left(N_{s,\gamma}(x,\vartheta_1,\vartheta_2) \left(\left|\mathbb{S}'(x)\right| + \left|\mathbb{S}'(\vartheta_1+\vartheta_2-x)\right| \right) \right. \\
&\quad \left. + 2M_{s,\gamma}(x,\vartheta_1,\vartheta_2) \left|\mathbb{S}'\left(\tfrac{\vartheta_1+\vartheta_2}{2}\right)\right| \right),
\end{aligned}
$$

where we have used

$$
\begin{aligned}
L_1(s,\lambda,\gamma) &= \int_0^1 \left| h - \tfrac{\lambda}{1+\gamma} \right| (1-h)^s\, dh = \int_0^1 \left| h - \tfrac{1+\gamma-\lambda}{1+\gamma} \right| h^s dh \\
&= \tfrac{\lambda(s+2)-(1+\gamma)}{(1+\gamma)(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)} \left(\tfrac{1+\gamma-\lambda}{1+\gamma} \right)^{s+2}, \qquad (2.6)
\end{aligned}
$$

$$
\begin{aligned}
L_2(s,\lambda,\gamma) &= \int_0^1 \left| h - \tfrac{\lambda}{1+\gamma} \right| h^s dh = \int_0^1 \left| h - \tfrac{1+\gamma-\lambda}{1+\gamma} \right| (1-h)^s\, dh \\
&= \tfrac{(1+\gamma-\lambda)(s+1)-\lambda}{(1+\gamma)(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)} \left(\tfrac{\lambda}{1+\gamma} \right)^{s+2}, \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
N_{s,\gamma}(x,\vartheta_1,\vartheta_2) &= \int_0^1 \left| h - \tfrac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)} \right| (1-h)^s\, dh \qquad (2.8) \\
&= \int_0^1 \left| \tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)} - h \right| h^s dh \\
&= \begin{cases} \tfrac{(s+1-\gamma)(\vartheta_2-x)-((1+2\gamma)(s+1)+\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)(s+1)(s+2)} \\ \quad + \tfrac{2}{(s+1)(s+2)} \left(\tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)} \right)^{s+2} & \text{if } \gamma \le \tfrac{\frac{\vartheta_1+\vartheta_2}{2}-x}{x-\vartheta_1}, \\ \tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)(s+1)} - \tfrac{1}{(s+2)} & \text{if } \gamma > \tfrac{\frac{\vartheta_1+\vartheta_2}{2}-x}{x-\vartheta_1}, \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
M_{s,\gamma}(x,\vartheta_1,\vartheta_2) &= \int_0^1 \left| h - \tfrac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)} \right| h^s dh \qquad (2.9) \\
&= \int_0^1 \left| \tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)} - h \right| (1-h)^s\, dh \\
&= \begin{cases} -\tfrac{1}{(s+1)(s+2)} + \tfrac{\gamma(\vartheta_2-\vartheta_1)}{(s+1)(1+\gamma)(\vartheta_1+\vartheta_2-2x)} \\ \quad + \tfrac{2}{(s+1)(s+2)} \left(1 - \tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)} \right)^{s+2} & \text{if } \gamma \le \tfrac{\frac{\vartheta_1+\vartheta_2}{2}-x}{x-\vartheta_1}, \\ \tfrac{\gamma(\vartheta_2-\vartheta_1)}{(s+1)(1+\gamma)(\vartheta_1+\vartheta_2-2x)} - \tfrac{1}{(s+1)(s+2)} & \text{if } \gamma > \tfrac{\frac{\vartheta_1+\vartheta_2}{2}-x}{x-\vartheta_1}. \end{cases}
\end{aligned}
$$

The proof is completed. □

Corollary 2.1. *In Theorem 2.1, if we take $\gamma = \frac{2}{13}, \lambda = \frac{14}{39}$ and we choose $x = \frac{3\vartheta_1+\vartheta_2}{4}$, we obtain the following Boole-type inequality*

$$\left| \frac{7\mathcal{S}(\vartheta_1)+32\mathcal{S}\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)+12\mathcal{S}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+32\mathcal{S}\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)+7\mathcal{S}(\vartheta_2)}{90} - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} \mathcal{S}(u)\,du \right|$$
$$\leq \frac{\vartheta_2-\vartheta_1}{16(s+1)(s+2)} \left(\left(\frac{14s-17}{45} + 2\left(\frac{31}{45}\right)^{s+2} \right) \left(|\mathcal{S}'(\vartheta_1)| + |\mathcal{S}'(\vartheta_2)| \right) \right.$$
$$+ \left(\frac{64s+38}{45} + 2\left(\frac{4}{15}\right)^{s+2} + 2\left(\frac{14}{45}\right)^{s+2} \right) \left(\left|\mathcal{S}'\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right| + \left|\mathcal{S}'\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right| \right)$$
$$\left. + 2\left(\frac{4s-7}{15} + 2\left(\frac{11}{15}\right)^{s+2} \right) \left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right| \right).$$

Corollary 2.2. *In Theorem 2.1, if we take $\gamma = \frac{1}{5}, \lambda = \frac{2}{5}$ and we choose $x = \frac{3\vartheta_1+\vartheta_2}{4}$, we obtain the following Bullen-Simpson-type inequality*

$$\left| \frac{\mathcal{S}(\vartheta_1)+4\mathcal{S}\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)+2\mathcal{S}\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+4\mathcal{S}\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)+\mathcal{S}(\vartheta_2)}{12} - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} \mathcal{S}(u)\,du \right|$$
$$\leq \frac{\vartheta_2-\vartheta_1}{16(s+1)(s+2)} \left(\left(\frac{s-1}{3} + 2\left(\frac{2}{3}\right)^{s+2} \right) \left(|\mathcal{S}'(\vartheta_1)| + 2\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right| + |\mathcal{S}'(\vartheta_2)| \right) \right.$$
$$\left. + 2\left(\frac{2s+1}{3} + 2\left(\frac{1}{3}\right)^{s+2} \right) \left(\left|\mathcal{S}'\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right| + \left|\mathcal{S}'\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right| \right) \right).$$

Corollary 2.3. *In Theorem 2.1, if we take $\gamma = \frac{1}{3}, \lambda = 0$ and choose $x = \frac{5\vartheta_1+\vartheta_2}{6}$, we obtain the following Maclaurin-type inequality*

$$\left| \frac{1}{8}\left(3\mathcal{S}\left(\frac{5\vartheta_1+\vartheta_2}{6}\right) + 2\mathcal{S}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 3\mathcal{S}\left(\frac{\vartheta_1+5\vartheta_2}{6}\right) \right) - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} \mathcal{S}(u)\,du \right|$$
$$\leq \frac{\vartheta_2-\vartheta_1}{36(s+1)(s+2)} \left(|\mathcal{S}'(\vartheta_1)| + |\mathcal{S}'(\vartheta_2)| + \left(3s - 2 + 10\left(\frac{5}{8}\right)^{s+1} \right) \left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right| \right.$$
$$\left. + \left(\frac{7s+4}{2} + 3\left(\frac{3}{8}\right)^{s+1} \right) \left(\left|\mathcal{S}'\left(\frac{5\vartheta_1+\vartheta_2}{6}\right)\right| + \left|\mathcal{S}'\left(\frac{\vartheta_1+5\vartheta_2}{6}\right)\right| \right) \right).$$

Corollary 2.4. *In Theorem 2.1, if we take $\gamma = \frac{13}{27}, \lambda = 0$ and choose $x = \frac{5\vartheta_1+\vartheta_2}{6}$, we obtain the following corrected Euler-Maclaurin-type inequality*

$$\left| \frac{1}{80}\left(27\mathcal{S}\left(\frac{5\vartheta_1+\vartheta_2}{6}\right) + 26\mathcal{S}\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + 27\mathcal{S}\left(\frac{\vartheta_1+5\vartheta_2}{6}\right) \right) - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} \mathcal{S}(u)\,du \right|$$
$$\leq \frac{\vartheta_2-\vartheta_1}{36(s+1)(s+2)} \left(|\mathcal{S}'(\vartheta_1)| + |\mathcal{S}'(\vartheta_2)| + \left(\frac{39s-2}{10} + \frac{41}{5}\left(\frac{41}{80}\right)^{s+1} \right) \left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right| \right.$$
$$\left. + \left(\frac{61s+22}{20} + \frac{39}{10}\left(\frac{39}{80}\right)^{s+1} \right) \left(|\mathcal{S}'(x)| + |\mathcal{S}'(\vartheta_1+\vartheta_2-x)| \right) \right).$$

Corollary 2.5. *In Theorem 2.1, if we take $\gamma = 0, \lambda = \frac{3}{8}$ and choose $x = \frac{2\vartheta_1+\vartheta_2}{3}$, we obtain the following $\frac{3}{8}$-Simpson-type inequality*

$$\left|\frac{\mathcal{S}(\vartheta_1)+3\mathcal{S}\left(\frac{2\vartheta_1+\vartheta_2}{3}\right)+3\mathcal{S}\left(\frac{\vartheta_1+2\vartheta_2}{3}\right)+\mathcal{S}(\vartheta_2)}{8} - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{S}(u)\,du\right|$$
$$\leq \frac{\vartheta_2-\vartheta_1}{36(s+1)(s+2)}\left(\left(\frac{3s-2}{2}+\frac{25}{8}\left(\frac{5}{8}\right)^s\right)\left(|\mathcal{S}'(\vartheta_1)|+|\mathcal{S}'(\vartheta_2)|\right)+\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|\right.$$
$$\left.+\left(\frac{7s+4}{2}+\frac{9}{8}\left(\frac{3}{8}\right)^s\right)\left(\left|\mathcal{S}'\left(\frac{2\vartheta_1+\vartheta_2}{3}\right)\right|+\left|\mathcal{S}'\left(\frac{\vartheta_1+2\vartheta_2}{3}\right)\right|\right)\right).$$

Corollary 2.6. *In Theorem 2.1, if we take $\gamma = \lambda = 0$, we obtain the following companion Ostrowski-type inequality*

$$\left|\frac{\mathcal{S}(x)+\mathcal{S}(\vartheta_1+\vartheta_2-x)}{2} - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{S}(u)\,du\right|$$
$$\leq \frac{(x-\vartheta_1)^2}{(s+1)(s+2)(\vartheta_2-\vartheta_1)}\left(|\mathcal{S}'(\vartheta_1)|+|\mathcal{S}'(\vartheta_2)|+(s+1)\left(|\mathcal{S}'(x)|+|\mathcal{S}'((\vartheta_1+\vartheta_2-x))|\right)\right)$$
$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(s+1)(s+2)(\vartheta_2-\vartheta_1)}\left((s+1)\left(|\mathcal{S}'(x)|+|\mathcal{S}'(\vartheta_1+\vartheta_2-x)|\right)+2\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|\right).$$

Theorem 2.2. *Let $\mathcal{S}$ be as in Lemma 2.1. If $|\mathcal{S}'|^q$ is extended s-convex on $[\vartheta_1, \vartheta_2]$ where $q > 1$ with $\frac{1}{q}+\frac{1}{p} = 1$, then we have*

$$\left|Q(\vartheta_1, x, \vartheta_2; \mathcal{S}) - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{S}(u)\,du\right|$$
$$\leq \frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\frac{1}{p+1}\left(\left(\frac{\lambda}{1+\gamma}\right)^{p+1}+\left(\frac{1+\gamma-\lambda}{1+\gamma}\right)^{p+1}\right)\right)^{\frac{1}{p}}$$
$$\times\left(\left(\frac{|\mathcal{S}'(\vartheta_1)|^q+|\mathcal{S}'(x)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{S}'(\vartheta_1+\vartheta_2-x)|^q+|\mathcal{S}'(\vartheta_2)|^q}{s+1}\right)^{\frac{1}{q}}\right)$$
$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\Delta(\gamma, \vartheta_1, \vartheta_2)\right)^{\frac{1}{p}}$$
$$\times\left(\left(\frac{|\mathcal{S}'(x)|^q+\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+|\mathcal{S}'(\vartheta_1+\vartheta_2-x)|^q}{s+1}\right)^{\frac{1}{q}}\right),$$

where $\Delta(\gamma, \vartheta_1, \vartheta_2)$ is given by (2.10).

Proof. From Lemma 2.1, absolute value, Hölder's inequality and the extended s-convexity of $|\mathcal{S}'|^q$, we get

$$\left|Q(\vartheta_1, x, \vartheta_2; \mathcal{S}) - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{S}(u)\,du\right|$$
$$\leq \frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\int_0^1\left|h-\frac{\lambda}{1+\gamma}\right|^p dh\right)^{\frac{1}{p}}\left(\int_0^1|\mathcal{S}'((1-h)\vartheta_1+hx)|^q\,dh\right)^{\frac{1}{q}}$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\int_0^1\left|h-\frac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|^p dh\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{S}'\left((1-h)x+h\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q dh\right)^{\frac{1}{q}}$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\int_0^1\left|h-\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|^p dh\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left|\mathcal{S}'\left((1-h)\frac{\vartheta_1+\vartheta_2}{2}+h(\vartheta_1+\vartheta_2-x)\right)\right|^q dh\right)^{\frac{1}{q}}$$

$$+\frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\int_0^1\left|h-\frac{1+\gamma-\lambda}{1+\gamma}\right|^p dh\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{S}'\left(((1-h)(\vartheta_1+\vartheta_2-x)+h\vartheta_2)\right)\right|^q dh\right)^{\frac{1}{q}}$$

$$\leq\frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\int_0^1\left|h-\frac{\lambda}{1+\gamma}\right|^p dh\right)^{\frac{1}{p}}\left(\left(\int_0^1\left((1-h)^s\left|\mathcal{S}'(\vartheta_1)\right|^q+h^s\left|\mathcal{S}'(x)\right|^q\right)dh\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left((1-h)^s\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|^q+h^s\left|\mathcal{S}'(\vartheta_2)\right|^q\right)dh\right)^{\frac{1}{q}}\right)$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\int_0^1\left|h-\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|^p dh\right)^{\frac{1}{p}}$$

$$\times\left(\left(\int_0^1\left((1-h)^s\left|\mathcal{S}'(x)\right|^q+h^s\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\right)dh\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left((1-h)^s\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+h^s\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|^q\right)dh\right)^{\frac{1}{q}}\right)$$

$$=\frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\frac{1}{p+1}\left(\left(\frac{\lambda}{1+\gamma}\right)^{p+1}+\left(\frac{1+\gamma-\lambda}{1+\gamma}\right)^{p+1}\right)\right)^{\frac{1}{p}}$$

$$\times\left(\left(\frac{\left|\mathcal{S}'(\vartheta_1)\right|^q+\left|\mathcal{S}'(x)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|^q+\left|\mathcal{S}'(\vartheta_2)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\Delta(\gamma,\vartheta_1,\vartheta_2)\right)^{\frac{1}{p}}$$

$$\times\left(\left(\frac{\left|\mathcal{S}'(x)\right|^q+\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|^q}{s+1}\right)^{\frac{1}{q}}\right),$$

where

$$\int_0^1\left|h-\frac{\lambda}{1+\gamma}\right|^p dh=\frac{1}{p+1}\left(\left(\frac{\lambda}{1+\gamma}\right)^{p+1}+\left(\frac{1+\gamma-\lambda}{1+\gamma}\right)^{p+1}\right)$$

and

$$\Delta(\gamma,\vartheta_1,\vartheta_2) \tag{2.10}$$
$$=\int_0^1 \left|h-\tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|^p dh$$
$$=\begin{cases} \frac{1}{p+1}\left(\left(\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right)^{p+1}+\left(1-\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right)^{p+1}\right) & \text{if } \gamma\le\frac{\frac{\vartheta_1+\vartheta_2}{2}-x}{x-\vartheta_1}, \\ \frac{1}{p+1}\left(\left(\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right)^{p+1}-\left(\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}-1\right)^{p+1}\right) & \text{if } \gamma>\frac{\frac{\vartheta_1+\vartheta_2}{2}-x}{x-\vartheta_1}. \end{cases}$$

The proof is over. □

Theorem 2.3. *Let $\mathcal{S}$ be as in Lemma 2.1. If $|\mathcal{S}'|^q$ is extended s-convex on $[\vartheta_1,\vartheta_2]$ where $q\ge 1$, then we have*

$$\left|Q(\vartheta_1,x,\vartheta_2;\mathcal{S})-\tfrac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{S}(u)\,du\right|$$
$$\le\tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\tfrac{\lambda^2+(1+\gamma-\lambda)^2}{2(1+\gamma)^2}\right)^{1-\frac{1}{q}}\Big(\left(L_1(s,\lambda,\gamma)\left|\mathcal{S}'(\vartheta_1)\right|^q+L_2(s,\lambda,\gamma)\left|\mathcal{S}'(x)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(L_2(s,\lambda,\gamma)\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|^q+L_1(s,\lambda,\gamma)\left|\mathcal{S}'(\vartheta_2)\right|^q\right)^{\frac{1}{q}}\Big)$$
$$+\tfrac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\Omega(\gamma,\vartheta_1,\vartheta_2)\right)^{1-\frac{1}{q}}$$
$$\times\Big(\left(N_{s,\gamma}(x,\vartheta_1,\vartheta_2)\left|\mathcal{S}'(x)\right|^q+M_{s,\gamma}(x,\vartheta_1,\vartheta_2)\left|\mathcal{S}'\left(\tfrac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(M_{s,\gamma}(x,\vartheta_1,\vartheta_2)\left|\mathcal{S}'\left(\tfrac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+N_{s,\gamma}(x,\vartheta_1,\vartheta_2)\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|^q\right)^{\frac{1}{q}}\Big),$$

where $L_1(s,\lambda,\gamma), L_2(s,\lambda,\gamma), N_{s,\gamma}(x,\vartheta_1,\vartheta_2)$ and $M_{s,\gamma}(x,\vartheta_1,\vartheta_2)$ are defined as in (2.6)-(2.9), respectively.

Proof. From Lemma 2.1, absolute value, power mean inequality and the extended s-convexity of $|\mathcal{S}'|^q$, we get

$$\left|Q(\vartheta_1,x,\vartheta_2;\mathcal{S})-\tfrac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\mathcal{S}(u)\,du\right|$$
$$\le\tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\int_0^1\left|h-\tfrac{\lambda}{1+\gamma}\right|dh\right)^{1-\frac{1}{q}}\left(\int_0^1\left|h-\tfrac{\lambda}{1+\gamma}\right|\left|\mathcal{S}'((1-h)\vartheta_1+hx)\right|^q dh\right)^{\frac{1}{q}}$$
$$+\tfrac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\int_0^1\left|h-\tfrac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|dh\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|h-\tfrac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|\left|\mathcal{S}'\left((1-h)\,x+h\tfrac{\vartheta_1+\vartheta_2}{2}\right)\right|^q dh\right)^{\frac{1}{q}}$$

$$+\tfrac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\int_0^1\left|h-\tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|dh\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|h-\tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|\left|\mathcal{S}'\left((1-h)\,\tfrac{\vartheta_1+\vartheta_2}{2}+h\,(\vartheta_1+\vartheta_2-x)\right)\right|^q dh\right)^{\frac{1}{q}}$$

$$+\tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\int_0^1\left|h-\tfrac{1+\gamma-\lambda}{1+\gamma}\right|dh\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|h-\tfrac{1+\gamma-\lambda}{1+\gamma}\right|\left|\mathcal{S}'\left(((1-h)\,(\vartheta_1+\vartheta_2-x)+h\vartheta_2)\right)\right|^q dh\right)^{\frac{1}{q}}$$

$$\leq\tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\int_0^1\left|h-\tfrac{\lambda}{1+\gamma}\right|dh\right)^{1-\frac{1}{q}}\left(\left(\int_0^1\left|h-\tfrac{\lambda}{1+\gamma}\right|\left((1-h)^s\left|\mathcal{S}'(\vartheta_1)\right|^q+h^s\left|\mathcal{S}'(x)\right|^q\right)dh\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|h-\tfrac{1+\gamma-\lambda}{1+\gamma}\right|\left((1-h)^s\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|^q+h^s\left|\mathcal{S}'(\vartheta_2)\right|^q\right)dh\right)^{\frac{1}{q}}\right)$$

$$+\tfrac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\int_0^1\left|h-\tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|dh\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(\int_0^1\left|h-\tfrac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|\left((1-h)^s\left|\mathcal{S}'(x)\right|^q+h^s\left|\mathcal{S}'\left(\tfrac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\right)dh\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|h-\tfrac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|\left((1-h)^s\left|\mathcal{S}'\left(\tfrac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+h^s\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|^q\right)dh\right)^{\frac{1}{q}}\right)$$

$$=\tfrac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}$$

$$\times\left(\tfrac{\lambda^2+(1+\gamma-\lambda)^2}{2(1+\gamma)^2}\right)^{1-\frac{1}{q}}\left(\left(\left|\mathcal{S}'(\vartheta_1)\right|^q\int_0^1\left|h-\tfrac{\lambda}{1+\gamma}\right|(1-h)^s\,dh+\left|\mathcal{S}'(x)\right|^q\int_0^1\left|h-\tfrac{\lambda}{1+\gamma}\right|h^s dh\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|\mathcal{S}'(\vartheta_1+\vartheta_2-x)\right|^q\int_0^1\left|h-\tfrac{1+\gamma-\lambda}{1+\gamma}\right|(1-h)^s\,dh+\left|\mathcal{S}'(\vartheta_2)\right|^q\int_0^1\left|h-\tfrac{1+\gamma-\lambda}{1+\gamma}\right|h^s dh\right)^{\frac{1}{q}}\right)$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\Omega\left(\gamma,\vartheta_1,\vartheta_2\right)\right)^{1-\frac{1}{q}}\left(\left(\left|\mathcal{S}'\left(x\right)\right|^q\int\limits_0^1\left|h-\frac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|(1-h)^s\,dh\right.\right.$$

$$\left.+\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\int\limits_0^1\left|h-\frac{(\vartheta_2-x)-(1+2\gamma)(x-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|h^s dh\right)^{\frac{1}{q}}$$

$$+\left(\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\int\limits_0^1\left|h-\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|(1-h)^s\,dh\right.$$

$$\left.\left.+\left|\mathcal{S}'\left(\vartheta_1+\vartheta_2-x\right)\right|^q\int\limits_0^1\left|h-\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|h^s dh\right)^{\frac{1}{q}}\right)$$

$$=\frac{(x-\vartheta_1)^2}{\vartheta_2-\vartheta_1}\left(\frac{\lambda^2+(1+\gamma-\lambda)^2}{2(1+\gamma)^2}\right)^{1-\frac{1}{q}}\left(\left(L_1\left(s,\lambda,\gamma\right)\left|\mathcal{S}'\left(\vartheta_1\right)\right|^q+L_2\left(s,\lambda,\gamma\right)\left|\mathcal{S}'\left(x\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(L_2\left(s,\lambda,\gamma\right)\left|\mathcal{S}'\left(\vartheta_1+\vartheta_2-x\right)\right|^q+L_1\left(s,\lambda,\gamma\right)\left|\mathcal{S}'\left(\vartheta_2\right)\right|^q\right)^{\frac{1}{q}}\right)$$

$$+\frac{(\vartheta_1+\vartheta_2-2x)^2}{4(\vartheta_2-\vartheta_1)}\left(\Omega\left(\gamma,\vartheta_1,\vartheta_2\right)\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(N_{s,\gamma}\left(x,\vartheta_1,\vartheta_2\right)\left|\mathcal{S}'\left(x\right)\right|^q+M_{s,\gamma}\left(x,\vartheta_1,\vartheta_2\right)\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(M_{s,\gamma}\left(x,\vartheta_1,\vartheta_2\right)\left|\mathcal{S}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|^q+N_{s,\gamma}\left(x,\vartheta_1,\vartheta_2\right)\left|\mathcal{S}'\left(\vartheta_1+\vartheta_2-x\right)\right|^q\right)^{\frac{1}{q}}\right),$$

where we have used (2.6)-(2.9) and

$$\Omega\left(\gamma,\vartheta_1,\vartheta_2\right)=\int\limits_0^1\left|h-\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right|dh$$

$$=\begin{cases}\frac{1}{2}\left(1-\frac{2\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}+2\left(\frac{\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}\right)^2\right) & \text{if } \gamma\leq\frac{\frac{\vartheta_1+\vartheta_2}{2}-x}{x-\vartheta_1},\\ \frac{1}{p+1}\left(\frac{2\gamma(\vartheta_2-\vartheta_1)}{(1+\gamma)(\vartheta_1+\vartheta_2-2x)}-1\right) & \text{if } \gamma>\frac{\frac{\vartheta_1+\vartheta_2}{2}-x}{x-\vartheta_1}.\end{cases}$$

the proof is finished. □

3. Applications

In this section, we present some applications of the obtained results. Let us consider the following means for arbitrary real numbers $\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4$.

The arithmetic mean: $A\left(\vartheta_1,\vartheta_2\right)=\frac{\vartheta_1+\vartheta_2}{2}$ and $A\left(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4\right)=\frac{\vartheta_1+\vartheta_2+\vartheta_3+\vartheta_4}{4}$.

The p-logarithmic mean: $L_p\left(\vartheta_1,\vartheta_2\right)=\left(\frac{\vartheta_2^{p+1}-\vartheta_1^{p+1}}{(p+1)(\vartheta_2-\vartheta_1)}\right)^{\frac{1}{p}}$, $\vartheta_1,\vartheta_2>0,\vartheta_1\neq\vartheta_2$ and $p\in\mathbb{R}\backslash\left\{-1,0\right\}$.

Proposition 3.1. *Let $0<\vartheta_1<\vartheta_2$ be tow real numbers, then we have*

$$\left|\frac{A\left(\vartheta_1^2,\vartheta_2^2\right)+2A^2(\vartheta_1,\vartheta_1,\vartheta_1,\vartheta_2)+A^2(\vartheta_1,\vartheta_2)+2A^2(\vartheta_1,\vartheta_2,\vartheta_2,\vartheta_2)-6L_2^2(\vartheta_1,\vartheta_2)}{5}\right|\leq\frac{1}{6}\left(\vartheta_2^2-\vartheta_1^2\right).$$

Proof. The assertion follows from Corollary 2.2 with $s = 0$, applied to the function $\mathcal{S}(z) = z^2$, which $|\mathcal{S}'(z)|$ is a P-function. □

Proposition 3.2. *Let $0 < \vartheta_1 < \vartheta_2$ be tow real numbers, then we have*

$$\left|\frac{3A^2(\vartheta_1,\vartheta_1,\vartheta_1,\vartheta_1,\vartheta_1,\vartheta_2)+2A^2(\vartheta_1,\vartheta_2)+3A^2(\vartheta_1,\vartheta_2,\vartheta_2,\vartheta_2,\vartheta_2,\vartheta_2)}{8} - L_2^2(\vartheta_1,\vartheta_2)\right| \leq \frac{25}{288}\left(\vartheta_2^2 - \vartheta_1^2\right).$$

Proof. The assertion follows from Corollary 2.3 with $s = 1$, applied to the function $\mathcal{S}(z) = z^2$, which $|\mathcal{S}'(z)|$ is a convex function. □

4. Conclusion

In this paper, we have proposed a new parametric integral identity that spans several quadrature formulas. We have established bounds on the error estimates of several quadrature formulas for functions whose derivatives are extended s-convex. We have also discussed some particularly cases that can be deduced. Applications to inequalities involving special means are provided. In the future, we hope to establish the fractional analogue as well as the quantum one under different classes of generalized convexity. We hope that the ideas and techniques developed in this paper will intrigue and inspire interested readers working in this area.

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