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INEQUALITIES FOR COMPLETELY MONOTONIC DEGREES OF FUNCTIONS INVOLVING GAMMA AND POLYGAMMA FUNCTIONS

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Abstract. We are interested in finding some completely monotonic functions containing the gamma, digamma and polygamma functions. As a consequence, we deduce some new bounds for the gamma, digamma and polygamma functions, which refine recent results.

1. Introduction

Special functions have numerous applications in fluid dynamics, electrical current, solutions of wave equations and heat conduction. At the heart of the theory of special functions lie the gamma and the psi functions. The gamma function was introduced by Euler [19] as:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x}{x\prod_{s=1}^{n}\left(1+\frac{x}{s}\right)}, \qquad x>0$$

which satisfies

$$\frac{\Gamma(1+x)}{x} = \Gamma(x), \qquad x\in(0,\infty) \tag{1.1}$$

and it has the following asymptotic formula [1]:

$$\ln\Gamma(x) \sim \ln\sqrt{2\pi} - x + \left(x-\frac{1}{2}\right)\ln x + \sum_{i=1}^{\infty}\frac{B_{2i}}{(2i)(2i-1)\,x^{2i-1}}, \qquad x\to\infty. \tag{1.2}$$

where B_{2i} are the Bernoulli numbers. The digamma $\psi(x)$ and polygamma $\psi^{(m)}(x)$ functions are given by [1]:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \sum_{i=1}^{\infty}\frac{x}{i(x+i)} - \gamma - \frac{1}{x} = \int_0^\infty\left(\frac{1}{te^t} - \frac{e^{(-x+1)t}}{e^t-1}\right)dt, \qquad x\in(0,\infty) \tag{1.3}$$

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where $\gamma \simeq 0.5772156649$ is Euler-Mascheroni's constant, and

$$\psi^{(m)}(x) = \sum_{i=0}^{\infty} \frac{(-1)^{1+m}\, m!}{(i+x)^{1+m}} = \int_0^{\infty} \frac{(-1)^{1+m}\, t^m\, e^{(-x+1)t}}{e^t - 1}\, dt, \qquad x > 0, \quad m \in \mathbb{N} \tag{1.4}$$

and they have the functional equation:

$$\psi^{(m)}(x+1) = \frac{(-1)^m\, m!}{x^{1+m}} + \psi^{(m)}(x), \qquad m = 0, 1, 2, \cdots . \tag{1.5}$$

The digamma $\psi(x)$ and polygamma $\psi^{(m)}(x)$ functions have the following asymptotic formulas [1]:

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)\, x^{2i}}, \qquad x \to \infty \tag{1.6}$$

and for $s \in \mathbb{N}$,

$$\psi^{(m)}(x) \sim (-1)^{m-1} \left(\frac{m!}{2x^{1+m}} + \frac{(m-1)!}{x^m} + \sum_{i=1}^{\infty} \frac{(m-1+2i)!\, B_{2i}}{(2i)!\, x^{2i+m}} \right), \quad x \to \infty. \tag{1.7}$$

For extra information about this topic, see [12, 13, 22]. A function F defined on an interval I is completely monotonic if it satisfies that

$$(-1)^m F^{(m)}(x) \geq 0 \qquad x \in I;\ m = 0, 1, 2, \dots .$$

The necessary and sufficient condition for $F(x)$ being completely monotonic for $x > 0$ is that [21]:

$$F(x) = \int_0^{\infty} e^{-xt} d\nu(t),$$

where $\nu(t)$ is non-negative measure on $t \geq 0$ such that the integral converges for $x > 0$. Let $F(x)$ be a completely monotonic function for $x > 0$ and suppose the notation $F(\infty) = \lim_{x\to\infty} F(x)$. If $x^{\varepsilon}[F(x) - F(\infty)]$ is a completely monotonic function for $x > 0$ if and only if $\varepsilon \in [0, \delta]$, then the number $\delta \in \mathbb{R}^+$ is called the completely monotonic degree of $F(x)$ for $x > 0$ and denoted by $\deg^x_{CM}[F(x)] = \delta$. For more information, see [6, 8, 14–18, 24]. The polygamma functions $\psi^{(m)}(x)$ are strictly completely monotonic on $(0, \infty)$ when $m = 1, 3, \cdots$, and so are $-\psi^{(m)}(x)$ for $m = 2, 4, \cdots$.

Alzer and Batir [2] showed that the function

$$H_{\varpi}(x) = x - x \ln x + \ln \Gamma(x) - \ln \sqrt{2\pi} + \frac{1}{2}\psi(\varpi + x), \qquad x > 0; \varpi \geq 0$$

is completely monotonic on $(0, \infty)$ for $\varpi \geq \frac{1}{3}$ and so is the function $-H_0(x)$. As a consequence, the following inequalities are deduced:

$$\exp\Big[-x - \frac{1}{2}\psi\Big(x + \frac{1}{3}\Big)\Big] < \frac{\Gamma(x)}{\sqrt{2\pi}\, x^x} < \exp\Big[-x - \frac{1}{2}\psi(x)\Big], \qquad x > 0 \tag{1.8}$$

$$\frac{1}{2}\psi'\left(x + \frac{1}{3}\right) < -\psi(x) + \ln x < \frac{1}{2}\psi'(x), \qquad x > 0 \tag{1.9}$$

and for $m = 2, 3, \cdots$,

$$\frac{(-1)^{m+1}}{2}\psi^{(m)}\left(x+\frac{1}{3}\right) < (-1)^m\psi^{(m-1)}(x) - \frac{(m-2)!}{x^{m-1}} < \frac{(-1)^{m+1}}{2}\psi^{(m)}(x), \qquad x > 0. \tag{1.10}$$

In 2008, Batir [4] refined (1.8) by:

$$\exp\left[-x-\frac{1}{2}\psi(x)-\frac{4}{6(4x-1)}\right] < \frac{\Gamma(x)}{\sqrt{2\pi}\,x^x} < \exp\left[-x-\frac{1}{2}\psi(x)-\frac{1}{6x}\right], \qquad x > 0. \tag{1.11}$$

In 2011, Şevli and Batir [20] proved that the function

$$F_\upsilon(x) = x - \left(x-\frac{1}{2}\right)\ln x + \ln\Gamma(x) - \ln\sqrt{2\pi} - \frac{1}{12}\psi'(x+\upsilon), \qquad x > 0, \upsilon \geq 0$$

is completely monotonic on $(0,\infty)$ for $\upsilon \geq \frac{1}{2}$ and so is the function $-F_0(x)$. As a consequence, the following inequalities are deduced:

$$\frac{1}{\sqrt{x}}\exp\left[-x+\frac{1}{12}\psi'\left(x+\frac{1}{2}\right)\right] < \frac{\Gamma(x)}{\sqrt{2\pi}\,x^x} < \frac{1}{\sqrt{x}}\exp\left[-x+\frac{1}{12}\psi'(x)\right], \qquad x > 0 \tag{1.12}$$

$$\frac{1}{2x} - \frac{1}{12}\psi''\left(x+\frac{1}{2}\right) < -\psi(x) + \ln x < \frac{1}{2x} - \frac{1}{12}\psi''(x), \qquad x > 0 \tag{1.13}$$

and for $m = 2, 3, \cdots$ and $x > 0$,

$$\frac{(m-1)!}{2x^m} + \frac{(-1)^m}{12}\psi^{(m+1)}\left(x+\frac{1}{2}\right) < (-1)^m\psi^{(m-1)}(x) - \frac{(m-2)!}{x^{m-1}} < \frac{(m-1)!}{2x^m} + \frac{(-1)^m}{12}\psi^{(m+1)}(x). \tag{1.14}$$

After that Batir [5] presented some sharp bounds for the psi function:

$$\frac{1}{2}\ln\left(x^2+x+e^{-2\gamma}\right) \leq \psi(1+x) < \frac{1}{2}\ln\left(x^2+x+\frac{1}{3}\right), \qquad x \in (0,\infty) \tag{1.15}$$

which can be written as:

$$\frac{1}{x} - \frac{1}{2}\ln\left(1+\frac{1}{x}+\frac{1}{3x^2}\right) < -\psi(x) + \ln x \leq \frac{1}{x} - \frac{1}{2}\ln\left(1+\frac{1}{x}+\frac{e^{-2\gamma}}{x^2}\right), \qquad x \in (0,\infty) \tag{1.16}$$

Recently many mathematicians studied the k- generalized gamma and polygamma functions, where $k > 0$, and the μ- generalized gamma and polygamma functions, where $\mu \in \mathbb{N}$. They deduced some new conclusions about the ordinary gamma and polygamma functions and some new proofs of theirs established conclusions when k approaches to one or μ approaches to infinity. The previous results were generalized in [7, 9–11, 23].

We will introduce three completely monotonic functions involving Γ, ψ and ψ'. Also, we will investigate their completely monotonic degrees on $(0,\infty)$. As a consequence, some new bounds for Γ and $\psi^{(m)}$ $(m \in \mathbb{N}\cup\{0\},)$ will be deduced, which refine the previous results.

2. Auxiliary Results

The following corollary [18] will be used in proving some next results:

Corollary 2.1. *Suppose that L is a real-valued function defined on $x > x_0$, $x_0 \in \mathbb{R}$ with $L(x)$ tends to zero as $x \to \infty$. Then for $r \in (0, \infty)$, $L(x) > 0$, if $L(r + x) - L(x) < 0$ for all $x > x_0$ and $L(x) < 0$, if $L(r + x) - L(x) > 0$ for all $x > x_0$.*

Lemma 2.1. *The following inequalities are true:*

$$\psi\left(x + \frac{1}{3}\right) + \frac{1}{12x^2} > \psi(x) + \frac{1}{3}\psi'(x), \qquad x \geq 2.2, \tag{2.1}$$

$$\psi'(x) - \frac{1}{x} > \frac{1}{4(x - \frac{3}{10})^2}, \qquad x \geq 0.9, \tag{2.2}$$

$$\psi'\left(x + \frac{1}{4}\right) > \frac{1}{x}, \qquad 8x > 1, \tag{2.3}$$

and

$$\psi'(x) + \frac{1}{3}\psi''\left(x + \frac{1}{4}\right) < \frac{2}{x} - \ln\left(1 + \frac{1}{x} + \frac{e^{-2\gamma}}{x^2}\right), \qquad x \geq 4. \tag{2.4}$$

Proof. Letting the function $H(x) = \psi\left(x + \frac{1}{3}\right) - \psi(x) - \frac{1}{3}\psi'(x) + \frac{1}{12x^2}$ and using the functional equation (1.5), we get

$$-H(x) + H(x+1) = \frac{\frac{-1}{25}\left(2 + 145(x - 2.2) + 50(x - 2.2)^2\right)}{12x^2(x+1)^2(1+3x)} < 0, \qquad x \geq 2.2.$$

Using the asymptotic expansions (1.6) and (1.7), we have $\lim_{x\to\infty} H(x) = 0$ and then Corollary 2.1 gives $H(x) > 0$ for all $x \geq 2.2$. Next, we set the function $M(x) = \psi'(x) - \frac{1}{x} - \frac{1}{4(x-\frac{3}{10})^2}$. Then

$$M(x+1) - M(x) = \frac{-m(x - 0.9)}{x^2(1+x)(-3+10x)^2(7+10x)^2} < 0, \qquad x \geq 0.9$$

where $2m(x) = 1503 + 22560x + 52200x^2 + 40000x^3 + 10000x^4$. By the same way as before, we have $M(x) > 0$ for all $x \geq 0.9$. After that, we suppose that the function $T(x) = \psi'\left(x + \frac{1}{4}\right) - \frac{1}{x}$ and then

$$-T(x) + T(x+1) = \frac{-(8x - 1)}{x(x+1)(1+4x)^2} < 0, \qquad 8x > 1.$$

Then, we get (2.3). Finally, we set $N(x) = \psi'(x) + \frac{1}{3}\psi''\left(x + \frac{1}{4}\right) - \frac{2}{x} + \ln\left(1 + \frac{1}{x} + \frac{e^{-2\gamma}}{x^2}\right)$ and by using (1.5), we get

$$N'(x+1) - N'(x) = \frac{-2n(x - 4)}{x^3(1+x)^2(1+4x)^4(e^{-2\gamma} + x + x^2)(e^{-2\gamma} + 2 + 3x + x^2)e^{4\gamma}} < 0, \qquad x \geq 4$$

where

$$n(x) = -5353349 - 401301050e^{2\gamma} + 128830600e^{4\gamma}$$

$$
\begin{aligned}
&+x\Big(-9045441-838790470e^{2\gamma}+272216390e^{4\gamma}\Big)\\
&+x^2\Big(-6552316-778752162e^{2\gamma}+254864409e^{4\gamma}\Big)\\
&+x^3\Big(-2637153-421471717e^{2\gamma}+138817176e^{4\gamma}\Big)\\
&+x^4\Big(-636816-146530514e^{2\gamma}+48488221e^{4\gamma}\Big)\\
&+x^5\Big(-92256-33935043e^{2\gamma}+11266512e^{4\gamma}\Big)\\
&+x^6\Big(-7424-5234928e^{2\gamma}+1741792e^{4\gamma}\Big)\\
&+x^7\Big(-256-518688e^{2\gamma}+172800e^{4\gamma}\Big)\\
&+x^8\Big(-29952e^{2\gamma}+9984e^{4\gamma}\Big)+x^9\Big(-768e^{2\gamma}+256e^{4\gamma}\Big)\\
>\ &0, \qquad x\geq 0
\end{aligned}
$$

and similary as before, we get (2.4). □

From (1.3) and (1.4), we have the lemma:

Lemma 2.2. *Let $x>0$. Then the following limits are correct:*

$$\lim_{x\to 0} x^{m+1}\psi^{(m)}(x)=(-1)^{m+1}\, m!, \qquad m=0,1,2,\cdots, \tag{2.5}$$

$$\lim_{x\to 0} x^{m}\psi^{(m)}(x+b)=0, \qquad m\in\mathbb{N}, \quad b>0, \tag{2.6}$$

and

$$\lim_{x\to 0} x^{m+2}\,\psi^{(m)}(x)=0, \qquad m=0,1,2,\cdots. \tag{2.7}$$

3. Main results

Theorem 3.1. *Assume that $x>0$. Then the function*

$$U_\xi(x)=-\ln x+\frac{1}{6}\psi'(x)+\frac{1}{3}\psi'(x+\xi)+\psi(x), \qquad \xi\geq 0$$

is completely monotonic on $(0,\infty)$ if and only if $\xi\leq\frac{1}{2}$. The function $U_{1/2}(x)$ satisfies that $2\leq \deg^{x}_{CM}\left[U_{1/2}(x)\right]<3$.

Proof. Using the identity [1]:

$$\ln\left(\frac{f}{d}\right)=\int_0^\infty \frac{e^{-dt}-e^{-ft}}{t}dt, \qquad f,d>0 \tag{3.1}$$

and the relations (1.3) and (1.4), we have $U_\xi(x)=\int_0^\infty \frac{e^{-xt}}{t(e^t-1)}\varphi_\xi(t)dt$, where

$$\varphi_\xi(t)=e^t-1-te^t+\frac{t^2}{6}e^t+\frac{t^2}{3}e^{(1-\xi)t}.$$

Let $\xi\leq\frac{1}{2}$, then we obtain

$$\varphi_\xi(t)\ \geq\ e^t-1-te^t+\frac{t^2}{6}e^t+\frac{t^2}{3}e^{\frac{t}{2}}$$

$$\begin{aligned} &= \frac{t^5}{720} + \sum_{n=4}^{\infty} \frac{(1+n)\Big[(-4+n)+2(2+n)2^{-n}\Big]t^{n+2}}{6\,(n+2)!} \\ &> 0, \qquad t>0. \end{aligned}$$

Consequently, $U_\xi(x)$ is completely monotonic on $(0,\infty)$ for $\xi \le \frac{1}{2}$. Conversely, if $U_\xi(x)$ is completely monotonic, then we get for $x>0$:

$$x^2\, U_\xi(x) = x^2\left[\frac{1}{6}\psi'(x) + \frac{1}{3}\psi'\,(x+\xi) + \psi(x) - \ln x\right] > 0.$$

Using the asymptotic expansions (1.6) and (1.7), we have

$$\lim_{x\to\infty} x^2\, U_\xi(x) = \lim_{x\to\infty} x^2\left[\frac{1}{3(x+\xi)} - \frac{1}{3x} + \frac{1}{6(x+\xi)^2} + O\left(\frac{1}{x^3}\right)\right] = -\frac{\xi}{3} + \frac{1}{6} \ge 0$$

and then $\xi \le \frac{1}{2}$. Furthermore, using the asymptotic expansions (1.6) and (1.7), we have

$$U_{1/2}(\infty) = \lim_{x\to\infty} U_{1/2}(x) = 0.$$

Now,

$$x^2\, U_{1/2}(x) = \int_0^\infty \frac{\Upsilon_1(t)}{12t^3(e^{\frac{t}{2}}-1)^3(e^{\frac{t}{2}}+1)^3} e^{-xt}dt, \qquad x>0$$

where

$$\begin{aligned} &\Upsilon_1(t) \\ =\ & 24\Big(-1+3e^t-3e^{2t}+e^{3t}\Big) + 4t^3\Big(e^{\frac{t}{2}} - 2e^t - 4e^{2t} - e^{\frac{5t}{2}}\Big) \\ & + t^4\Big(e^{\frac{t}{2}} + 2e^t + 6e^{\frac{3t}{2}} + 2e^{2t} + e^{\frac{5t}{2}}\Big) \\ =\ & \frac{504t^7}{7!} + \frac{7056t^8}{8!} + \frac{56088t^9}{9!} + \frac{335610t^{10}}{10!} + \frac{1688049t^{11}}{11!} + \frac{7554789t^{12}}{12!} + \frac{62219313t^{13}}{2(13!)} \\ & + \frac{1926695043t^{14}}{16(14!)} + \frac{14224269663t^{15}}{32(15!)} + \frac{12645653247t^{16}}{8(16!)} + \frac{87278151555t^{17}}{16(17!)} \\ & + \frac{1175490520893t^{18}}{64(18!)} + \sum_{r=15}^{\infty} \frac{L(r)}{(r+4)!} t^{r+4} > 0 \end{aligned}$$

with

$$\begin{aligned} L(r) &= 24\Big(3^{r+4} - 3(2^{r+4}) + 3\Big) \\ &\quad + 4(r+4)(r+3)(r+2)\left[\left(\frac{1}{2}\right)^{r+1} - 2 - 2^{r+3} - \left(\frac{5}{2}\right)^{r+1}\right] \\ &\quad + (r+4)(r+3)(r+2)(r+1)\left[\left(\frac{1}{2}\right)^{r} + 2 + 6\left(\frac{3}{2}\right)^{r} + 2^{r+1} + \left(\frac{5}{2}\right)^{r}\right] \\ &= 24\Big((1.5)^{r+4} - 3\Big)2^{r+4} + 72 + (r+4)(r+3)(r+2)\Bigg[(r+3)\left(\frac{1}{2}\right)^{r} + 2(r-3) \end{aligned}$$

$$+2^{r+1}(r-15)+(r-9)\left(\frac{5}{2}\right)^r+6(r+1)\left(\frac{3}{2}\right)^r\Bigg]$$
$$> \quad 0, \qquad r\geq 15.$$

Then, $2\leq \deg_{CM}^{x}\Big[U_{1/2}(x)\Big]$. However,

$$x^3\, U_{1/2}(x)=\int_0^\infty \frac{\Upsilon_2(t)}{24t^4(e^{\frac{t}{2}}-1)^4(e^{\frac{t}{2}}+1)^4}e^{-xt}dt, \qquad x>0$$

where

$$\begin{aligned}&\Upsilon_2(t)\\ =\ &-144(e^{4t}+1)-e^{\frac{7t}{2}}\Big(-6+t\Big)t^4-e^{\frac{t}{2}}\Big(6+t\Big)t^4-e^{\frac{5t}{2}}\Big(-30+23t\Big)t^4-e^{\frac{3t}{2}}\Big(30+23t\Big)t^4\\ &-4e^{3t}\Big(-144-9t^4+t^5\Big)-16e^{2t}\Big(54-6t^4+t^5\Big)-4e^{t}\Big(-144-3t^4+t^5\Big)\end{aligned}$$

with $\Upsilon_2(3.5)=686947$ and $\Upsilon_2(3.6)=-1.29751(10^6)$. Then $x^3\, U_{1/2}(x)$ is not a completely monotonic function and hence $\deg_{CM}^{x}\Big[U_{1/2}(x)\Big]<3$. □

Theorem 3.2. *Suppose that* $x>0$. *Then the function*

$$V_\alpha(x)=\ln\sqrt{2\pi}-\ln\Gamma(x)-\frac{1}{2}\psi(x)-x+x\ln x-\frac{1}{6}\psi'(x+\alpha), \quad \alpha\geq 0$$

is completely monotonic on $(0,\infty)$ *if and only if* $\alpha\geq\frac{1}{4}$. *Also,* $-V_\alpha(x)$ *is CM on* $(0,\infty)$ *if and only if* $\alpha=0$. *The functions* $V_{1/4}(x)$ *and* $-V_0(x)$ *satisfy that* $1\leq\deg_{CM}^{x}\Big[V_{1/4}(x)\Big]<2$ *and* $2\leq\deg_{CM}^{x}[-V_0(x)]<3$.

Proof. Using Binet's first formula [3]:

$$\ln\Gamma(x)=\left(x-\frac{1}{2}\right)\ln x-x+\ln\sqrt{2\pi}+\int_0^\infty\left[\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right]\frac{e^{-xt}}{t}dt, \qquad x>0 \tag{3.2}$$

we get

$$V_\alpha(x)=\frac{1}{2}\Big(\ln x-\psi(x)\Big)-\frac{1}{6}\psi'(x+\alpha)-\int_0^\infty\left[\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right]\frac{e^{-xt}}{t}dt$$

and by using (1.3), (1.4) and (3.1), we have $V_\alpha(x)=\int_0^\infty\frac{e^{-xt}}{t^2(e^t-1)}\varphi_\alpha(t)dt$, where

$$\varphi_\alpha(t)=e^t-1-te^t+\frac{t^2}{2}e^t-\frac{t^3}{6}e^{(1-\alpha)t}.$$

Let $\alpha\geq\frac{1}{4}$, then we obtain

$$\varphi_\alpha(t)\geq e^t-1-te^t+\frac{t^2}{2}e^t-\frac{t^3}{6}e^{\frac{3}{4}t}=\sum_{r=2}^{\infty}\frac{f(r)}{6\,(r+3)!}t^{r+3}>0,$$

where

$$\frac{4^r\, f(r)}{3(r+2)(r+1)}=4^r-3^r-r3^{r-1}=\sum_{s=2}^{r}\tbinom{r}{s}\,3^{r-s}>0.$$

Consequently, $V_\alpha(x)$ is completely monotonic on $(0,\infty)$ for $\alpha \geq \frac{1}{4}$. Conversely, if $V_\alpha(x)$ is completely monotonic, then we get for $x > 0$:

$$x^2\, V_\alpha(x) = x^2\left[V_0(x) + \frac{1}{6}\Big(\psi'(x) - \psi'(x+\alpha)\Big)\right] > 0. \tag{3.3}$$

Using the asymptotic expansions (1.2), (1.6) and (1.7), we have $\lim_{x\to\infty} x^2\, V_0(x) = -\frac{1}{24}$ and $\lim_{x\to\infty} x^2\left[\psi'(x) - \psi'(x+\alpha)\right] = \alpha$. From (3.3), we conclude that $-\frac{1}{24} + \frac{\alpha}{6} \geq 0$ and then $\alpha \geq \frac{1}{4}$. Now for $\alpha = 0$, we have

$$V_0(x) = -\int_0^\infty \frac{e^{-xt}}{t^2(e^t-1)}\left(\sum_{r=1}^{\infty} \frac{r(r+1)(r+2)t^{r+3}}{6\,(r+3)!}\right) dt$$

and consequently, $-V_0(x)$ is completely monotonic on $(0,\infty)$. Conversely, we assume that $-V_\alpha(x)$ is completely monotonic on $(0,\infty)$ with $\alpha > 0$, then

$$x\, V_\alpha(x) < 0, \qquad x > 0, \quad \alpha > 0. \tag{3.4}$$

Using functional equation(1.1) and the relations (2.5) and (2.6), we get $\lim_{x\to 0} x\, V_\alpha(x) = \frac{1}{2} > 0$ and this contradicts with (3.4) and hence $\alpha = 0$. Furthermore, using the asymptotic expansions (1.2), (1.6) and (1.7), we have

$$V_{1/4}(\infty) = \lim_{x\to\infty} V_{1/4}(x) = 0.$$

Now,

$$x\, V_{1/4}(x) = \int_0^\infty \frac{\Upsilon_3(t)}{24t^3(e^{\frac{t}{4}}-1)^2(e^{\frac{t}{4}}+1)^2(e^{\frac{t}{2}}+1)^2} e^{-xt} dt, \qquad x > 0$$

where

$$\begin{aligned}
\Upsilon_3(t) &= 48\Big(-1 + 2e^t - e^{2t}\Big) + 24t\Big(-e^t + e^{2t}\Big) + 24t^2e^t \\
&\quad +4t^3\Big(e^{\frac{3t}{4}} - 3e^t - e^{\frac{7t}{4}}\Big) + t^4\Big(3e^{\frac{3t}{4}} + e^{\frac{7t}{4}}\Big) \\
&= \frac{3t^6}{20} + \frac{31t^7}{160} + \frac{341t^8}{2688} + \frac{328311t^9}{16(9!)} + \frac{2245743t^{10}}{32(10!)} + \sum_{r=7}^{\infty} \frac{S(r)}{(r+4)!} t^{r+4} > 0
\end{aligned}$$

with

$$\begin{aligned}
S(r) &= 24r\Big(2^{r+3} - 1\Big) + 24(r+4)(3+r) + 3(4+r)(3+r)(2+r)^2\left(\frac{3}{4}\right)^r \\
&\quad +(r+4)(r+3)(r+2)\left[(r-6)\left(\frac{7}{4}\right)^r - 12\right] \\
&> 0, \qquad r \geq 7.
\end{aligned}$$

Then, $1 \leq \deg^x_{CM}\left[V_{1/4}(x)\right]$. However,

$$x^2\, V_{1/4}(x) = \int_0^\infty \frac{\Upsilon_4(t)}{96t^4(e^{\frac{t}{4}}-1)^3(e^{\frac{t}{4}}+1)^3(e^{\frac{t}{2}}+1)^3} e^{-xt} dt, \qquad x > 0$$

where

$$\begin{aligned}
&\Upsilon_4(t)\\
=\;&-576-192e^{3t}\Big(-3+t\Big)-e^{\frac{11t}{4}}\Big(-8+t\Big)t^4-3e^{\frac{3t}{4}}t^4\Big(8+3t\Big)-2e^{\frac{7t}{4}}t^4\Big(-8+11t\Big)\\
&+48e^{2t}\Big(-36+8t-4t^2-2t^3+t^4\Big)+48e^{t}\Big(36-4t+4t^2-2t^3+t^4\Big)
\end{aligned}$$

with $\Upsilon_4(6)\simeq 4.24(10^8)$ and $\Upsilon_4(6.1)\simeq -1.49(10^9)$. Then $x^2\ V_{1/4}(x)$ is not a completely monotonic function and hence $\deg_{CM}^{x}\Big[V_{1/4}(x)\Big]<2$. We also have

$$-V_0(\infty)=-\lim_{x\to\infty}V_0(x)=0$$

and

$$-x^2\ V_0(x)=\int_0^\infty\frac{\Upsilon_5(t)}{6t^4(e^t-1)^3}e^{-xt}dt,\qquad x>0$$

where

$$\begin{aligned}
\Upsilon_5(t)&=36\Big(1-3e^t+3e^{2t}-e^{3t}\Big)+12t\Big(e^t-2e^{2t}+e^{3t}\Big)+12t^2\Big(e^{2t}-e^t\Big)+6t^3\Big(e^t+e^{2t}\Big)\\
&\quad -t^4\Big(e^t+5e^{2t}\Big)+t^5\Big(e^t+e^{2t}\Big)\\
&=\frac{3t^7}{20}+\frac{t^8}{4}+\sum_{r=4}^{\infty}\frac{D(r)}{(r+5)!}t^{r+5}>0
\end{aligned}$$

with

$$\begin{aligned}
&D(r)\\
=\;&\Bigg(3888+606(-4+r)+647(-4+r)^2+209(-4+r)^3+25(-4+r)^4+(-4+r)^5\Bigg)2^r\\
&+4(-4+r)3^{r+5}+72+306r+214r^2+77r^3+14r^4+r^5\\
>\;&0,\qquad r\geq 4.
\end{aligned}$$

Then, $2\leq\deg_{CM}^{x}\left[-V_0(x)\right]$. However,

$$-x^3\ V_0(x)=\int_0^\infty\frac{\Upsilon_6(t)}{6t^5(e^t-1)^4}e^{-xt}dt,\qquad x>0$$

where

$$\begin{aligned}
\Upsilon_6(t)&=144-36e^{4t}\Big(-4+t\Big)-e^t\Big(576-36t+36t^2-18t^3+6t^4+t^6\Big)\\
&\quad -e^{3t}\Big(576-108t+36t^2+18t^3+6t^4-6t^5+t^6\Big)\\
&\quad -4e^{2t}\Big(-216+27t-18t^2+6t^4-3t^5+t^6\Big)
\end{aligned}$$

with $\Upsilon_6(0.8)=0.000141185$ and $\Upsilon_6(0.9)=-0.00773072$. Then $-x^3\ V_0(x)$ is not a completely monotonic function and hence $\deg_{CM}^{x}\left[V_0(x)\right]<3$. □

Theorem 3.3. *For $x>0$ and $\beta\geq 0$, the function*

$$W_\beta(x)=\ln\sqrt{2\pi}-\ln\Gamma(x+\beta)-(x+\beta)+(x+\beta)\ln(x+\beta)-\frac{1}{2}\psi(x+\beta)-\frac{1}{6}\psi'(x+\beta)+\frac{1}{24x^2}$$

is completely monotonic on $(0,\infty)$ *if and only if* $\beta \geq \frac{3}{10}$. *Also,* $-W_\beta(x)$ *is completely monotonic on* $(0,\infty)$ *if and only if* $\beta = 0$. *The functions* $W_{3/10}(x)$ *and* $-W_0(x)$ *satisfy that* $1 \leq \deg_{CM}^x\left[W_{3/10}(x)\right] < 2$ *and* $2 \leq \deg_{CM}^x\left[-W_0(x)\right] < 3$.

Proof. Using (1.3), (1.4), (3.1), (3.2) and the identity $\frac{1}{x^r} = \frac{1}{(r-1)!}\int_0^\infty t^{r-1}e^{-xt}dt$ for $x > 0$, (see [1]), we have

$$W_\beta(x) = \int_0^\infty \frac{e^{-(x+\beta)t}}{t^2(e^t-1)}\Omega_\beta(t)dt,$$

where

$$\Omega_\beta(t) = e^t - 1 - te^t + \frac{t^2}{2}e^t - \frac{t^3}{6}e^t + \frac{t^3}{24}\left(e^t-1\right)e^{\beta t}.$$

Let $\beta \geq \frac{3}{10}$, then we obtain

$$\Omega_\beta(t) \geq e^t - 1 - te^t + \frac{t^2}{2}e^t - \frac{t^3}{6}e^t + \frac{t^3}{24}\left(e^{\frac{13t}{10}} - e^{\frac{3t}{10}}\right)$$

and hence

$$\Omega_\beta(t) \geq \frac{17t^6}{14400} + \frac{41t^7}{42000} + \sum_{r=5}^\infty \frac{B(r)}{24\,(3+r)!}t^{r+3} > 0,$$

where

$$\begin{aligned}\frac{B(r)}{(r+2)(r+1)} &= -4r + (r+3)\left[\left(\frac{13}{10}\right)^r - \left(\frac{3}{10}\right)^r\right]\\ &= \frac{3(r-5)(r-2)}{10} + (r+3)\sum_{s=1}^{r-2}\binom{r}{s}\left(\frac{3}{10}\right)^{r-s}\\ &> 0, \qquad r \geq 5.\end{aligned}$$

Consequently, $W_\beta(x)$ is completely monotonic on $(0,\infty)$ for $\beta \geq \frac{3}{10}$. Conversely, if $W_\beta(x)$ is completely monotonic, then we have $W_\beta(x) > 0$ for $x > 0$ and by using the asymptotic expansions (1.2), (1.6) and (1.7), we have

$$\begin{aligned}\lim_{x\to\infty} x^3\, W_\beta(x) &= \lim_{x\to\infty} x^3\left(\frac{1}{24x^2} - \frac{1}{24(x+\beta)^2} - \frac{1}{40(x+\beta)^3} + O\left(\frac{1}{x^4}\right)\right)\\ &= \frac{\beta}{12} - \frac{1}{40}\\ &\geq 0\end{aligned}$$

and then $\beta \geq \frac{3}{10}$. Now for $\beta = 0$, we have

$$W_0(x) = -\int_0^\infty \frac{e^{-xt}}{t^2(e^t-1)}\left(\sum_{r=2}^\infty \frac{(-1+r)(1+r)(2+r)}{8\,(3+r)!}t^{3+r}\right)dt.$$

Consequently, $-W_0(x)$ is CM on $(0,\infty)$. Conversely, we assume that $-W_\beta(x)$ is CM on $(0,\infty)$ with $\beta > 0$, then

$$x^2\, W_\beta(x) < 0, \qquad x > 0, \quad \beta > 0. \tag{3.5}$$

Using the relations (1.3) and(1.4), we get $\lim_{x\to 0} x^2\, W_\beta(x) = \frac{1}{24} > 0$ and this contradicts with (3.5) and hence $\beta = 0$. Furthermore, using the asymptotic expansions (1.2), (1.6) and (1.7), we have

$$W_{3/10}(\infty)=\lim_{x\to\infty}W_{3/10}(x)=0.$$

Now,

$$x\,W_{3/10}(x)=\int_0^\infty \frac{\frac{\varrho_1(t)}{120t^3e^{\frac{3t}{10}}(e^{\frac{t}{10}}-1)^2(e^{\frac{t}{10}}+1)^2}}{\left(1-e^{\frac{t}{10}}+e^{\frac{t}{5}}-e^{\frac{3t}{10}}+e^{\frac{2t}{5}}\right)^2\left(1+e^{\frac{t}{10}}+e^{\frac{t}{5}}+e^{\frac{3t}{10}}+e^{\frac{2t}{5}}\right)^2}\,e^{-xt}dt,\ x>0$$

where

$$\begin{aligned}\varrho_1(t) &= 240\Big(-1+2e^t-e^{2t}\Big)+12t\Big(-3-4e^t+7e^{2t}\Big)+12t^2\Big(7e^t+3e^{2t}\Big)\\ &\quad +t^3\Big(5e^{\frac{3t}{10}}-22e^t-10e^{\frac{13t}{10}}-38e^{2t}+5e^{\frac{23t}{10}}\Big)+2t^4\Big(7e^t+3e^{2t}\Big)\\ &= \frac{17t^7}{40}+\frac{4769t^8}{8400}+\frac{3584529t^9}{25(9!)}+\frac{86234877t^{10}}{125(10!)}+\frac{7069861887t^{11}}{2500(11!)}+\frac{65056576899t^{12}}{6250(12!)}\\ &\quad +\frac{17736838749981t^{13}}{500000(13!)}+\sum_{r=10}^{\infty}\frac{L(r)}{(r+4)!}t^{r+4}>0\end{aligned}$$

with

$$\begin{aligned}L(r) &= 2^{r+1}\Big(-552+2r-165r^2-8r^3+3r^4\Big)+2\Big(552+334r+188r^2+59r^3+7r^4\Big)\\ &\quad +\frac{5(r+4)(r+3)(r+2)}{10^{r+1}}\Big(3^{r+1}-2(13^{r+1})+23^{r+1}\Big)\\ &= 2^{r+1}\Big[4968+6302(r-10)+1395(r-10)^2+112(r-10)^3+3(r-10)^4\Big]\\ &\quad +\frac{5(r+4)(r+3)(r+2)}{10^{r+1}}\Big[3^{r+1}+13^r(10r-3)+\sum_{s=2}^{r+1}\binom{r+1}{s}(13^{r+1-s})10^s\Big]\\ &\quad +2\Big(552+334r+188r^2+59r^3+7r^4\Big)\\ &> 0,\qquad r\geq 10.\end{aligned}$$

Then, $1\leq \deg_{CM}^{x}\Big[W_{3/10}(x)\Big]$.

But,

$$x^2\,W_{3/10}(x)=\int_0^\infty \frac{\frac{\varrho_2(t)}{600t^4e^{\frac{3t}{10}}(e^{\frac{t}{10}}-1)^3(e^{\frac{t}{10}}+1)^3}}{\left(1-e^{\frac{t}{10}}+e^{\frac{t}{5}}-e^{\frac{3t}{10}}+e^{\frac{2t}{5}}\right)^3\left(1+e^{\frac{t}{10}}+e^{\frac{t}{5}}+e^{\frac{3t}{10}}+e^{\frac{2t}{5}}\right)^3}\,e^{-xt}dt,\ x>0$$

where

$$\begin{aligned}\varrho_2(t) &= -18\Big(200+40t+3t^2\Big)+e^t\Big(10800+960t+1002t^2-294t^3+7t^4-49t^5\Big)\\ &\quad +e^{3t}\Big(3600-480t-306t^2-54t^3+87t^4-9t^5\Big)\\ &\quad -2e^{2t}\Big(5400-120t+321t^2+426t^3-253t^4+71t^5\Big)\end{aligned}$$

with $\varrho_2(8.4) = 1.61227(10^{14})$ and $\varrho_2(8.5) = -2.14035(10^{14})$. Then $x^2\, W_{3/10}(x)$ is not a completely monotonic function and hence $\deg_{CM}^{x}\left[W_{3/10}(x)\right] < 2$.

Also,

$$W_0(\infty) = \lim_{x\to\infty} W_0(x) = 0.$$

And

$$-x^2\, W_0(x) = \int_0^\infty \frac{\Lambda_1(t)}{6t^4(e^t-1)^3} e^{-xt} dt, \qquad x > 0$$

where

$$\begin{aligned}
\Lambda_1(t) &= 36\Big(1-3e^t+3e^{2t}-e^{3t}\Big)+12t\Big(e^t-2e^{2t}+e^{3t}\Big)+12t^2\Big(-e^t+e^{2t}\Big)\\
&\quad +6t^3\Big(e^t+e^{2t}\Big)-t^4\Big(e^t+5e^{2t}\Big)+t^5\Big(e^t+e^{2t}\Big)\\
&= \frac{3t^7}{20}+\frac{t^8}{4}+\frac{59t^9}{280}+\sum_{r=5}^{\infty}\frac{A(r)}{(r+5)!}t^{r+5} > 0
\end{aligned}$$

with

$$\begin{aligned}
A(r) &= 2^r\Big[3888+606(-4+r)+647(-4+r)^2+209(-4+r)^3+25(-4+r)^4+(-4+r)^5\Big]\\
&\quad +4(-4+r)3^{r+5}+72+306r+214r^2+77r^3+14r^4+r^5\\
&> 0, \qquad r \geq 4.
\end{aligned}$$

Then, $2 \leq \deg_{CM}^{x}\left[-W_0(x)\right]$. However,

$$-x^3\, W_0(x) = \int_0^\infty \frac{\Lambda_2(t)}{6t^5(e^t-1)^4} e^{-xt} dt, \qquad x > 0$$

where

$$\begin{aligned}
\Lambda_2(t) &= 144\Big(-1+e^t\Big)^4-36e^t\Big(-1+e^t\Big)^3 t-36e^t\Big(-1+e^t\Big)^2 t^2-18e^t\Big(-1+e^{2t}\Big)t^3\\
&\quad -6e^t\Big(1+4e^t+e^{2t}\Big)t^4+6e^{2t}\Big(2+e^t\Big)t^5-e^t\Big(1+4e^t+e^{2t}\Big)t^6
\end{aligned}$$

with $\Lambda_2(0.8) = 0.000141185$ and $\Lambda_2(0.9) = -0.00773072$. Then $-x^3\, W_0(x)$ is not a completely monotonic function and hence $\deg_{CM}^{x}\left[-W_0(x)\right] < 3$. □

4. Some new inequalities for the Γ, ψ and $\psi^{(m)}$ functions

From Theorem 3.1, we have the corollary:

Corollary 4.1. *For $x > 0$ and $0 \leq \xi \leq \frac{1}{2}$, we have*

$$-\psi(x) + \ln x < \frac{1}{6}\psi'(x) + \frac{1}{3}\psi'(x+\xi), \tag{4.1}$$

and for $m = 2, 3, \cdots$, we have

$$(-1)^m \psi^{(m-1)}(x) - \frac{(m-2)!}{x^{m-1}} < \frac{(-1)^{m+1}}{6}\psi^{(m)}(x) + \frac{(-1)^{m+1}}{3}\psi^{(m)}(x+\xi). \tag{4.2}$$

Remark 4.1.
- Using the decreasing property of the function $\psi'(x)$ on $(0,\infty)$, we get $\psi'(x) \geq \psi'(x+\xi)$ for $0 \leq \xi \leq \frac{1}{2}$ and then the upper bound of (4.1) refines the upper bound of (1.9) for every $x > 0$.
- Using the completely monotonic property of $\psi'(x)$ on $(0,\infty)$, we get $(-1)^{m+1}\ \psi^{(m+1)}(x) < 0$ for all $m \in \mathbb{N} \cup \{0\}$ and then $(-1)^{m+1}\psi^{(m)}(x+\xi) \leq (-1)^{m+1}\psi^{(m)}(x)$ for $0 \leq \xi \leq \frac{1}{2}$ and this proves that the upper bound of (4.2) refines the upper bound of (1.10) for every $x > 0$.

From Theorem 3.2, we get the following results:

Corollary 4.2. *Set $a, b \in [0,\infty)$ and $x \in (0,\infty)$. Then*

$$\exp\Big[-x - \frac{1}{2}\psi(x) - \frac{1}{6}\psi'(x+b)\Big] < \frac{\Gamma(x)}{\sqrt{2\pi}\ x^x} < \exp\Big[-x - \frac{1}{2}\psi(x) - \frac{1}{6}\psi'(x+a)\Big] \tag{4.3}$$

with the best constants $a = \frac{1}{4}$ and $b = 0$.

Proof. The right-hand side of (4.3) is equivalent to $x^2\ V_a(x) > 0$ and this leads to $a \geq \frac{1}{4}$ as stated in the proof of Theorem 3.2. Using the decreasing property of the function $\psi'(x)$ on $(0,\infty)$, we get $-\psi'(x+\frac{1}{4}) \leq -\psi'(x+a)$ for $a \geq \frac{1}{4}$ and then $a = \frac{1}{4}$ is the best constant in (4.3). Also, Theorem 3.2 gives the left-hand side of inequality (4.3) for $b = 0$. If there exist $b > 0$ such that left-hand side of (4.3) is valid for $x \in (0,\infty)$, then we would have

$$\lim_{h\to 0} x \ln\Gamma(x) > \lim_{x\to 0}\left[x^2 \ln x - x^2 + x\ln\sqrt{2\pi} - \frac{x}{2}\psi(x) - \frac{x}{6}\psi'(x+b)\right]$$

and using (2.5) and (2.6), we have $\lim_{x\to 0} x\ln\Gamma(x) > \frac{1}{2}$, which contradicts that $\lim_{x\to 0} x\ln\Gamma(x) = 0$ and then $b = 0$ is the best constant in (4.3). ☐

Remark 4.2. Using the completely monotonic property of $\psi'(x)$ on $(0,\infty)$, we deduce that the upper bound of (4.3) refines the upper bound of (1.8) for every $x > 0$.

Using (4.1) at $\xi = \frac{1}{4}$, we conclude that the upper bound of (4.3) refines the upper bound of (1.12) for every $x > 0$.

Using (2.3), we deduce that the upper bound of (4.3) at $a = \frac{1}{4}$ refines the upper bound of (1.11) for every $x > \frac{1}{8}$.

Corollary 4.3. *Set $a, b \in [0,\infty)$ and $x \in (0,\infty)$. Then*

$$\frac{1}{2}\psi'(x) + \frac{1}{6}\psi''(x+b) < -\psi(x) + \ln x < \frac{1}{2}\psi'(x) + \frac{1}{6}\psi''(x+a) \tag{4.4}$$

with the best constants $a = \frac{1}{4}$ and $b = 0$.

Proof. The right-hand side of (4.4) is equivalent to $V_a'(x) < 0$ and hence

$$x^3\ V_a'(x) = x^3\left[V_0'(x) + \frac{1}{6}\Big(\psi''(x) - \psi''(x+a)\Big)\right] < 0. \tag{4.5}$$

Using the asymptotic expansions (1.6) and (1.7), we have $\lim_{x\to\infty} x^3\ V_0'(x) = \frac{1}{12}$ and

$$\lim_{x\to\infty} x^3\left[\psi''(x) - \psi''(x+a)\right] = -2a.$$

From (4.5), we conclude that $\frac{1}{12}(-4a+1)\le 0$ and then $a\ge\frac{1}{4}$. Using the increasing property of the function $\psi''(x)$ on $(0,\infty)$, we deduce that $a=\frac{1}{4}$ is the best possible constant in (4.4). Also, Theorem 3.2 gives the left-hand side of inequality (4.4) for $b=0$. If there exist $b>0$ such that the left-hand side of (4.4) is valid for $x\in(0,\infty)$, then we would have

$$\lim_{x\to 0} x^2[-\psi(x)+\ln x] > \frac{1}{2}\lim_{x\to 0} x^2\psi'(x)+\frac{1}{6}\lim_{x\to 0} x^2\psi''(x+b)$$

and this leads to

$$-\lim_{x\to 0} x^2\psi(x) > \frac{1}{2}\lim_{x\to 0} x^2\psi'(x)+\frac{1}{6}\lim_{x\to 0} x^2\psi''(x+b). \tag{4.6}$$

From (2.5), (2.6) and (2.7), we have $\lim_{x\to 0} x^2\psi(x)=0$ and

$$\frac{1}{2}\lim_{x\to 0} x^2\psi'(x)+\frac{1}{6}\lim_{x\to 0} x^2\psi''(x+b)=\frac{1}{2},$$

which contradict with the inequality (4.6). Hence the best constant is $b=0$. □

Remark 4.3. Using the completely monotonic property of $-\psi''(x)$ on $(0,\infty)$, we deduce that the upper bound of (4.4) refines the upper bound of (1.9) for every $x>0$.

Using (4.2) at $\xi=\frac{1}{4}$ and $s=2$, we conclude that the upper bound of (4.4) at $a=\frac{1}{4}$ refines the upper bound of (1.13) for every $x>0$.

Using (2.4), we conclude that the upper bound of (4.4) at $a=\frac{1}{4}$ refines the upper bound of (1.16) for every $x\ge 4$.

Corollary 4.4. *Set $a,b\in[0,\infty), x\in(0,\infty)$ and $m=2,3,\cdots$. Then*

$$\begin{aligned}\frac{(-1)^{m+1}}{2}\psi^{(m)}(x)+\frac{(-1)^{m+1}}{6}\psi^{(m+1)}(x+b) &< -\frac{(m-2)!}{x^{m-1}}+(-1)^m\psi^{(m-1)}(x)\\ &< \frac{(-1)^{m+1}}{2}\psi^{(m)}(x)+\frac{(-1)^{m+1}}{6}\psi^{(m+1)}(x+a)\end{aligned} \tag{4.7}$$

with the best constants $a=\frac{1}{4}$ and $b=0$.

Proof. The right-hand side of (4.4) is equivalent to $(-1)^m V_a^{(m)}(x)>0$ for $m=2,3,\cdots$, and hence

$$\begin{aligned}(-1)^m\, x^{m+2}V_a^{(m)}(x) &= \frac{(-1)^m\, x^{m+2}\left[\psi^{(m+1)}(x)-\psi^{(m+1)}(x+a)\right]}{6}+(-1)^m\, x^{m+2}\, V_0^{(m)}(x)\\ &> 0.\end{aligned} \tag{4.8}$$

Using the asymptotic expansion (1.7), we have $\lim_{x\to\infty}(-1)^m\, x^{m+2}\, V_0^{(m)}(x)=-\frac{(m+1)!}{24}$ and

$$\lim_{x\to\infty}(-1)^m\, x^{m+2}\left[\psi^{(m+1)}(x)-\psi^{(m+1)}(x+a)\right]=(m+1)!\, a.$$

From (4), we conclude that $\frac{(m+1)!}{24}(-1+4a)\ge 0$ and then $a\ge\frac{1}{4}$. Using the completely monotonic property of $-\psi''(x)$ on $(0,\infty)$, we deduce that $(-1)^{m+1}\,\psi^{(m+2)}(x)>0$ on $(0,\infty)$, and then $(-1)^{m+1}\,\psi^{(m+1)}(x)$ is increasing on $(0,\infty)$, and hence $a=\frac{1}{4}$ is the best constant in (4.4). Also, Theorem 3.2 gives the left-hand side of inequality (4.4) for $b = 0$. If there exist $b > 0$ such that the left-hand side of (4.4) is valid for $x \in (0, \infty)$, then we would have

$$\lim_{x\to 0} x^{m+1}\Big[(-1)^m \psi^{(m-1)}(x) - \frac{(m-2)!}{x^{m-1}}\Big] > \frac{(-1)^{m+1}}{2} \lim_{x\to 0} x^{m+1}\, \psi^{(m)}(x) + \frac{(-1)^{m+1}}{6} \lim_{x\to 0} x^{m+1}\, \psi^{(m+1)}(x+b)$$

which leads to

$$(-1)^m \lim_{x\to 0} x^{m+1} \psi^{(m-1)}(x) > \frac{(-1)^{m+1}}{2} \lim_{x\to 0} x^{m+1}\, \psi^{(m)}(x) + \frac{(-1)^{m+1}}{6} \lim_{x\to 0} x^{m+1}\, \psi^{(m+1)}(x+b). \tag{4.9}$$

By using (2.5), (2.6) and (2.7), we have $\lim_{x\to 0} x^{m+1}\psi^{(m-1)}(x) = 0$ and

$$\frac{(-1)^{m+1}}{2} \lim_{x\to 0} x^{m+1}\, \psi^{(m)}(x) + \frac{(-1)^{m+1}}{6} \lim_{x\to 0} x^{m+1}\, \psi^{(m+1)}(x+b) = \frac{m!}{2}, \qquad m = 2, 3, \cdots$$

which contradict with the inequality (4). Hence the best constant is $b = 0$. ☐

Remark 4.4.
- Using the completely monotonic property of $\psi'(x)$ on $(0, \infty)$, we get $(-1)^{m+1}\, \psi^{(m+1)}(x) < 0$ and then the upper bound of (4.4) refines the upper bound of (1.10) for every $m = 2, 3, \cdots$ and $x > 0$.
- The relation (4.2) can be written as:

$$(-1)^{m+1}\psi^{(m)}(x) - \frac{(m-1)!}{x^m} < \frac{(-1)^m}{6}\psi^{(m+1)}(x) + \frac{(-1)^m}{3}\psi^{(m+1)}\left(x + \frac{1}{4}\right), \qquad m \in \mathbb{N}$$

  and this proves that the upper bound of (4.4) at $a = \frac{1}{4}$ improves the upper bound of (1) for all $m = 2, 3, \cdots$ and $x > 0$.

From Theorem 3.3, we get the following results:

Corollary 4.5. *Set $a \in [0, \infty)$ and $x \in (0, \infty)$. Then*

$$\exp\Big[-x - \frac{1}{2}\psi(x) - \frac{1}{6}\psi'(x) + \frac{1}{24x^2}\Big] < \frac{\Gamma(x)}{\sqrt{2\pi}\, x^x} < \exp\Big[-x - \frac{1}{2}\psi(x) - \frac{1}{6}\psi'(x) + \frac{1}{24(x-a)^2}\Big] \tag{4.10}$$

with the best constant $a = \frac{3}{10}$, where the upper bound is valid for $x > a$ and the lower bound is valid for $x > 0$.

Proof. The right-hand side of (4.10) is equivalent to $x^3\, W_a(x) > 0$ for $x > 0$ and this leads to $a \geq \frac{3}{10}$ as mentioned in the proof of Theorem 3.3. Using the decreasing property of the function $\frac{1}{y^2}$ on $(0, \infty)$, we deduce that $a = \frac{3}{10}$ is the best possible constant in (4.10). The left-hand side of (4.10) is equivalent to $W_0(x) < 0$ in Theorem 3.3. ☐

Remark 4.5.
- The lower bound of (4.10) refines the lower bound of (4.3) at $b = 0$ for all $x > 0$.
- Using (2.1), we deduce that the lower bound of (4.10) refines the lower bound of (1.8) for every $x \geq 2.2$.

- Using (2.2), we deduce that the upper bound of (4.10) at $a = \frac{3}{10}$ refines the upper bound of (1.11) for every $x \geq 0.9$.

Corollary 4.6. *Set $a \in [0, \infty)$ and $x \in (0, \infty)$. Then*

$$\frac{1}{2}\psi'(x) + \frac{1}{6}\psi''(x) + \frac{1}{12x^3} < -\psi(x) + \ln x < \frac{1}{2}\psi'(x) + \frac{1}{6}\psi''(x) + \frac{1}{12(x-a)^3} \tag{4.11}$$

with the best possible constants $a = \frac{3}{10}$, where the upper bound is valid for $x > a$ and the lower bound is valid for $x > 0$.

Proof. The right-hand side of (4.11) is equivalent to $W_a'(x) < 0$ for $x > 0$ and hence

$$x^4\, W_a'(x) = x^4 \left[\ln(a+x) - \psi(a+x) - \frac{1}{2}\psi'(a+x) - \frac{1}{6}\psi''(a+x) - \frac{1}{12x^3} \right] < 0. \tag{4.12}$$

Using the asymptotic expansions (1.6) and (1.7), we have

$$\lim_{h\to\infty} x^4\, W_a'(x) = \lim_{x\to\infty} x^4 \left[\frac{1}{12(x+a)^3} - \frac{1}{12x^3} + \frac{3}{40(x+a)^4} + O\left(\frac{1}{x^5}\right) \right] = -\frac{a}{4} + \frac{3}{40}.$$

From (4.12), we conclude that $a \geq \frac{3}{10}$. Using the decreasing property of the function $\frac{1}{y^3}$ on $(0, \infty)$, we deduce that $a = \frac{3}{10}$ is the best possible constant in (4.11). The left-hand side of (4.11) is equivalent to $W_0'(x) > 0$ in Theorem 3.3.

□

Remark 4.6. The lower bound of (4.11) refines the lower bound of (4.4) at $b = 0$ for all $x > 0$.

Corollary 4.7. *Set $a \in [0, \infty), x \in (0, \infty)$ and $m = 2, 3, \cdots$. Then*

$$\begin{aligned}\frac{(-1)^{m+1}}{2}\psi^{(m)}(x) + \frac{(-1)^{m+1}}{6}\psi^{(m+1)}(x) + \frac{(m+1)!}{24x^{m+2}} &< (-1)^m\psi^{(m-1)}(x) - \frac{(m-2)!}{x^{m-1}} \\ &< \frac{(-1)^{m+1}}{2}\psi^{(m)}(x) + \frac{(-1)^{m+1}}{6}\psi^{(m+1)}(x) + \frac{(m+1)!}{24(x-a)^{m+2}}\end{aligned} \tag{4.13}$$

with the best possible constants $a = \frac{3}{10}$, where the upper bound is valid for $x > a$ and the lower bound is valid for $x > 0$.

Proof. The right-hand side of (4.7) is equivalent to

$$\begin{aligned}(-1)^m W_a^{(m)}(x) = \frac{(m-2)!}{(x+a)^{m-1}} + (-1)^{m+1}\psi^{(m-1)}(x+a) + \frac{(-1)^{m+1}}{2}\psi^{(m)}(x+a) \\ + \frac{(-1)^{m+1}}{6}\psi^{(m+1)}(x+a) + \frac{(m+1)!}{24x^{m+2}} > 0, \qquad m = 2, 3, \cdots,\ x > 0\end{aligned}$$

and by using the asymptotic expansion (1.7), we have

$$\begin{aligned}\lim_{x\to\infty} x^{m+3}\,(-1)^m W_a^{(m)}(x) &= x^{m+3}\left[\frac{(1+m)!}{24x^{m+2}} - \frac{(1+m)!}{24(x+a)^{m+2}} - \frac{(2+m)!}{80(x+a)^{m+3}} + O\left(\frac{1}{x^{m+4}}\right)\right] \\ &= \frac{(m+2)!(-3+10a)}{240} \\ &\geq 0\end{aligned}$$

and then we have $a \geq \frac{3}{10}$. Using the decreasing property of the function $\frac{1}{y^{m+2}}$ for $m \in \mathbb{N}$ on $(0, \infty)$, we deduce that $a = \frac{3}{10}$ is the best possible constant in (4.7). The left-hand side of (4.7) is equivalent to $(-1)^m W_0^{(m)}(x) < 0$ in Theorem 3.3. □

Remark 4.7. The lower bound of (4.7) refines the lower bound of (4.4) at $b = 0$ for all $x > 0$.

5. Conclusion

The main conclusions of this paper are mentioned in Theorems 3.1, 3.2 and 3.3. The author proved the completely monotonic and the completely monotonic degree of three functions containing the gamma, digamma and polygamma functions, and derived some new bounds for Γ and $\psi^{(m)}$ $(m \in \mathbb{N} \cup \{0\})$. These bounds refine some recent results.

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