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EXPONENTIALLY $s-$CONVEX FUNCTIONS IN THE SECOND SENSE ON THE CO-ORDINATES AND SOME NOVEL INTEGRAL INEQUALITIES

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Abstract. This article describes new classes of convexity, namely exponentially $s-$convex functions in the second sense on the co-ordinates. Then, some new integral inequalities are proved by using some classical inequalities and properties of exponentially $s-$convex functions in the second sense on the co-ordinates.

1. Introduction

The concept of convex function, which has an important place in inequality theory, is a structure that stands out among the function classes known in mathematics with its properties, geometric interpretation, wide usage areas and aesthetic structure. Because of this feature, it has been used by many researchers and has been widely used, especially in the field of inequality theory. Convex functions whose definition is given by an inequality have a structure expressed with the help of the mean function. Moreover, the basic idea in the definition of the convex function is to make a comparison between the appearance of the linear combination of two points under the function and the linear combination of the images of these points. Let's start with the definition of this function class.

Definition 1.1. (See [18]) Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

The main purpose of the studies on different types of convexity is to optimize the bounds and generalize some known classical inequalities. An important class of convex functions, the definition of which has been given with the motivation of this main purpose, is exponentially convex functions, and the definition is given as follows.

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Definition 1.2. (See [12]) A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be exponentially convex function, if

$$f((1-t)x+ty) \leq (1-t)\frac{f(x)}{e^{\alpha x}} + t\frac{f(y)}{e^{\alpha y}}$$

for all $x, y \in I, \alpha \in \mathbb{R}$ and $t \in [0,1]$.

If the above inequality holds in the reversed sense, then f is said to be exponentially concave function. Note that if $\alpha = 0$, then the class of exponentially convex functions reduce to class of classical convex function. However, the converse is not true.

In [14], Dragomir mentioned an expansion of the concept of convex function, which is used in many inequalities in the field of inequality theory and has applications in different fields of mathematics, especially convex programming.

Definition 1.3. (See [14]) Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

Expressing convex functions in co-ordinates brought up the question that it is possible for Hermite-Hadamard inequality to expand into co-ordinates. The answer to this motivating question has been found in Dragomir's paper (see [14]) and has taken its place in the literature as the expansion of Hermite-Hadamard inequality to a rectangle from the plane $\mathbb{R}^2$ stated below.

Theorem 1.1. *Suppose that $f : \Delta = [a,b] \times [c,d] \to [0,\infty)$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp.

In [13], W.W. Breckner defined the s-convex in the second sense class as follows

Definition 1.4. A function $F : \mathbb{R}^+ \to \mathbb{R}$, is said to be $s-$convex in the second sense if

$$F(\beta_1 u_1 + \beta_2 u_2) \leq \beta_1^s F(u_1) + \beta_2^s F(u_2)$$

for all $\beta_1, \beta_2 \geq 0$, $u_1, u_2 \geq 0$ with $\beta_1 + \beta_2 = 1$ and for some fixed $s \in (0,1]$. This class of functions is denoted by K_s^2.

Aslan and Akdemir gave the definition of exponentially convex function in co-ordinates in 2022 as follows.

Definition 1.5. (See [8]) Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \longrightarrow \mathbb{R}$ is exponentially convex on the co-ordinates on Δ, if the following inequality holds,

$$f(tx + (1-t)z, ty + (1-t)w) \leq t\frac{f(x,y)}{e^{\alpha(x+y)}} + (1-t)\frac{f(z,w)}{e^{\alpha(z+w)}}$$

for all $(x, y), (z, w) \in \Delta, \alpha \in \mathbb{R}$ and $t \in [0, 1]$.

In [8], Aslan and Akdemir gave a definition equivalent to the definition of the exponentially convex function in co-ordinates as follows.

Definition 1.6. The mapping $f : \Delta \longrightarrow \mathbb{R}$ is exponentially convex on the co-ordinates on Δ, if the following inequality holds,

$$\begin{aligned} & f(ta + (1-t)b, sc + (1-s)d) \\ \leq \ & ts\frac{f(a,c)}{e^{\alpha(a+c)}} + t(1-s)\frac{f(a,d)}{e^{\alpha(a+d)}} + (1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t)(1-s)\frac{f(b,d)}{e^{\alpha(b+d)}} \end{aligned}$$

for all $(a, c), (a, d), (b, c), (b, d) \in \Delta, \alpha \in \mathbb{R}$ and $t, s \in [0, 1]$.

In [8], Aslan and Akdemir proved a general result of exponentially convexity in co-ordinates as follows.

Theorem 1.2. *Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be partial differentiable mapping on $\Delta = [a, b] \times [c, d]$ and $f \in L(\Delta)$, $\alpha \in \mathbb{R}$. If f is exponentially convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \leq \frac{1}{4}\left[\frac{f(a,c)}{e^{\alpha(a+c)}} + \frac{f(a,d)}{e^{\alpha(a+d)}} + \frac{f(b,c)}{e^{\alpha(b+c)}} + \frac{f(b,d)}{e^{\alpha(b+d)}}\right].$$

Theorem 1.3. (See [4]) *Let $F : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$ and $\partial^2 F/\partial t\partial s \in L(\Delta)$, $\alpha, \beta \in (0, 1], \alpha_1 \in \mathbb{R}$. If $\left|\partial^2 F/\partial t\partial s\right|$ is exponentially convex function on the co-ordinates on Δ, then one has*

$$\begin{aligned} & \left|\frac{F(a,c) + F(a,d) + F(b,c) + F(b,d)}{4} + A\right| \\ \leq \ & \frac{(b-a)(d-c)}{4(\alpha+1)(\beta+1)} \times \left[\frac{\partial^2 F/\partial t\partial s\,(a,c)}{e^{\alpha_1(a+c)}} + \frac{\partial^2 F/\partial t\partial s\,(a,d)}{e^{\alpha_1(a+d)}} + \frac{\partial^2 F/\partial t\partial s\,(b,c)}{e^{\alpha_1(b+c)}} + \frac{\partial^2 F/\partial t\partial s\,(b,d)}{e^{\alpha_1(b+d)}}\right] \end{aligned}$$

where

$$A = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b [F(x,c) + F(x,d)]\,dx + \frac{1}{d-c}\int_c^d [F(a,y) + F(b,y)]\,dy\right].$$

Theorem 1.4. (See [4]) *Let $F : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$ and $\partial^2 F/\partial t\partial s \in L(\Delta)$, $\alpha, \beta \in (0, 1], \alpha_1 \in \mathbb{R}$. If $\left|\partial^2 F/\partial t\partial s\right|, q > 1,$ is exponentially convex function on the co-ordinates on Δ, then one has*

$$\begin{aligned} & \left|\frac{F(a,c) + F(a,d) + F(b,c) + F(b,d)}{4} + A\right| \\ \leq \ & \frac{(b-a)(d-c)}{4} \end{aligned}$$

$$\times\left[\frac{4}{p(\alpha+1)^p(\beta+1)^p}+\frac{\left|\partial^2 F/\partial t\partial s\,(a,c)\right|^q}{qe^{\alpha_1(a+c)}}+\frac{\left|\partial^2 F/\partial t\partial s\,(a,d)\right|^q}{qe^{\alpha_1(a+d)}}\right.$$
$$\left.+\frac{\left|\partial^2 F/\partial t\partial s\,(b,c)\right|^q}{qe^{\alpha_1(b+c)}}+\frac{\left|\partial^2 F/\partial t\partial s\,(b,d)\right|^q}{qe^{\alpha_1(b+d)}}\right],$$

where $p^{-1}+q^{-1}=1$ *and*

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[F(x,c)+F(x,d)\right]dx+\frac{1}{d-c}\int_c^d\left[F(a,y)+F(b,y)\right]dy\right].$$

For information on types of exponentially convexity in co-ordinates and $s-$convex functions, we recommend readers the following articles ([1–3, 5–12, 15–17, 19, 20]).

2. Main result

In this study, exponentially s-convex function ın the second sense on the co-ordinates have been introduced and a fundamental integral inequality of Hadamard-type has been proved for exponentially s-convex function in the second sense on the co-ordinates.

Definition 2.1. Let us consider the bidimensional interval $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. The mapping $f:\Delta\longrightarrow\mathbb{R}$ is exponentially s-convex in the second sense on Δ, if the following inequality holds,

$$f\left(tx+(1-t)\,z,ty+(1-t)\,w\right)\le t^s\frac{f(x,y)}{e^{\alpha(x+y)}}+(1-t)^s\frac{f(z,w)}{e^{\alpha(z+w)}}$$

for all $(x,y),(z,w)\in\Delta,\alpha\in R,s\in(0,1]$ and $t\in[0,1]$.

A definition equivalent to the exponentially $s-$ convex function definition in the second sense can be made as follows.

Definition 2.2. Let us consider the bidimensional interval $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. The mapping $f:\Delta\longrightarrow\mathbb{R}$ is exponentially s_1-convex in the second sense on the co-ordinates on Δ, if the following inequality holds,

$$\begin{aligned}&f\left(tx+(1-t)y,sz+(1-s)w\right)\\ \le\;& t^{s_1}s^{s_1}\frac{f(x,z)}{e^{\alpha(x+z)}}+t^{s_1}(1-s)^{s_1}\frac{f(x,w)}{e^{\alpha(x+w)}}+(1-t)^{s_1}s^{s_1}\frac{f(y,z)}{e^{\alpha(y+z)}}+(1-t)^{s_1}(1-s)^{s_1}\frac{f(y,w)}{e^{\alpha(y+w)}}\end{aligned}$$

for all $(x,z),(x,w),(y,z),(y,w)\in\Delta,\alpha\in R,s_1\in(0,1]$ and $t,s\in[0,1]$.

Lemma 2.1. *A function* $f:\Delta\longrightarrow\mathbb{R}$ *will be called exponentially s-convex in the second sense on the co-ordinates on* Δ, *if the partial mappings* $f_y:[a,b]\longrightarrow\mathbb{R}$, $f_y(u)=e^{\alpha y}f(u,y)$ *and* $f_x:[c,d]\longrightarrow\mathbb{R}$, $f_x(v)=e^{\alpha x}f(x,v)$ *are exponentially s-convex in the second sense on the co-ordinates on* Δ, *where defined for all* $y\in[c,d]$ *and* $x\in[a,b]$.

Proof. From the definition of partial mapping f_x, we can write

$$\begin{aligned}f_x(tv_1+(1-t)\,v_2)&=e^{\alpha x}f(x,tv_1+(1-t)\,v_2)\\&=e^{\alpha x}f(tx+(1-t)x,tv_1+(1-t)\,v_2)\end{aligned}$$

$$\begin{aligned} &\leq e^{\alpha x}\left[t^{s}\frac{f(x,v_1)}{e^{\alpha(x+v_1)}}+(1-t)^{s}\frac{f(x,v_2)}{e^{\alpha(x+v_2)}}\right] \\ &= t^{s}\frac{f(x,v_1)}{e^{\alpha v_1}}+(1-t)^{s}\frac{f(x,v_2)}{e^{\alpha v_2}} \\ &= t^{s}\frac{f_x(v_1)}{e^{\alpha v_1}}+(1-t)^{s}\frac{f_x(v_2)}{e^{\alpha v_2}}. \end{aligned}$$

Similarly,

$$\begin{aligned} f_y(tu_1+(1-t)\,u_2) &= e^{\alpha y}f(tu_1+(1-t)\,u_2,y) \\ &= e^{\alpha y}f(tu_1+(1-t)\,u_2,ty+(1-t)y) \\ &\leq e^{\alpha y}\left[t^{s}\frac{f(u_1,y)}{e^{\alpha(u_1+y)}}+(1-t)^{s}\frac{f(u_2,y)}{e^{\alpha(u_2+y)}}\right] \\ &= t^{s}\frac{f(u_1,y)}{e^{\alpha u_1}}+(1-t)^{s}\frac{f(u_2,y)}{e^{\alpha u_2}} \\ &= t^{s}\frac{f_y(u_1)}{e^{\alpha u_1}}+(1-t)^{s}\frac{f_y(u_2)}{e^{\alpha u_2}}. \end{aligned}$$

Proof is completed. □

Theorem 2.1. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be partial differentiable mapping on $\Delta=[a,b]\times[c,d]$ and $f\in L(\Delta)$, $\alpha\in\mathbb{R}$ and $s_1\in(0,1]$. If f is exponentially s_1-convex function in the second sense on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned} &\frac{1}{(b-a)\,(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\ &\leq \frac{1}{(s_1+1)^2}\left[\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{f(b,d)}{e^{\alpha(b+d)}}\right]. \end{aligned}$$

Proof. By the definition of the exponentially s_1-convex function in the second sense on the co-ordinates on Δ, we can write

$$\begin{aligned} &f\left(ta+(1-t)b,sc+(1-s)d\right) \\ &\leq t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{s_1}(1-s)^{s_1}\frac{f(a,d)}{e^{\alpha(a+d)}}+(1-t)^{s_1}s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t)^{s_1}(1-s)^{s_1}\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

By integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we have

$$\begin{aligned} &\int_0^1\int_0^1 f\left(ta+(1-t)b,sc+(1-s)d\right)dtds \\ &\leq \int_0^1\int_0^1 t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}}dtds+\int_0^1\int_0^1 t^{s_1}(1-s)^{s_1}\frac{f(a,d)}{e^{\alpha(a+d)}}dtds \\ &\quad+\int_0^1\int_0^1(1-t)^{s_1}s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}}dtds+\int_0^1\int_0^1(1-t)^{s_1}(1-s)^{s_1}\frac{f(b,d)}{e^{\alpha(b+d)}}dtds. \end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Remark 2.1. If we choose $\alpha=0$ and $s_1=1$ in Theorem (2.1), the result agrees with the Hadamard-type inequality proved by Dragomir (See [14]).

Remark 2.2. If we choose $\alpha = 0$ in Theorem (2.1), the result coincides with the Hadamard-type inequality proved by Alomari and Darus (See [7]).

Remark 2.3. If we choose $s_1 = 1$ in Theorem (2.1), the result will match Theorem (1.2).

Theorem 2.2. *Let $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ and $f \in L(\Delta)$, $\alpha \in \mathbb{R}$ and $s_1 \in (0,1]$. If $|f|$ is exponentially s_1-convex function in the second sense on the co-ordinates on Δ, $p > 1$ then the following inequality holds;*

$$\begin{aligned} &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy \right| \\ \leq\ & \left(\frac{1}{(ps_1+1)^2} \right)^{\frac{1}{p}} \left(\frac{|f(a,c)|}{e^{\alpha(a+c)}} + \frac{|f(a,d)|}{e^{\alpha(a+d)}} + \frac{|f(b,c)|}{e^{\alpha(b+c)}} + \frac{|f(b,d)|}{e^{\alpha(b+d)}} \right). \end{aligned}$$

Proof. By the definition of the exponentially s_1-convex function in the second sense on the co-ordinates on Δ, we can write

$$\begin{aligned} & f(ta+(1-t)b, sc+(1-s)d) \\ \leq\ & t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}} + t^{s_1}(1-s)^{s_1}\frac{f(a,d)}{e^{\alpha(a+d)}} + (1-t)^{s_1}s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t)^{s_1}(1-s)^{s_1}\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

By using the property of absolute value in integrals, the both sides of inequality above are integrated with respect to t, s on $[0,1]^2$, we can write

$$\begin{aligned} & \left| \int_0^1 \int_0^1 f(ta+(1-t)b, sc+(1-s)d)\, dtds \right| \\ \leq\ & \int_0^1 \int_0^1 \left| t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}} \right| dtds + \int_0^1 \int_0^1 \left| t^{s_1}(1-s)^{s_1}\frac{f(a,d)}{e^{\alpha(a+d)}} \right| dtds \\ & + \int_0^1 \int_0^1 \left| (1-t)^{s_1}s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}} \right| dtds + \int_0^1 \int_0^1 \left| (1-t)^{s_1}(1-s)^{s_1}\frac{f(b,d)}{e^{\alpha(b+d)}} \right| dtds. \end{aligned}$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\begin{aligned} & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy \right| \\ \leq\ & \left(\int_0^1 \int_0^1 t^{ps_1}s^{ps_1}dtds \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{f(a,c)}{e^{\alpha(a+c)}} \right|^q dtds \right)^{\frac{1}{q}} \\ & + \left(\int_0^1 \int_0^1 t^{ps_1}(1-s)^{ps_1}dtds \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{f(a,d)}{e^{\alpha(a+d)}} \right|^q dtds \right)^{\frac{1}{q}} \\ & + \left(\int_0^1 \int_0^1 (1-t)^{ps_1}s^{ps_1}dtds \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{f(b,c)}{e^{\alpha(b+c)}} \right|^q dtds \right)^{\frac{1}{q}} \\ & + \left(\int_0^1 \int_0^1 (1-t)^{ps_1}(1-s)^{ps_1}dtds \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{f(b,d)}{e^{\alpha(b+d)}} \right|^q dtds \right)^{\frac{1}{q}}. \end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Theorem 2.3. *Let $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ and $f \in L(\Delta)$, $\alpha \in \mathbb{R}$ and $s_1 \in (0,1]$. If $|f|$ is exponentially s_1-convex function* *in the second sense on the co-ordinates on* Δ*,* $p, q > 1$*,* $\frac{1}{p}+\frac{1}{q}=1$*, then the following inequality holds;*

$$\begin{aligned} &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\ \leq\ & \left(\frac{1}{p\left(ps_1+1\right)^2}\right)+\frac{1}{q}\left(\frac{|f(a,c)|^q}{e^{\alpha q(a+c)}}+\frac{|f(a,d)|^q}{e^{\alpha q(a+d)}}+\frac{|f(b,c)|^q}{e^{\alpha q(b+c)}}+\frac{|f(b,d)|^q}{e^{\alpha q(b+d)}}\right).\end{aligned}$$

Proof. By the definition of the exponentially s_1-convex function in the second sense on the co-ordinates on Δ, we can write

$$\begin{aligned} & f\left(ta+(1-t)b, sc+(1-s)d\right) \\ \leq\ & t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{s_1}(1-s)^{s_1}\frac{f(a,d)}{e^{\alpha(a+d)}}+(1-t)^{s_1}s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t)^{s_1}(1-s)^{s_1}\frac{f(b,d)}{e^{\alpha(b+d)}}.\end{aligned}$$

By using the property of absolute value in integrals, the both sides of inequality above are integrated with respect to t, s on $[0,1]^2$, we can write

$$\begin{aligned} & \left|\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+(1-s)d\right)dtds\right| \\ \leq\ & \int_0^1\int_0^1\left|t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|t^{s_1}(1-s)^{s_1}\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds \\ & +\int_0^1\int_0^1\left|(1-t)^{s_1}s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|(1-t)^{s_1}(1-s)^{s_1}\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds\end{aligned}$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\begin{aligned} & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\ \leq\ & \frac{1}{p}\left(\int_0^1\int_0^1 t^{ps_1}s^{ps_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right) \\ & +\frac{1}{p}\left(\int_0^1\int_0^1 t^{ps_1}(1-s)^{ps_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right) \\ & +\frac{1}{p}\left(\int_0^1\int_0^1 (1-t)^{ps_1}s^{ps_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right) \\ & +\frac{1}{p}\left(\int_0^1\int_0^1 (1-t)^{ps_1}(1-s)^{ps_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right).\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

The property of the sum of two functions related to exponentially s-convexity in the second sense on the co-ordinates and the multiplication of a function with a constant is proven below.

Proposition 2.1. *If* $f, g : \Delta \to \mathbb{R}$ *are two exponentially* s_1*-convex function in the second sense on the co-ordinates on* Δ*, then* $f+g$ *are exponentially* s_1*-convex function in the second sense on the co-ordinates on* Δ*.*

Proof. By the definition of the exponentially s_1-convex function in the second sense on the co-ordinates on Δ, we can write

$$\begin{aligned} & f\left(ta+(1-t)b, sc+(1-s)d\right)+g\left(ta+(1-t)b, sc+(1-s)d\right) \\ \leq\ & t^{s_1}s^{s_1}\left(\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{g(a,c)}{e^{\alpha(a+c)}}\right)+t^{s_1}(1-s)^{s_1}\left(\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{g(a,d)}{e^{\alpha(a+d)}}\right) \\ & +(1-t)^{s_1}s^{s_1}\left(\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{g(b,c)}{e^{\alpha(b+c)}}\right)+(1-t)^{s_1}(1-s)^{s_1}\left(\frac{f(b,d)}{e^{\alpha(b+d)}}+\frac{g(b,d)}{e^{\alpha(b+d)}}\right). \end{aligned}$$

Namely,

$$\begin{aligned} & (f+g)\left(ta+(1-t)b, sc+(1-s)d\right) \\ \leq\ & t^{s_1}s^{s_1}\frac{(f+g)(a,c)}{e^{\alpha(a+c)}}+t^{s_1}(1-s)^{s_1}\frac{(f+g)(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t)^{s_1}s^{s_1}\frac{(f+g)(b,c)}{e^{\alpha(b+c)}}+(1-t)^{s_1}(1-s)^{s_1}\frac{(f+g)(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore $(f+g)$ is exponentially s_1-convex in the second sense on the co-ordinates on Δ. □

Proposition 2.2. *If $f:\Delta\to\mathbb{R}$ is exponentially s_1-convex function in the second sense on the co-ordinates on Δ and $k\geq 0$ then kf is exponentially s_1-convex function in the second sense on the co-ordinates on Δ.*

Proof. By the definition of the exponentially s_1-convex function in the second sense on the co-ordinates on Δ, we can write

$$\begin{aligned} & f\left(ta+(1-t)b, sc+(1-s)d\right) \\ \leq\ & t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{s_1}(1-s)^{s_1}\frac{f(a,d)}{e^{\alpha(a+d)}}+(1-t)^{s_1}s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t)^{s_1}(1-s)^{s_1}\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

If both sides are multiplied by k, we have,

$$\begin{aligned} & (kf)\left(ta+(1-t)b, sc+(1-s)d\right) \\ \leq\ & t^{s_1}s^{s_1}\frac{(kf)(a,c)}{e^{\alpha(a+c)}}+t^{s_1}(1-s)^{s_1}\frac{(kf)(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t)^{s_1}s^{s_1}\frac{(kf)(b,c)}{e^{\alpha(b+c)}}+(1-t)^{s_1}(1-s)^{s_1}\frac{(kf)(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore (kf) is exponentially s_1-convex function in the second sense on the co-ordinates on Δ. □

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