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LOGARITHMIC SEMI P-FUNCTION AND SOME NEW INEQUALITIES

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Abstract. In this study, we introduce and study the concept of logarithmic semi P-functions and their some algebraic properties. Then, we obtain the Hermite-Hadamard integral inequality for the log semi-P-functions. After that, we obtain some new inequalities by using Hölder, Hölder-İşcan and power-mean integral inequalities and show that the result obtained with Hölder-İşcan inequality gives better approach than the Hölder integral inequality. Some applications to special means of real numbers are also given.

1. Preliminaries and fundamentals

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. See articles [3, 6, 8, 9, 13] and the references therein. Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave. This double integral inequality is well known as the Hermite-Hadamard integral inequality [4]. Hermite-Hadamard integral inequality is one of the most studied result in inequalities via convex functions. It provides a necessary and sufficient condition for a function to be convex. Some refinements of the Hermite-Hadamard integral inequality for convex functions have been obtained [2, 15].

In [3], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

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Definition 1.1 ([3]). *A nonnegative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if the inequality*

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$. We will denote by $P(I)$ the set of P-functions on the interval I.

Theorem 1.1. *Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)\,dx \leq 2[f(a) + f(b)]. \tag{1.2}$$

In [10], Kadakal et al. introduced and studied the concept of semi P-functions and their some algebraic properties. Also, the authors compared the results obtained with both Hölder, Hölder-İşcan inequalities and power-mean, improved-power-mean integral inequalities and show that the result obtained with Hölder-İşcan and improved power-mean inequalities give better approach than the others.

Definition 1.2 ([10]). A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called semi P-function if for every $x, y \in I$ and $t \in [0,1]$,

$$f(tx + (1-t)y) \leq (tx + (1-t)y)[f(x) + f(y)].$$

We note that if the non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is semi P-function then

$$f(x) \leq 2xf(x)$$

for all $x \in I$, i.e. $(2x-1)f(x) \geq 0$ for all $x \in I$. In this case, we can say that either "$x \geq 1/2$ and $f(x) \geq 0$" or "$x \leq 1/2$ and $f(x) \leq 0$".

Theorem 1.2 ([10]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a semi P-function. If $a < b$ and $f \in L[a,b]$, then the following Hermite-Hadamard type inequalities hold:*

$$\frac{1}{a+b}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq (a+b)\frac{f(a)+f(b)}{2}.$$

In [7], the author introduced and studied the concept of multiplicatively P-functions (or log-P-function) and their some algebraic properties. After that she obtain some new Hermite-Hadamard type integral inequalities for this class functions. Then some applications to special means of real numbers are given.

Definition 1.3 ([7]). Let $I \neq \emptyset$ be an interval in $\mathbb{R}$. The function $f : I \to [0,\infty)$ is said to be multiplicatively P-function (or log-P-function), if the inequality

$$f(tx + (1-t)y) \leq f(x)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In [5], İşcan establishes new refinements for integral and sum forms of Hölder integral inequality. Many existing inequalities related to the Hölder inequality can be improved via newly obtained inequalities, which we illustrate by an application. The Hölder integral inequality is as follows:

Theorem 1.3 (Hölder-İşcan integral inequality [5])**.** *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned}
&\int_a^b |f(x)g(x)|\,dx \\
\leq\ & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\} \qquad (1.3)
\end{aligned}$$

This article is organized as follows: In chapter 2, we introduce a new concept, which is called semi P-function, and give some algebraic properties of semi P-function. We examine the connections of semi P-function with other types of convexity. Then, we obtain the Hermite-Hadamard inequality for the semi P-function. In chapter 3, by using an identity, we obtain some refinements of the Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is a semi P-function. After that, we compare the results obtained with both Hölder and Hölder-İşcan integral inequalities. We prove that the Hölder-İşcan integral inequality gives a better approximation than the Hölder integral inequality. In the last chapter, we give some applications to special means of real numbers.

2. The definition of logarithmic semi P-function

In recent years, many function classes such as P-function, geometric P-function, semi harmonically P-function, semi P-geometric-arithmetically function, etc. have been studied by many authors, and integral inequalities belonging to these function classes have been studied in the literature (see [3, 11, 12, 14]).

The main purpose of this manuscript is to introduce the concept of logarithmic semi P-function and establish some results connected with the Hermite-Hadamard integral inequality for logarithmic semi P-function. Then, a new function class, logarithmic semi P-function definition will be given, and the relations of this function class with the above-mentioned function classes will also be given. Moreover, we prove two Hermite-Hadamard type inequalities for the logarithmic semi P-functions. Moreover, we obtain some refinements of the Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is logarithmic semi P-functions. Also, some applications to special means of positive real numbers are also given.

Definition 2.1. Let $f: I\subset\mathbb{R}\to[1,\infty)$ be a function. If $\ln f$ is semi P-function, then the function f is called logarithmic semi P-function (or log semi P-function). That is, if for every $a,b\in I$ and $t\in[0,1]$

$$f\left(ta+(1-t)b\right)\leq\left[f\left(a\right)f\left(b\right)\right]^{ta+(1-t)b}. \qquad (2.1)$$

We will denote by $LSP(I)$ the class of all log semi P-functions on interval I. If this inequality reverses, then the function is called log semi P-concave.

Remark 2.1. We note that if the function $f : I \subseteq \mathbb{R} \to [1, \infty)$ is log semi P-function then

$$f(x) \leq [f(x) f(x)]^{tx+(1-t)x} = [f(x)]^{2x}$$

for all $x \in I$, i.e. $[f(x)]^{2x-1} \geq 1$ for all $x \in I$. In this case, we can say that either "$x \geq \frac{1}{2}$ and $f(x) \geq 1$" or "$x \leq \frac{1}{2}$ and $f(x) \leq 1$". Therefore, it must be $x \geq \frac{1}{2}$.

Example 2.1. The function $f : [1, \infty) \to \mathbb{R}$, $f(x) = e^x$ is a log semi P-function.

Example 2.2. The function $f : [1, \infty) \to \mathbb{R}$, $f(x) = e^{x^n}$, $n \in N$ is a log semi P-function.

Example 2.3. The function $f : [1, \infty) \to \mathbb{R}$, $f(x) = x$ is a log semi P-function.

Example 2.4. For every $c \in \mathbb{R}\,(c \geq 0)$, the function $f : \left[\frac{1}{2}, \infty\right) \subset \mathbb{R} \to \mathbb{R}$, $f(x) = c$ is a log semi P-function.

Remark 2.2. Let $f : I \subset [1, \infty) \to [1, \infty)$ be a multiplicatively P-function (or log-P-function). Then, every multiplicatively P-function is also log semi P-function. Indeed, since $1 \leq tx + (1-t)y$ for $1 \leq x < y$, we have

$$f(tx + (1-t)y) \leq f(x)f(y) \leq [f(x) f(y)]^{tx+(1-t)y}.$$

Theorem 2.1. *Let $f, g : I \subset [1/2, \infty) \to [1, \infty)$. If f and g are log semi P-functions, then fg is a log semi P-function.*

Proof. Let f, g be log semi P-functions, then

$$\begin{aligned}
(fg)(ta + (1-t)b) &= [f(ta + (1-t)b)]\,[g(ta + (1-t)b)] \\
&\leq [f(a)f(b)]^{ta+(1-t)b}\,[g(a)g(b)]^{ta+(1-t)b} \\
&= [f(a)g(a)]^{ta+(1-t)b}\,[f(b)g(b)]^{ta+(1-t)b} \\
&= [(fg)(a)]^{ta+(1-t)b}\,[(fg)(b)]^{ta+(1-t)b},
\end{aligned}$$

for every $a, b \in I$ and $t \in [0, 1]$. This completes the proof of theorem. $\square$

Theorem 2.2. *Let $f_\alpha : I \subset [1/2, \infty) \to [1, \infty)$ be an arbitrary family of log semi P-functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in I : f(u) < \infty\}$ is nonempty, then J is an interval and f is a log semi P-function on interval J.*

Proof. Let $t \in [0, 1]$ and $a, b \in J$ be arbitrary. Then

$$\begin{aligned}
f(ta + (1-t)b) &= \sup_\alpha f_\alpha(ta + (1-t)b) \\
&\leq \sup_\alpha [f_\alpha(a)f_\alpha(b)]^{ta+(1-t)b} \\
&\leq \sup_\alpha [f_\alpha(a)]^{ta+(1-t)b} \sup_\alpha [f_\alpha(b)]^{ta+(1-t)b} \\
&= [f_\alpha(a)]^{ta+(1-t)b}\,[f_\alpha(b)]^{ta+(1-t)b} \\
&= [f(a)]^{ta+(1-t)b}\,[f(b)]^{ta+(1-t)b} < \infty.
\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is a log semi P-function on J. This completes the proof of theorem. $\square$

Now, we establish Hermite-Hadamard integral inequality for log semi P-functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on interval $[a,b]$.

Theorem 2.3. *Let $f : I \subset [1/2,\infty) \to [1,\infty)$ be a log semi P-function. If $a<b$ and $f \in L[a,b]$, then the following Hermite-Hadamard type integral inequalities hold:*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{2}{a+b}} \leq \frac{1}{b-a}\int_a^b f(u)\, f(a+b-u)\, du \leq [f(a)f(b)]^{a+b}. \tag{2.2}$$

Proof. From the propery of the log semi P-function of f, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}[ta+(1-t)b]+\frac{1}{2}[(1-t)a+tb]\right) \\ &\leq [f(ta+(1-t)b)\, f((1-t)a+tb)]^{\frac{a+b}{2}}, \end{aligned}$$

thus, we have

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{2}{a+b}} \leq f(ta+(1-t)b)\, f((1-t)a+tb).$$

By changing the variable $u = ta+(1-t)b$ and taking integral in the last inequality with respect to $t \in [0,1]$, we deduce that

$$\begin{aligned} \left[f\left(\frac{a+b}{2}\right)\right]^{\frac{2}{a+b}} &\leq \int_0^1 f(ta+(1-t)b)\, f((1-t)a+tb)\, dt \\ &= \frac{1}{b-a}\int_a^b f(u)\, f(a+b-u)\, du. \end{aligned}$$

By using the property of the log semi P-function of f, if the variable is changed as $u = ta+(1-t)b$, then

$$\begin{aligned} \frac{1}{b-a}\int_a^b [f(u)\, f(a+b-u)]\, du &\leq \int_0^1 [f(a)f(b)]^{a+b}\, dt \\ &= [f(a)f(b)]^{a+b}. \end{aligned}$$

This completes the proof of theorem. $\square$

3. Some new inequalities for the log semi P-functions

The main purpose of this section is to obtain new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is log semi P-function. In this section, for shortness, we will denote by $A(a,b)$ the arithmetic mean of real numbers a and b. Dragomir and Agarwal [1] used the following lemma:

Lemma 3.1. *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt \tag{3.1}$$

Theorem 3.1. *Let $f : [1/2, \infty) \to [1, \infty)$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$ and assume that $f' \in L[a, b]$. If $|f'|$ is a log semi P-function on the interval $[a, b]$, then the following inequality holds*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (3.2)\\ \leq\ & \frac{1}{2}\frac{(|f'(a)|\,|f'(b)|)^a + (|f'(a)|\,|f'(b)|)^b}{\ln(|f'(a)|\,|f'(b)|)} + \frac{1}{2}\frac{4(|f'(a)|\,|f'(b)|)^{\frac{a+b}{2}} - 2(|f'(a)|\,|f'(b)|)^b}{(b-a)\ln^2(|f'(a)|\,|f'(b)|)}. \end{aligned}$$

Proof. By using the Lemma 3.1 and $|f'(ta+(1-t)b)| \leq [|f'(a)|\,|f'(b)|]^{ta+(1-t)b}$, we get

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\ \leq\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|\,[|f'(a)|\,|f'(b)|]^{ta+(1-t)b}\,dt\right) \\ =\ & \frac{1}{2}\frac{(|f'(a)|\,|f'(b)|)^a + (|f'(a)|\,|f'(b)|)^b}{\ln(|f'(a)|\,|f'(b)|)} + \frac{1}{2}\frac{4(|f'(a)|\,|f'(b)|)^{\frac{a+b}{2}} - 2(|f'(a)|\,|f'(b)|)^b}{(b-a)\ln^2(|f'(a)|\,|f'(b)|)}. \end{aligned}$$

This completes the proof of theorem. □

Theorem 3.2. *Let $f : [1/2, \infty) \to [1, \infty)$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$, $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and assume that the function $f' \in L[a, b]$. If $|f'|^q$ is a log semi P-function on the interval $[a, b]$, then the following inequality holds*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (3.3)\\ \leq\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{[|f'(a)|^q\,|f'(b)|^q]^b - [|f'(a)|^q\,|f'(b)|^q]^a}{(b-a)\ln[|f'(a)|^q\,|f'(b)|^q]}\right)^{\frac{1}{q}} \end{aligned}$$

Proof. By using the Lemma 3.1, Hölder's integral inequality and $|f'(ta+(1-t)b)|^q \leq [|f'(a)|^q\,|f'(b)|^q]^{ta+(1-t)b}$ which is the log semi P-function of the function $|f'|^q$, we obtain

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\ \leq\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[|f'(a)|^q\,|f'(b)|^q\right]^{ta+(1-t)b}\, dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left[|f'(a)|^q\,|f'(b)|^q\right]^b-\left[|f'(a)|^q\,|f'(b)|^q\right]^a}{(b-a)\ln\left[|f'(a)|^q\,|f'(b)|^q\right]}\right)^{\frac{1}{q}},
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1 |1-2t|^p\, dt &= \frac{1}{p+1}, \\
\int_0^1 \left[|f'(a)|^q\,|f'(b)|^q\right]^{ta+(1-t)b}\, dt &= \frac{\left[|f'(a)|^q\,|f'(b)|^q\right]^b-\left[|f'(a)|^q\,|f'(b)|^q\right]^a}{(b-a)\ln\left[|f'(a)|^q\,|f'(b)|^q\right]},
\end{aligned}
$$

This completes the proof of theorem. $\square$

Theorem 3.3. *Let $f:[1/2,\infty)\to[1,\infty)$ be a differentiable function on I°, $a,b\in I^{\circ}$ with $a<b$, $q\geq 1$ and assume that $f'\in L[a,b]$. If the function $|f'|^q$ is a log semi P-function on the interval $[a,b]$, then the following inequality holds*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (3.4)\\
&\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{\left(|f'(a)|^q\,|f'(b)|^q\right)^a+\left(|f'(a)|^q\,|f'(b)|^q\right)^b}{(b-a)\ln\left(|f'(a)|^q\,|f'(b)|^q\right)} \\
&\quad+\frac{4\left(|f'(a)|^q\,|f'(b)|^q\right)^{\frac{a+b}{2}}-2\left(|f'(a)|^q\,|f'(b)|^q\right)^b}{(b-a)^2\ln^2\left(|f'(a)|^q\,|f'(b)|^q\right)}.
\end{aligned}
$$

Proof. Assume first that $q>1$. From the Lemma 3.1, well known Hölder integral inequality and the property of the log semi P-function of the function $|f'|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{2}\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right| dt \\
&\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|\, dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left[|f'(a)|^q\,|f'(b)|^q\right]^{ta+(1-t)b} dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left(|f'(a)|^q\,|f'(b)|^q\right)^a+\left(|f'(a)|^q\,|f'(b)|^q\right)^b}{(b-a)\ln\left(|f'(a)|^q\,|f'(b)|^q\right)}\right. \\
&\quad\left.+\frac{4\left(|f'(a)|^q\,|f'(b)|^q\right)^{\frac{a+b}{2}}-2\left(|f'(a)|^q\,|f'(b)|^q\right)^b}{(b-a)^2\ln^2\left(|f'(a)|^q\,|f'(b)|^q\right)}\right)^{\frac{1}{q}}
\end{aligned}
$$

where

$$
\int_0^1 |1-2t|dt=\frac{1}{2}
$$

$$\begin{aligned}
&\int_0^1 |1-2t| \left[|f'(a)|^q |f'(b)|^q\right]^{ta+(1-t)b} dt \\
&= \frac{\left(|f'(a)|^q |f'(b)|^q\right)^a + \left(|f'(a)|^q |f'(b)|^q\right)^b}{(b-a)\ln\left(|f'(a)|^q |f'(b)|^q\right)} + \frac{4\left(|f'(a)|^q |f'(b)|^q\right)^{\frac{a+b}{2}} - 2\left(|f'(a)|^q |f'(b)|^q\right)^b}{(b-a)^2 \ln^2\left(|f'(a)|^q |f'(b)|^q\right)}.
\end{aligned}$$

For $q=1$ we use the estimates from the proof of Theorem 3.1, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 3.1. *Under the assumption of Theorem 3.3 with $q=1$, we get the conclusion of Theorem 3.1.*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{1}{2}\frac{\left(|f'(a)|\,|f'(b)|\right)^a + \left(|f'(a)|\,|f'(b)|\right)^b}{\ln\left(|f'(a)|\,|f'(b)|\right)} + \frac{1}{2}\frac{4\left(|f'(a)|\,|f'(b)|\right)^{\frac{a+b}{2}} - 2\left(|f'(a)|\,|f'(b)|\right)^b}{\ln^2\left(|f'(a)|\,|f'(b)|\right)}.
\end{aligned}$$

Now, we will prove the Theorem 3.2 by using Hölder-İşcan integral inequality. Then we will show that the result we have obtained in this theorem gives a better approach than that obtained in the Theorem 3.2.

Theorem 3.4. *Let $f:[1/2,\infty)\to[1,\infty)$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $q>1, \frac{1}{p}+\frac{1}{q}=1$ and assume that $f'\in L[a,b]$. If the function $|f'|^q$ is a log semi P-function on interval $[a,b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[(K_1)^{\frac{1}{q}} + (K_2)^{\frac{1}{q}}\right], \tag{3.5}$$

where

$$\begin{aligned}
K_1 &= \int_0^1 (1-t)\left[|f'(a)|^q |f'(b)|^q\right]^{ta+(1-t)b} \\
&= \frac{\left(|f'(a)|^q |f'(b)|^q\right)^b}{(b-a)\ln\left[|f'(a)|^q |f'(b)|^q\right]} - \frac{\left(|f'(a)|^q |f'(b)|^q\right)^b - \left(|f'(a)|^q |f'(b)|^q\right)^a}{(b-a)^2\ln^2\left[|f'(a)|^q |f'(b)|^q\right]}, \\
K_2 &= \int_0^1 t\left[|f'(a)|^q |f'(b)|^q\right]^{ta+(1-t)b} dt \\
&= \frac{\left(|f'(a)|^q |f'(b)|^q\right)^b - \left(|f'(a)|^q |f'(b)|^q\right)^a}{(b-a)^2\ln^2\left[|f'(a)|^q |f'(b)|^q\right]} - \frac{\left(|f'(a)|^q |f'(b)|^q\right)^a}{(b-a)\ln\left[|f'(a)|^q |f'(b)|^q\right]}.
\end{aligned}$$

Proof. By using Lemma 3.1, Hölder-İşcan integral inequality and $|f'(ta+(1-t)b)|^q \le \left[|f'(a)|^q |f'(b)|^q\right]^{ta+(1-t)b}$ which is the log semi P-function of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\
&\le \frac{b-a}{2}\left(\int_0^1 (1-t)\,|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{b-a}{2}\left(\int_0^1 t\left|1-2t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 (1-t)\left|1-2t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]^{ta+(1-t)b} dt\right)^{\frac{1}{q}} \\
&+\frac{b-a}{2}\left(\int_0^1 t\left|1-2t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]^{ta+(1-t)b} dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{2\left(p+1\right)}\right)^{\frac{1}{p}}\left[\left(K_1\right)^{\frac{1}{q}}+\left(K_2\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

where

$$
\int_0^1 (1-t)\left|1-2t\right|^p dt=\int_0^1 t\left|1-2t\right|^p dt=\frac{1}{2\left(p+1\right)}.
$$

This completes the proof of theorem. □

Remark 3.1. The inequality (3.5) gives better results than the inequality (3.3). Let us show that

$$
\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\left(K_1\right)^{\frac{1}{q}}+\left(K_2\right)^{\frac{1}{q}}\right]\leq\left(\frac{\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]^b-\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]^a}{(b-a)\ln\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]}\right)^{\frac{1}{q}}.
$$

Using concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^\lambda, 0<\lambda\leq 1$ by simple calculation we get

$$
\begin{aligned}
&\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\left(K_1\right)^{\frac{1}{q}}+\left(K_2\right)^{\frac{1}{q}}\right] \\
=\ & \left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\frac{\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)^b}{(b-a)\ln\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]}-\frac{\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)^b-\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)^a}{(b-a)^2\ln^2\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]}\right]^{\frac{1}{q}} \\
&+\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\frac{\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)^b-\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)^a}{(b-a)^2\ln^2\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]}-\frac{\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)^a}{(b-a)\ln\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]}\right]^{\frac{1}{q}} \\
\leq\ & 2\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\frac{1}{2}\frac{\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)^b-\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)^a}{(b-a)\ln\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]}\right]^{\frac{1}{q}} \\
=\ & \left(\frac{\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)^b-\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)^a}{(b-a)\ln\left[\left|f'(a)\right|^q\left|f'(b)\right|^q\right]}\right)^{\frac{1}{q}}
\end{aligned}
$$

which is the required result.

4. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers a,b with $b>a$:

1. The arithmetic mean

$$
A:=A(a,b)=\frac{a+b}{2},\quad a,b\geq 0,
$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \quad a,b \geq 0$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b}, \quad a,b > 0,$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a = b \end{cases} ; \quad a,b > 0$$

5. The p-logaritmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R} \backslash \{-1,0\} \\ a, & a = b \end{cases} ; \quad a,b > 0.$$

6.The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b > 0,$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature: $H \leq G \leq L \leq I \leq A$. It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 4.1. *Let $a, b \in [1, \infty)$ with $a < b$ and $n \in \mathbb{N}$. Then, the following inequalities are obtained:*

$$\exp\left\{A^{n-1}(a,b)\right\} \leq \frac{1}{b-a}\int_a^b e^{u^n + (a+b-u)^n} du \leq \exp\left\{4A(a,b)A(a^n,b^n)\right\}$$

or

$$A^{n-1}(a,b) \leq \ln\left\{\frac{1}{b-a}\int_a^b e^{u^n + (a+b-u)^n} du\right\} \leq 4A(a,b)A(a^n,b^n)$$

Proof. The assertion follows from the inequalities (2.2) for the function

$$f(x) = e^{x^n}, \quad x \in [1, \infty).$$

□

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