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TWO INEQUALITIES FOR THE MEDIANS OF A TRIANGLE

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ABSTRACT. With the help of software Maple, we establish two geometric inequalities involving medians, circumradius and inradius of a triangle. One of them is the best possible inequality in the strong sense. We also propose four related conjectures checked by the computer.

1. INTRODUCTION

Let ABC be a triangle with circumradius R and inradius r. Let m_a, m_b, m_c be its three medians.

We have known a few inequalities involving m_a, m_b, m_c, R and r in the literature. F. Leuenberger established the following linear inequality (see [1], inequality 8.2):

$$\sum m_a \leq 4R + r. \tag{1.1}$$

where $\sum$ denotes the cyclic sum. Recently, the author [12] pointed that there exist a flaw in the proof given in [1], where the proof satisfies only the case for the acute triangle. We also gave a simple proof of this inequality in [12].

In [3], the author established the following inequality involving the reciprocal sum of medians for a triangle:

$$\sum \frac{1}{m_a} \leq \frac{2}{3}\left(\frac{1}{R} + \frac{1}{r}\right). \tag{1.2}$$

In [9], the author proved the reverse inequality of (1.2):

$$\sum \frac{1}{m_a} \geq \frac{5}{2R + r}. \tag{1.3}$$

In [13], the author established an inequality chain including inequalities (1.2) and (1.3), that is

$$\frac{1}{3}\sum \frac{1}{w_a - 2r} \geq 2\sum \frac{1}{h_a + 3r} \geq \frac{2}{3}\sum \frac{1}{w_a - r} \geq \frac{2}{3}\sum \frac{w_a + r_a}{bc}$$

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$$\geq \frac{2}{3}\left(\frac{1}{R}+\frac{1}{r}\right) \geq \sum \frac{1}{m_a} \geq \frac{5}{s}\sqrt{\frac{\sum h_a + 18r}{\sum h_a + 16r}} \geq \frac{5}{s}\sqrt{\frac{2R+23r}{2R+21r}}$$
$$\geq \frac{5}{s} + (6\sqrt{3}-10)\frac{r}{sR} \geq \frac{5}{2R+r}, \tag{1.4}$$

where a, b, c are the sides of triangle ABC, h_a, h_b, h_c the altitudes, w_a, w_b, w_c the angle-bisectors, r_a, r_b, r_c the radii of excircles and $s = (a+b+c)/2$.

For the sum $\sum \frac{1}{m_b + m_c}$, what is an inequality similar to (1.2) ? After studying, the author first found that the following inequality holds:

$$\sum \frac{1}{m_b + m_c} \leq \frac{1}{2R} + \frac{1}{4r}. \tag{1.5}$$

This bring us to obtain the following stronger result:

Theorem 1.1. *In any triangle ABC the following inequality holds:*

$$\sum \frac{1}{(m_b + m_c)^2} \leq \frac{1}{12}\left(\frac{1}{R} + \frac{1}{2r}\right)^2, \tag{1.6}$$

with equality if and only if triangle ABC is equilateral.

Inequality (1.5) can be obtained immediately from (1.6) by using the power mean inequality. Consequently, inequality (1.6) is better than (1.5).

For the sum $\sum \frac{1}{m_b + m_c}$, we also find the following result similar to (1.3):

Theorem 1.2. *In any triangle ABC the following inequality holds:*

$$\sum \frac{1}{m_b + m_c} \geq \frac{7}{6R + 2r}, \tag{1.7}$$

with equality if and only if triangle ABC is equilateral.

In fact, inequality (1.7) is a sharp result in the strong sense (see Remark 4.1 below).

Our goal of this paper is to prove Theorem 1.1 and 1.2. We shall also propose four related conjectures.

2. Preliminaries

In order to prove Theorem 1.1 and 1.2, we shall give some lemmas in this section.

Lemma 2.1. *For any triangle ABC with the area S, if the following inequality holds:*

$$f(a, b, c, m_a, m_b, m_c, S) \geq 0, \tag{2.1}$$

then it is equivalent to

$$f\left(m_a, m_b, m_c, \frac{3}{4}a, \frac{3}{4}b, \frac{3}{4}c, \frac{3}{4}S\right) \geq 0. \tag{2.2}$$

The above lemma is the central conclusion of "the Median-Dual Transformation" (see the monograph [14], p.109).

Lemma 2.2. *In any triangle ABC the following inequality holds:*

$$\sum m_a \geq \frac{1}{2R}\sum a^2, \tag{2.3}$$

with equality if and only if triangle ABC is equilateral.

We can find inequality (2.3) in [14] (p.213). In fact, inequality (2.3) can be obtained from the following inequality:

$$m_a \geq \frac{b^2+c^2}{4R}, \tag{2.4}$$

which is equivalent to the following inequality (see [14], p.223):

$$\frac{m_a}{h_a} \geq \frac{b^2+c^2}{2bc}. \tag{2.5}$$

Lemma 2.3. *In any triangle ABC the following inequality holds:*

$$(m_b+m_c)^2 \geq h_a^2 + \frac{9}{4}a^2, \tag{2.6}$$

with equality if and only if $b=c$.

Inequality (2.6) was noticed by the author many years ago. A proof of it can be found in my recent paper [11].

Lemma 2.4. *In any triangle ABC the following inequality holds:*

$$2\sum m_b m_c \geq \sum m_a^2 + \sum h_a^2, \tag{2.7}$$

with equality if and only if triangle ABC is equilateral.

To the author's knowledge, inequality (2.7) was appeared in a Chinese paper [2] at the earliest. It can be proved by using Lemma 2.3 as follows:

Proof. According to Lemma 2.3, we have

$$\sum (m_b+m_c)^2 \geq \sum h_a^2 + \frac{9}{4}\sum a^2.$$

Hence

$$2\sum m_a^2 + 2\sum m_b m_c \geq \sum h_a^2 + \frac{9}{4}\sum a^2.$$

Using the following known identity:

$$\sum m_a^2 = \frac{3}{4}\sum a^2, \tag{2.8}$$

the desired inequality follows immediately. Also, it is easily seen that equality in (2.7) occurs only when $a=b=c$. This completes the proof of Lemma 2.4. □

Remark 2.1. Inequality (2.7) can also be obtained by using the following inequality:

$$m_a(m_b+m_c-m_a) \geq h_b^2 + h_c^2, \tag{2.9}$$

which was established by the author in [4].

Lemma 2.5. *In any triangle ABC the following inequality holds:*

$$m_a m_b m_c \geq \frac{1}{2} R \sum h_a^2, \tag{2.10}$$

with equality if and only if triangle ABC is isosceles.

Note that identities $bc = 2Rh_a$ and $abc = 4SR$, one sees that inequality (2.10) is equivalent to

$$8Rm_a m_b m_c \geq \sum b^2 c^2, \tag{2.11}$$

and

$$abcm_a m_b m_c \geq \frac{1}{2} S \sum b^2 c^2. \tag{2.12}$$

The latter was first proved by X.Z. Yang in [15].

For any triangle ABC, we have the following three basic identities:

$$\sum a = 2s, \tag{2.13}$$

$$\sum bc = s^2 + 4Rr + 3r^2, \tag{2.14}$$

$$abc = 4Rrs. \tag{2.15}$$

By applying these identities, one can obtain the expressions of $\sum a^n$ and $\sum (bc)^n$ (k being natural number) in terms of R, r and s. For example, the identities are given in the following Lemma 2.6 and 2.7.

Lemma 2.6. *In any triangle ABC the following identities hold:*

$$\sum a^2 = 2s^2 - 8Rr - 2r^2, \tag{2.16}$$

$$\sum a^3 = 2s^3 - (12Rr + 6r^2)s, \tag{2.17}$$

$$\sum a^4 = 2s^4 - 4(4R + 3r)rs^2 + 2(4R + r)^2 r^2, \tag{2.18}$$

$$\sum a^5 = 2s^5 - 20(R + r)rs^3 + 10(2R + r)(4R + r)r^2 s, \tag{2.19}$$

$$\begin{aligned}\sum a^6 =& 2s^6 - 6(4R + 5r)rs^4 + 6(24R^2 + 24Rr + 5r^2)r^2 s^2 \\ &- 2(4R + r)^3 r^3,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\sum a^7 =& 2s^7 - 14(2R + 3r)rs^5 + 14(16R^2 + 20Rr + 5r^2)r^2 s^3 \\ &- 14(2R + r)(4R + r)^2 r^3 s,\end{aligned} \tag{2.21}$$

$$\begin{aligned}\sum a^8 =& 2s^8 - 8(4R + 7r)rs^6 + 20(16R^2 + 24Rr + 7r^2)r^2 s^4 \\ &- 8(4R + r)(32R^2 + 32Rr + 7r^2)r^3 s^2 + 2(4R + r)^4 r^4,\end{aligned} \tag{2.22}$$

$$\begin{aligned}\sum a^9 =& 2s^9 - 36(R + 2r)rs^7 + 36(12R^2 + 21Rr + 7r^2)r^2 s^5 \\ &- 12(160R^3 + 240R^2 r + 105Rr^2 + 14r^3)r^3 s^3 \\ &+ 18(2R + r)(4R + r)^3 r^4 s,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\sum a^{10} =& 2s^{10} - 10(4R + 9r)rs^8 + 140(2R + 3r)(2R + r)r^2 s^6 \\ &- 20(160R^3 + 280R^2 r + 140Rr^2 + 21r^3)r^3 s^4\end{aligned}$$

$$+10(40R^2+40Rr+9r^2)(4R+r)^2r^4s^2-2(4R+r)^5r^5, \tag{2.24}$$

$$\begin{aligned}\sum a^{11} =& 2s^{11}-22(2R+5r)rs^9+44(16R^2+36Rr+15r^2)r^2s^7\\ &-308(2R+r)(8R^2+12Rr+3r^2)r^3s^5+22(4R+r)(160R^3\\ &+240R^2r+108Rr^2+15r^3)r^4s^3-22(2R+r)(4R+r)^4r^5s,\end{aligned} \tag{2.25}$$

$$\begin{aligned}\sum a^{12} =& 2s^{12}-12(4R+11r)rs^{10}+18(48R^2+120Rr+55r^2)r^2s^8\\ &-56(128R^3+288R^2r+180Rr^2+33r^3)r^3s^6+6(4480R^4\\ &+8960R^3r+6048R^2r^2+1680Rr^3+165r^4)r^4s^4\\ &-12(48R^2+48Rr+11r^2)(4R+r)^3r^5s^2+2(4R+r)^6r^6,\end{aligned} \tag{2.26}$$

$$\begin{aligned}\sum a^{13} =& 2s^{13}-52(R+3r)rs^{11}+130(8R^2+22Rr+11r^2)r^2s^9\\ &-312(32R^3+80R^2r+55Rr^2+11r^3)r^3s^7+26(1792R^4\\ &+4032R^3r+3024R^2r^2+924Rr^3+99r^4)r^4s^5\\ &-52(112R^3+168R^2r+77Rr^2+11r^3)(4R+r)^2r^5s^3\\ &+26(2R+r)(4R+r)^5r^6s,\end{aligned} \tag{2.27}$$

$$\begin{aligned}\sum a^{14} =& 2s^{14}-14(4R+13r)rs^{12}+154(8R^2+24Rr+13r^2)r^2s^{10}\\ &-42(320R^3+880R^2r+660Rr^2+143r^3)r^3s^8+42(1792R^4\\ &+4480R^3r+3696R^2r^2+1232Rr^3+143r^4)r^4s^6\\ &-14(4R+r)(3584R^4+7168R^3r+4928R^2r^2+1408Rr^3\\ &+143r^4)r^5s^4+14(56R^2+56Rr+13r^2)(4R+r)^4r^6s^2\\ &-2(4R+r)^7r^7.\end{aligned} \tag{2.28}$$

The above identities (2.16)-(2.18) can be found in [14]. In [5] and [6], the author proved identities (2.19)-(2.28) and identities (2.29)-(2.31) below.

Lemma 2.7. *In any triangle triangle ABC the following identities hold:*

$$\sum b^2c^2 = s^4-2(4R-r)rs^2+(4R+r)^2r^2, \tag{2.29}$$

$$\sum b^3c^3 = s^6-3(4R-r)rs^4+3r^4s^2+(4R+r)^3r^3, \tag{2.30}$$

$$\begin{aligned}\sum b^6c^6 =& s^{12}-6(4R-r)rs^{10}+3(48R^2-24Rr+5r^2)r^2s^8\\ &-4(32R^3-24R^2r+12Rr^2-5r^3)r^3s^6+3(16R+5r)r^7s^4\\ &+6(4R+r)^3r^7s^2+(4R+r)^6r^6.\end{aligned} \tag{2.31}$$

3. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1.

Proof. Using the formulas

$$r=\frac{S}{s} \tag{3.1}$$

and

$$R = \frac{abc}{4S}, \tag{3.2}$$

one sees that inequality (1.6) is equivalent to

$$\sum \frac{1}{(m_b + m_c)^2} \leq \frac{1}{12}\left(\frac{4S}{abc} + \frac{\sum a}{4S}\right)^2.$$

By Lemma 2.1, we must only prove the following dual inequality:

$$\frac{16}{9}\sum \frac{1}{(b+c)^2} \leq \frac{1}{12}\left(\frac{3S}{m_a m_b m_c} + \frac{\sum m_a}{3S}\right)^2,$$

which is equivalent to

$$\begin{aligned}&81S^4 + 18S^2 m_a m_b m_c \sum m_a + (m_a m_b m_c)^2 \left(\sum m_a\right)^2\\ &\geq 192S^2 (m_a m_b m_c)^2 \sum \frac{1}{(b+c)^2}.\end{aligned} \tag{3.3}$$

Now, we note that it follows from Lemma 2.2 and Lemma 2.5 that

$$m_a m_b m_c \sum m_a \geq \frac{1}{4}\sum a^2 \sum h_a^2. \tag{3.4}$$

Also, by Lemma 2.4 and identity (2.8), it is easy to get

$$\left(\sum m_a\right)^2 \geq \frac{3}{2}\sum a^2 + \sum h_a^2. \tag{3.5}$$

Consequently, for proving inequality (3.3) we need to show that

$$\begin{aligned}&81S^4 + \frac{9}{2}S^2 \sum a^2 \sum h_a^2 + (m_a m_b m_c)^2 \left(\frac{3}{2}\sum a^2 + \sum h_a^2\right)\\ &\geq 192S^2 (m_a m_b m_c)^2 \sum \frac{1}{(b+c)^2},\end{aligned}$$

Multiplying both sides by $64R^2$ and using $2Rh_a = bc$ and $abc = 4SR$, one sees that the above inequality is equivalent to

$$\begin{aligned}L_0 \equiv\, &324(abc)^2 S^2 + 72S^2 \sum a^2 \sum b^2c^2 + 16(m_a m_b m_c)^2 \left(6R^2 \sum a^2 + \sum b^2c^2\right)\\ &\geq 768(abc)^2 (m_a m_b m_c)^2 \sum \frac{1}{(b+c)^2}.\end{aligned} \tag{3.6}$$

With the help of Maple software, using $S = rs$, identities (2.15), (2.16), (2.29) and the following known identity (see [3]):

$$\begin{aligned}&16\sum (m_a m_b m_c)^2\\ &= s^6 - 3(4R - 11r)rs^4 - 3(20R^2 + 40Rr + 11r^2)r^2s^2 - (4R + r)^3 r^3,\end{aligned} \tag{3.7}$$

we easily obtain

$$\begin{aligned}L_0 =\, &s^{10} + (12R^2 - 20Rr + 179r^2)s^8 - 2(96R^3 - 218R^2r + 1064Rr^2\\ &- 89r^3)rs^6 - 2(72R^4 + 1328R^3r - 6264R^2r^2 - 708Rr^3 + 89r^4)r^2s^4\\ &+ (4R + r)(528R^4 + 1232R^3r - 2428R^2r^2 - 1412Rr^3 - 179r^4)r^3s^2\end{aligned}$$

$$+ (6R + r)(2R - r)(4R + r)^4 r^4. \tag{3.8}$$

Again, using identities (2.13)-(2.15), we easily get

$$\sum \frac{1}{(b+c)^2} = \frac{9s^4 + 2(4R - 3r)rs^2 + (4R + r)^2 r^2}{4s^2(s^2 + 2Rr + r^2)^2}, \tag{3.9}$$

Thus, by (3.8) and (3.9), one sees that inequality (3.6) is equivalent to

$$\begin{aligned} &s^2(s^2 + 2Rr + r^2)^2 L_0 \\ &- 192(abc)^2(m_a m_b m_c)^2 \left[9s^4 + 2(4R - 3r)rs^2 + (4R + r)^2 r^2\right] \geq 0, \end{aligned} \tag{3.10}$$

Since $abc = 4Rrs$, we know again that the above inequality is equivalent to

$$\begin{aligned} D_0 \equiv\, &(s^2 + 2Rr + r^2)^2 L_0 \\ &- 3072(Rr)^2(m_a m_b m_c)^2 \left[9s^4 + 2(4R - 3r)rs^2 + (4R + r)^2 r^2\right] \geq 0, \end{aligned} \tag{3.11}$$

Using identities (3.7) and (3.8), we obtain

$$\begin{aligned} D_0 =\, &s^{14} + (12R^2 - 16Rr + 181r^2)s^{12} - (144R^3 + 1344R^2 r + 1448Rr^2 \\ &- 537r^3)rs^{10} - (864R^4 - 17872R^3 r + 50336R^2 r^2 + 1432Rr^3 \\ &- 357r^4)r^2 s^8 + (768R^5 + 114560R^4 r + 170672R^3 r^2 + 110128R^2 r^3 \\ &- 1424Rr^4 - 357r^5)r^3 s^6 + (10944R^6 + 256512R^5 r + 131408R^4 r^2 \\ &- 111792R^3 r^3 - 49956R^2 r^4 - 4288Rr^5 - 537r^6)r^4 s^4 \\ &+ (4R + r)(5184R^6 + 80512R^5 r + 93472R^4 r^2 + 26016R^3 r^3 \\ &- 3840R^2 r^4 - 2164Rr^5 - 181r^6)r^5 s^2 + (48R^4 + 800R^3 r \\ &+ 184R^2 r^2 - 8Rr^3 - r^4)(4R + r)^4 r^6. \end{aligned}$$

It remains to show that $D_0 \geq 0$.

We recall that for any triangle ABC the following Gerretsen's inequality (see [1] and [14]) holds:

$$g_1 \equiv s^2 - 16Rr + 5r^2 \geq 0. \tag{3.12}$$

According to this, we can rewrite D_0 in the form:

$$D_0 = g_1^7 + m_6 g_1^6 + m_5 g_1^5 + m_4 g_1^4 + m_3 g_1^3 + m_2 g_1^2 + m_1 g_1 + m_0, \tag{3.13}$$

where

$$\begin{aligned} m_6 =\, &12R^2 + 96Rr + 146r^2, \\ m_5 =\, &8(126R^3 + 283R^2 r + 1631Rr^2 - 546r^3)r, \\ m_4 =\, &4(8424R^4 - 6312R^3 r + 117549R^2 r^2 - 80168Rr^3 + 12608r^4)r^2, \\ m_3 =\, &16(34992R^5 - 117072R^4 r + 529773R^3 r^2 - 536880R^2 r^3 \\ &+ 183304Rr^4 - 18992r^5)r^3, \\ m_2 =\, &144(32076R^6 - 166320R^5 r + 504657R^4 r^2 - 684000R^3 r^3 \\ &+ 387876R^2 r^4 - 91136Rr^5 + 7072r^6)r^4, \end{aligned}$$

$$\begin{aligned}m_1 =&576(26244R^7 - 139725R^6r + 300024R^5r^2 - 641262R^4r^3\\&+ 634920R^3r^4 - 272728R^2r^5 + 50368Rr^6 - 3136r^7),\\m_0 =&3456(R-2r)(65610R^6 - 86751R^5r - 11556R^4r^2 + 48684R^3r^3\\&- 21568R^2r^4 + 3584Rr^5 - 192r^6)r^7.\end{aligned}$$

By Euler's inequality:

$$R \geq 2r \tag{3.14}$$

we see that $m_5 > 0$ and $m_4 > 0$ hold. Putting $R = 2r + e$ and substituting it into m_3, we easily get

$$\begin{aligned}m_3 =&16(34992e^5 + 232848e^4r + 992877e^3r^2 + 2631390e^2r^3 &(3.15)\\&+ 3446116er^4 + 1684872er^5)r^3, &(3.16)\end{aligned}$$

As $e \geq 0$, so that $m_3 > 0$. Similarly, we know $m_2 > 0$. Consequently, by (3.12) and (3.13), to prove inequality $D_0 \geq 0$ it remains to show that

$$m_1g_1 + m_0 \geq 0. \tag{3.17}$$

Note that for any triangle ABC we have another Gerretsen inequality (see [1] and [14]):

$$g_2 \equiv 4R^2 + 4Rr + 3r^2 - s^2 \geq 0. \tag{3.18}$$

Putting $e = R - 2r$, we easily obtain

$$m_1g_1 + m_0 = 576(x_1g_1 + x_2g_2 + x_3)r^5, \tag{3.19}$$

where

$$\begin{aligned}g_1 =&s^2 - 16Rr + 5r^2,\\g_2 =&4R^2 + 4Rr + 3r^2 - s^2,\\x_1 =&3e^4(8748e^3 + 75897e^2r + 275940er^2 + 441266r^3),\\x_2 =&8(19197e^3 + 468701e^2r + 626232er^2 + 269568r^3)r^4,\\x_3 =&2e(196830e^6 + 2101707e^5r + 8865450e^4r^2 + 13145020e^3r^3\\&+ 8879608e^2r^4 + 3033504er^5 + 331776r^6)r^2.\end{aligned}$$

Clearly, Euler's inequality $e \geq 0$ shows that $x_1 \geq 0, x_2 > 0$ and $x_3 \geq 0$ hold. Finally, by identity (3.19), Gerretsen's inequalities (3.12) and (3.18), we conclude that inequality (3.17) holds. We thus finish the proofs of inequalities (3.3) and (1.6). Furthermore, it is easy to determine that equality in (1.6) occurs if and only if $\triangle ABC$ is equilateral. This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Proof. We set

$$q_a = \frac{1}{4}(r_b + r_c) + \frac{m_a^2}{r_b + r_c},$$

$$q_b = \frac{1}{4}(r_c + r_a) + \frac{m_b^2}{r_c + r_a},$$
$$q_c = \frac{1}{4}(r_a + r_b) + \frac{m_c^2}{r_a + r_b}.$$

By the the arithmetic-geometric mean inequality, we have $q_a \geq m_a, q_b \geq m_b$ and $q_c \geq m_c$. Consequently, for proving inequality (1.7), it suffices to prove that

$$\sum \frac{1}{q_b + q_c} \geq \frac{7}{6R + 2r}. \tag{4.1}$$

With the help of software Maple, using $s = (a + b + c)/2$ and following formulas:

$$r_a = \frac{S}{s - a}, \tag{4.2}$$
$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}, \tag{4.3}$$
$$S = \sqrt{s(s-a)(s-b)(s-c)}. \tag{4.4}$$

we easily obtain the following identity:

$$\sum \frac{1}{q_b + q_c} = \frac{8rsN_1}{M_1}, \tag{4.5}$$

where

$$\begin{aligned}
M_1 =& M_a M_b M_c,\\
M_a =& -a^5 + (-b + 2c)a^4 + 2(b - c)(b + c)a^3 + (2b^3 + 2bc^2 + 2c^3)a^2\\
&- (b - c)^2(b + c)^2 a - (b^2 + c^2)(b - c)^2 b,\\
M_b =& -b^5 + (-c + 2a)b^4 + 2(c - a)(c + a)b^3 + (2c^3 + 2ca^2 + 2a^3)b^2\\
&- (c - a)^2(c + a)^2 b - (c^2 + a^2)(c - a)^2 c,\\
M_c =& -c^5 + (-a + 2b)c^4 + 2(a - b)(a + b)c^3 + (2a^3 + 2ab^2 + 2b^3)c^2\\
&- (a - b)^2(a + b)^2 c - (a^2 + b^2)(a - b)^2 a,
\end{aligned}$$

$$\begin{aligned}
N_1 =& 8a^8c^4 + 8a^8b^4 + 8b^8c^4 - 12b^5c^7 + b^{10}c^2 - 4b^9c^3 - 12b^7c^5\\
&+ 14b^6c^6 + 8b^4c^8 - 4b^3c^9 + b^2c^{10} + 36a^6b^3c^3 - 10a^5b^3c^4\\
&+ 22a^5b^5c^2 + 22a^5b^2c^5 - 10a^5b^4c^3 - 10a^4b^5c^3 + 2a^4b^6c^2\\
&- 10a^4b^3c^5 + 6a^4c^7b + 6a^4cb^7 + 72a^4b^4c^4 - 6a^3bc^8\\
&- 16a^3b^2c^7 + 2a^4b^2c^6 + 36a^3b^3c^6 - 10a^3b^5c^4 - 6a^3b^8c\\
&+ 36a^3b^6c^3 - 10a^3b^4c^5 - 16a^3b^7c^2 + 6a^7cb^4 + 2a^6b^2c^4\\
&- 3a^2bc^9 - 16a^2b^3c^7 - 3a^2b^9c + 22a^2b^5c^5 + 5a^2b^2c^8\\
&+ 5a^2b^8c^2 + 2a^2b^4c^6 - 16a^2b^7c^3 - 3ab^9c^2 + 3abc^{10}\\
&+ 3ab^{10}c + 2a^2b^6c^4 + a^{10}b^2 + 6ab^4c^7 - 6ab^8c^3 + 6ab^7c^4\\
&- 3ab^2c^9 - 6ab^3c^8 - 16a^7b^3c^2 + 6a^7c^4b + 2a^6b^4c^2\\
&+ a^{10}c^2 - 4a^9c^3 - 4a^9b^3 - 12a^7b^5 - 12a^7c^5 + 14a^6b^6
\end{aligned}$$

$$+14a^6c^6 - 12a^5b^7 - 12a^5c^7 + 8a^4b^8 + 8a^4c^8 - 4a^3b^9$$
$$- 4a^3c^9 + a^2b^{10} + a^2c^{10} - 3a^9c^2b - 3a^9cb^2 + 3a^{10}bc$$
$$- 16a^7b^2c^3 - 6a^8c^3b - 6ca^8b^3 + 5a^8b^2c^2.$$

Putting $d = abc$, we obtain the following identity:

$$N_1 = 72d^4 + a_3d^3 + a_2d^2 + a_1d + a_0, \tag{4.6}$$

where

$$\begin{aligned} a_3 =& 46\sum a^3 - 10\sum a\sum a^2,\\ a_2 =& 2\sum a^2\sum a^4 + 19\sum a^6 - 16\sum a\sum a^5 + 22\sum b^3c^3,\\ a_1 =& 6\sum a^9 - 6\sum a^2\sum a^7 - 3\sum a\sum a^8 + 6\sum a^3\sum a^6,\\ a_0 =& 7\sum a^{12} + \sum a^2\sum a^{10} - 4\sum a^3\sum a^9\\ &+ 8\sum a^4\sum a^8 - 12\sum a^5\sum a^7 + 14\sum b^6c^6. \end{aligned}$$

With the help of Maple, from (4.6), making use of identity (2.13), (2.15), Lemma 2.6 and Lemma 2.7, one obtains the following identity

$$N_1 = 64r^4N_2, \tag{4.7}$$

where

$$\begin{aligned} N_2 =& s^8 + 8(4R + 5r)Rs^6 + (192R^4 + 192R^3r + 16R^2r^2 - 40Rr^3 - 2r^4)s^4\\ &- 8(4R + 3r)(4R + r)^3Rrs^2 + (4R + r)^6r^2. \end{aligned}$$

We now still set $d = abc$. Expanding the product $M_aM_bM_c$ and arranging gives

$$M_1 = -60d^5 + b_4d^4 + b_3d^3 + b_2d^2 + b_1d + b_0, \tag{4.8}$$

where

$$\begin{aligned} b_4 =& 66\sum a\sum a^2 - 138\sum a^3,\\ b_3 =& 18\sum a\sum a^5 - 114\sum a^6 + 56\sum a^2\sum a^4 - 72\sum b^3c^3,\\ b_2 =& 8\sum a^9 + 40\sum a^4\sum a^5 - 55\sum a^3\sum a^6 - 12\sum a\sum a^8\\ &+ 23\sum a^2\sum a^7,\\ b_1 =& 16\sum a^2\sum a^{10} + 19\sum a^{12} - 33\sum a^3\sum a^9 - 27\sum a^5\sum a^7\\ &+ 32\sum b^6c^6 - 3\sum a\sum a^{11} + 30\sum a^4\sum a^8,\\ b_0 =& -\sum a\sum a^{14} + 5\sum a^5\sum a^{10} - 7\sum a^6\sum a^9\\ &+ 4\sum a^7\sum a^8 - 2\sum a^4\sum a^{11} - 2\sum a^3\sum a^{12}\\ &+ 3\sum a^2\sum a^{13}. \end{aligned}$$

Using $s = (a + b + c)/2$, Lemma 2.6 and Lemma 2.7, we can further obtain

$$M_1 = 1024sr^5M_2, \tag{4.9}$$

where

$$M_2 = (R+2r)s^8 + (16R^3 + 32R^2r + 12Rr^2 - 2r^3)s^6 \\ + 2(R+r)(32R^4 - 32R^3r - 56R^2r^2 - 20Rr^3 - r^4)s^4 \\ - 2(8R^3 + 8R^2r - 2Rr^2 - r^3)(4R+r)^3rs^2 + (4R+r)^6Rr^2.$$

It follows from (4.5),(4.7) and (4.9) that

$$\sum \frac{1}{q_b + q_c} = \frac{N_2}{2M_2}. \tag{4.10}$$

Thus, to prove inequality (4.1) we need to prove

$$\frac{N_2}{2M_2} \geq \frac{7}{6R+2r}. \tag{4.11}$$

From identity (4.10), it is easy to show that $M_2 > 0$. So we have to prove

$$E_0 \equiv (3R+r)N_2 - 7M_2 \geq 0. \tag{4.12}$$

Using the expressions of M_2 and N_2, we easily obtain

$$E_0 = -(4R+13r)s^8 - (16R^3 + 72R^2r + 44Rr^2 - 14r^3)s^6 \\ + (128R^5 + 768R^4r + 1472R^3r^2 + 960R^2r^3 + 248Rr^4 \\ + 12r^5)s^4 + 2(8R^3 + 4R^2r - 26Rr^2 - 7r^3)(4R+r)^3rs^2 \\ - (4R-r)(4R+r)^6r^2.$$

It remains to show that $E_0 \geq 0$.

We now recall that for any triangle ABC the following fundamental inequality (see [1, 10, 14]) holds:

$$t_0 \equiv -s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R+r)^3 \geq 0. \tag{4.13}$$

According to this inequality and Gerretsen's inequalities (3.12) and (3.18), we can write E_0 as follows:

$$E_0 = (c_1s^4 + c_2s^2 + c_3)t_0 + 4r(c_4g_1 + c_5g_2 + c_6), \tag{4.14}$$

where

$$c_1 = 4R + 13r,$$
$$c_2 = 32R^3 + 204R^2r + 296Rr^2 - 40r^3,$$
$$c_3 = (432R^4 + 2704R^3r + 3720R^2r^2 - 1800Rr^3 + 55r^4)r,$$
$$c_4 = 336R^6 + 10942R^2r^4 + 14r^6,$$
$$c_5 = 4Rr(224R^4 + 2608R^3r + 3184R^2r^2 + 319r^4),$$
$$c_6 = 4(R-2r)(1152R^6 + 1388R^5r + 1676R^4r^2 - 11273R^3r^3 \\ + 8193R^2r^4 + 489Rr^5 + 7r^6)r.$$

By Euler's inequality, it is easy to know that $c_2 > 0, c_3 > 0$ and $c_6 > 0$ hold. Thus, form identity (4.14) we conclude that $E_0 \geq 0$ holds. Therefore, we finish the proof of inequality

(1.7). Also, it is easy to determine that equality in (1.7) holds if and only if $\triangle ABC$ is equilateral. This completes the proof of Theorem 1.2. $\square$

Remark 4.1. For any triangle ABC, inequality (1.7) is the best possible inequality in the form:

$$\sum \frac{1}{m_b + m_c} \geq \frac{1}{R - k(R - 2r)}, \tag{4.15}$$

where k is a constant such that $0 \leq k < 1$. That is given by (4.15) for $k = \frac{1}{7}$. This conclusion can be proved as follows: Let us consider an isosceles triangle with sides $x, 1, 1 (0 < x < 2)$. Putting $a = x, b = c = 1$ and using the previous formulas (3.1), (3.2), (4.3), (4.4), we get

$$r = \frac{x\sqrt{4 - x^2}}{2x + 4}, \ R = \frac{1}{\sqrt{4 - x^2}}, \ m_a = \frac{1}{2}\sqrt{4 - x^2}, \ m_b = m_c = \frac{1}{2}\sqrt{1 + 2x^2}.$$

Letting $x \to 0$, then $r = 0, R = \frac{1}{2}, m_a = 1, m_b = m_c = \frac{1}{2}$. In this setting (4.15) becomes

$$\frac{7}{3} \geq \frac{2}{1 - k},$$

and it follows that $k \leq \frac{1}{7}$. Therefore, the above statement is true.

5. Four conjectures

In this section, we shall present three related conjectures as open problems.

We first give an inequality similar to the previous inequality (1.6):

Conjecture 1. In any triangle ABC, the following inequality holds:

$$\sum \frac{1}{(m_b + m_c)^3} \leq \frac{1}{72}\left(\frac{1}{R} + \frac{1}{2r}\right)^3. \tag{5.1}$$

By the power mean inequality, the above inequality is stronger than inequality (1.6).

For inequality (1.5), we propose the following exponential generalization:

Conjecture 2. If $0 < k \leq 4.6$, then for any triangle ABC the following inequality holds:

$$\sum \frac{1}{(m_b + m_c)^k} \leq \frac{1}{2 \cdot 3^{k-1}}\left(\frac{1}{R^k} + \frac{1}{2^k r^k}\right) \tag{5.2}$$

If $k < 0$, then the above inequality holds reversely for the acute triangle ABC.

Remark 5.1. For the two cases $k = -1$ and $k = -2$, the inequalities could be easily proved by using the following known acute triangle inequalities (see [8]), respectively:

$$\sum m_a \geq \frac{5}{2}R + 4r \tag{5.3}$$

and

$$\sum (m_b + m_c)^2 \geq 4s^2. \tag{5.4}$$

In [7], the author proposed the following conjecture related to Lemma 2.2:

$$\frac{R_1 + R_2 + R_3}{r_1 + r_2 + r_3} \geq \frac{4R(m_a + m_b + m_c)}{a^2 + b^2 + c^2}, \tag{5.5}$$

where R_1, R_2, R_3 and r_1, r_2, r_3 are the distances from an interior point P inside triangle ABC to the vertices A, B, C and sides BC, CA, AB respectively.

We present here two new conjectures related to Lemma 2.3 and 2.4:

Conjecture 3. For an interior point P inside triangle ABC, the following inequality holds:

$$\frac{R_1 + R_2 + R_3}{r_1 + r_2 + r_3} \geq \frac{4(m_b + m_c)}{\sqrt{9a^2 + 4h_a^2}}. \tag{5.6}$$

Conjecture 4. For an interior point P inside triangle ABC, the following inequality holds:

$$\frac{R_1 + R_2 + R_3}{r_1 + r_2 + r_3} \geq \frac{4(m_b m_c + m_c m_a + m_a m_b)}{m_a^2 + m_b^2 + m_c^2 + h_a^2 + h_b^2 + h_c^2}. \tag{5.7}$$

Lemma 2.3 and 2.4 show that both inequalities (5.6) and (5.7) are sharpened versions of the following famous Erdös-Mordell inequality (see [1], inequality 12.13):

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3). \tag{5.8}$$

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