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ON SOME COEFFICIENT INEQUALITIES INVOLVING LEGENDRE POLYNOMIALS IN THE CLASS OF BI-UNIVALENT FUNCTIONS

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ABSTRACT. In the study of geometric function theory, Legendre polynomials and other uncommon polynomials have recently gained increased importance. Using these polynomials, subordination, and the Al-Oboudi differential operator, we create a new class of bi-univalent functions and obtain coefficient estimates and Fekete-Szegö inequalities for this new class.

1. INTRODUCTION

Let $\mathcal{A}$ represent the category of functions with the form

$$u(\mathfrak{z}) = \mathfrak{z} + \sum_{k=2}^{\infty} a_k \mathfrak{z}^k, \tag{1.1}$$

which, in the open unit disc, analytically $\mathcal{U} = \{\mathfrak{z} : |\mathfrak{z}| < 1\}$, and let $\mathcal{S} = \{u \in \mathcal{A} : u \text{ is univalent in } \mathcal{U}\}$.

The Koebo one quarter theorem states that any function has a range that includes the disc's radius [8]. There is a satisfying inverse for each of these functions.

$$u^{-1}(u(\mathfrak{z})) = \mathfrak{z} \quad (\mathfrak{z} \in \mathcal{U})$$

and

$$u(u^{-1}(w)) = w \quad \left(|w| < r_0(u),\ r_0(u) \geq \frac{1}{4}\right)$$

where

$$u^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \ . \tag{1.2}$$

If both u and u^{-1} are univalent in then a function is said to be bi-univalent in $\mathcal{U}$. We state for such a function that it belongs to the class Σ.

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u is said to be subordinate to v in the case of the analytical functions u and v indicated by

$$u(\mathfrak{z}) \prec v(\mathfrak{z}), \tag{1.3}$$

if analytical function w exists such that

$$w(0) = 0, \ |w(\mathfrak{z})| < 1, \ \text{and } u(\mathfrak{z}) = v(w(\mathfrak{z})).$$

The Al-Oboudi differential operator, also known as the Al-Oboudi operator, was developed by Al-Oboudi [1] for a function $u(\mathfrak{z}) \in \mathcal{A}$

$$\mathcal{D}_\mu^0 u(\mathfrak{z}) = u(\mathfrak{z}) \tag{1.4}$$

$$\mathcal{D}_\mu^1 u(\mathfrak{z}) = (1-\mu)u(\mathfrak{z}) + \mu \mathfrak{z} u'(\mathfrak{z}) = D_\mu u(\mathfrak{z}), \ \mu \geq 0 \tag{1.5}$$

$$\mathcal{D}_\mu^m u(\mathfrak{z}) = \mathcal{D}_\mu(\mathcal{D}_\mu^{m-1} u(\mathfrak{z})). \tag{1.6}$$

If u is determined by (1.1), then from (1.4) and (1.5) show that

$$\mathcal{D}_\mu^m u(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} [1+(n-1)\mu]^m a_n \mathfrak{z}^n, \quad m \in \mathbb{N}_0 = \{0,1,2,\dots\} \tag{1.7}$$

with $\mathcal{D}_\mu^m u(0) = 0$. When $\mu = 1$, we get Salagean's differential operator [23].

Legendre polynomials, which Adrien-Marie Legendre discovered in 1782, have numerous uses in physical research. The precise answers to the Legendre differential equation are the Legendre polynomials $P_n(x)$, commonly referred to as Legendre functions of the first class.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad n \in \mathbb{N}_0, \quad |x| < 1.$$

Let $\mathbb{C}$ and $\mathbb{N}$ stand for a set of complex numbers and positive integers, respectively, in this section and the one that follows. The Legendre polynomials are defined using the Rodrigues formula.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n \quad (n \in \mathbb{N}_0). \tag{1.8}$$

Any arbitrary real or complex value may be used for x. The Legendre polynomials $P_n(x)$ are produced using the following function

$$(1-2xt+t^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

where the particular branch of $(1-2xt+t^2)^{\frac{-1}{2}}$ is taken to be 1 as $t \longrightarrow 0$. The first few Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \quad P_3(x) = \frac{1}{2}(5x^3-3x). \tag{1.9}$$

General information about the Legendre polynomials and their applications is provided in [16–20]. The objective

$$\phi(\mathfrak{z}) = \frac{1-\mathfrak{z}}{\sqrt{1-2\mathfrak{z}\cos\alpha + \mathfrak{z}^2}},$$

is in β for every real α (see [13], Page 102, [21]), where β is the Caratheodory class defined by

$$\beta = \{p(\mathfrak{z}) \in U : p(0) = 1, R(p(\mathfrak{z})) > 0, \mathfrak{z} \in \mathbb{U}\},$$

$p(\mathfrak{z})a + c_1\mathfrak{z} + c_2\mathfrak{z}^2 + \ldots$. By using (1.8), it is easy to check that

$$\phi(\mathfrak{z}) = 1 + \sum_{n=1}^{\infty}[P_n(\cos\alpha) - P_{n-1}(\cos\alpha)]\mathfrak{z}^n = \sum_{n=1}^{\infty} B_n\mathfrak{z}^n \tag{1.10}$$

where

$$\mathfrak{B}_n = P_n(\cos\alpha) - P_{n-1}(\cos\alpha).$$

In particular by using (1.9), we get

$$\mathfrak{B}_1 = \cos\alpha - 1, \quad \mathfrak{B}_2 = \frac{1}{2}(\cos\alpha - 1)(1 + 3\cos\alpha) \tag{1.11}$$

If we consider

$$\frac{1}{(\phi(\mathfrak{z}))^2} = \frac{1 - 2\mathfrak{z}\cos\alpha + \mathfrak{z}^2}{(1-\mathfrak{z})^2} = 1 + 2(1 - \cos\alpha)\frac{\mathfrak{z}}{(1-\mathfrak{z})^2}.$$

The function ϕ transfers the unit disc onto the right plane $R(w) > 0$, minus the slit along the positive real axis from $\frac{1}{|\cos\frac{\alpha}{2}|}$ to ∞. The function $\phi(\mathbb{U})$ is univalent, symmetric with respect to the real axis, and starlike with respect to $\phi(0) = 1$.

In this study, we provide new subclasses of the functions of the function class Σ using the Al-Oboudi differential operator associated with the legendre polynomial and find estimates on the coefficients $|a_2|$ and $|a_3|$. The past research on bi-univalent functions [2, 7, 9, 10, 12, 14, 25–30], the current study of bi-univalent functions connected to different polynomials, and other [3–6, 11, 15, 22, 24, 31] recent publications on the Fekete all served as inspiration for these researches. Additionally, a number of classes are considered, and linkages to previously published data are made.

Definition 1.1. If the following criteria are met, the function u is considered to belong to the class The function u is said to be in the class $\mathcal{Q}^{\Sigma,\mu}(\xi, m; x)$:

$$(1-\xi)\frac{\mathcal{D}_\mu^m u(\mathfrak{z})}{\mathfrak{z}} + \xi(\mathcal{D}_\mu^m u(\mathfrak{z}))' \prec \phi(\mathfrak{z}) \tag{1.12}$$

and

$$(1-\xi)\frac{\mathcal{D}_\mu^m u(w)}{w} + \xi(\mathcal{D}_\mu^m u(w))' \prec \phi(w) \tag{1.13}$$

where $v = u^{-1}$ is determined by (1.2) and function $\mathcal{D}_\mu^m$ is the Al-Oboudi differential operator.

The methods Deniz initially employed in [7] are utilised in the section that follows to obtain estimates for the coefficients $|a_2|$ and $|a_3|$ for functions in the previously mentioned subclasses of the function class Σ, $\mathcal{Q}^{\Sigma,\mu}(\xi, m; x)$.

To get our primary findings, we need the following lemma.

Lemma 1.1. *If $h \in \beta$, then $|c_k| \le 2$ for each k, where β is the family of all functions h, analytic in $\mathbb{U}$, for which*

$$R\{h(\mathfrak{z})\} > 0 \quad (\mathfrak{z} \in \mathbb{U}),$$

where

$$h(\mathfrak{z}) = 1 + c_1\mathfrak{z} + c_2\mathfrak{z}^2 + \ldots \quad (\mathfrak{z} \in \mathbb{U}).$$

2. The class $\mathfrak{Q}^{\Sigma,\mu}(\xi, m; x)$ and the Fekete-Szegö inequality

For functions in the class $\mathfrak{Q}^{\Sigma,\mu}(\xi, m; x)$, we start by locating estimates for the coefficients $|a_2|$ and $|a_3|$. Define $p(\mathfrak{z})$ and $q(\mathfrak{z})$ functions by

$$p(\mathfrak{z}) := \frac{1+u(\mathfrak{z})}{1-u(\mathfrak{z})} = 1 + p_1\mathfrak{z} + p_2\mathfrak{z}^2 + \dots$$

and

$$q(\mathfrak{z}) := \frac{1+v(\mathfrak{z})}{1-v(\mathfrak{z})} = 1 + q_1\mathfrak{z} + q_2\mathfrak{z}^2 + \dots$$

or, equivalently

$$u(\mathfrak{z}) := \frac{p(\mathfrak{z})-1}{p(\mathfrak{z})+1} = \frac{1}{2}\left[p_1\mathfrak{z} + \left(p_2 - \frac{p_1^2}{2}\right)\mathfrak{z}^2 + \dots\right]$$

and

$$v(\mathfrak{z}) := \frac{q(\mathfrak{z})-1}{q(\mathfrak{z})+1} = \frac{1}{2}\left[q_1\mathfrak{z} + \left(q_2 - \frac{q_1^2}{2}\right)\mathfrak{z}^2 + \dots\right].$$

If $p(0) = 1 = q(0)$, then $p(\mathfrak{z})$ and $q(\mathfrak{z})$ are analytic in $\mathbb{U}$. Given that $u, v : \mathbb{U} \to \mathbb{U}$, the functions $p(\mathfrak{z})$ and $q(\mathfrak{z})$ have positive real parts. Then $|p_i| \leq 2$ and $|q_i| \leq 2$.

Theorem 2.1. *Let u given by 1.1 be in the class $\mathfrak{Q}^{\Sigma,\mu}(\xi, m; x)$. Then*

$$|a_2| \leq \frac{\sqrt{2}|\cos\alpha - 1|\sqrt{|\cos\alpha - 1|}}{\sqrt{|\{2(1+2\mu)^m(1+2\xi)(\cos\alpha-1)^2 - (1+\mu)^{2m}(1+\xi)^2(\cos\alpha-1)(1-3\cos\alpha)\}|}} \tag{2.1}$$

and

$$|a_3| \leq \frac{(\cos\alpha-1)^2}{(1+\mu)^{2m}(1+\xi)^2} + \frac{|\cos\alpha - 1|}{(1+2\mu)^m(1+2\xi)}. \tag{2.2}$$

Proof. It follows from (1.12) and (1.13) that

$$(1-\xi)\frac{\mathcal{D}_\mu^m u(\mathfrak{z})}{\mathfrak{z}} + \xi(\mathcal{D}_\mu^m u(\mathfrak{z}))' = \phi(u(\mathfrak{z})) \tag{2.3}$$

$$(1-\xi)\frac{\mathcal{D}_\mu^m u(w)}{w} + \xi(\mathcal{D}_\mu^m u(w))' = \phi(u(w)) \tag{2.4}$$

where $p(\mathfrak{z})$ and $q(w)$ in and have the following forms:

$$\phi(u(\mathfrak{z})) = 1 + \frac{1}{2}\mathfrak{B}_1 p_1\mathfrak{z} + \left(\frac{1}{2}\mathfrak{B}_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}\mathfrak{B}_2 p_1^2\right)\mathfrak{z}^2 + \dots \tag{2.5}$$

and

$$\phi(u(w)) = 1 + \frac{1}{2}\mathfrak{B}_1 q_1 w + \left(\frac{1}{2}\mathfrak{B}_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}\mathfrak{B}_2 q_1^2\right)w^2 + \dots \tag{2.6}$$

or equivalently

$$(1-\xi)\frac{\mathcal{D}_\mu^m u(\mathfrak{z})}{\mathfrak{z}} + \xi(\mathcal{D}_\mu^m u(\mathfrak{z}))' = 1 + \frac{1}{2}\mathfrak{B}_1 p_1\mathfrak{z} + \left(\frac{1}{2}\mathfrak{B}_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}\mathfrak{B}_2 p_1^2\right)\mathfrak{z}^2 + \dots \tag{2.7}$$

$$(1-\xi)\frac{\mathcal{D}_\mu^m u(w)}{w} + \xi(\mathcal{D}_\mu^m u(w))' = 1 + \frac{1}{2}\mathfrak{B}_1 q_1 w + \left(\frac{1}{2}\mathfrak{B}_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}\mathfrak{B}_2 q_1^2\right)w^2 + \dots. \tag{2.8}$$

Now, equating the corresponding coefficients in (2.7) and (2.8), we get

$$(1+\xi)(1+\mu)^m a_2 = \frac{1}{2}\mathfrak{B}_1 p_1 \tag{2.9}$$

$$(1+2\xi)(1+2\mu)^m a_3 = \frac{1}{2}\mathfrak{B}_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}\mathfrak{B}_2 p_1^2, \tag{2.10}$$

$$-(1+\xi)(1+\mu)^m a_2 = \frac{1}{2}\mathfrak{B}_1 q_1 \tag{2.11}$$

$$(1+2\xi)(1+2\mu)^m(2a_2^2 - a_3) = \frac{1}{2}\mathfrak{B}_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}\mathfrak{B}_2 q_1^2. \tag{2.12}$$

From (2.9) and (2.11),

$$a_2 = \frac{\mathfrak{B}_1 p_1}{2(1+\xi)(1+\mu)^m} = \frac{-\mathfrak{B}_1 q_1}{2(1+\xi)(1+\mu)^m} \tag{2.13}$$

which implies

$$p_1 = -q_1 \tag{2.14}$$

and

$$8(1+\xi)^2(1+2\mu)^{2m} a_2^2 = \mathfrak{B}_1^2(p_1^2 + q_1^2) \tag{2.15}$$

adding (2.10) and (2.12),

$$2(1+2\xi)(1+2\mu)^m a_2^2 = \frac{1}{2}\mathfrak{B}_1(p_2+q_2) - \frac{1}{4}(p_1^2+q_1^2)(\mathfrak{B}_1 - \mathfrak{B}_2). \tag{2.16}$$

By using (2.13) and (2.14), we have

$$4[\mathfrak{B}_1^2(1+2\xi)(1+2\mu)^m - (\mathfrak{B}_1 - \mathfrak{B}_2)(1+\xi)^2(1+\mu)^{2m}]a_2^2 = \mathfrak{B}_1^3(p_2+q_2). \tag{2.17}$$

Thus, by using (1.11)

$$|a_2| \leq \frac{\sqrt{2}|\cos\alpha - 1|\sqrt{|\cos\alpha - 1|}}{\sqrt{|\{2(1+2\mu)^m(1+2\xi)(\cos\alpha-1)^2 - (1+\mu)^{2n}(1+\xi)^2(\cos\alpha - 1)(1-3\cos\alpha)\}|}}.$$

Also, by subtracting (2.12) from (2.10), we get

$$(1+2\xi)(1+2\mu)^m(a_3 - a_2^2) = \frac{1}{4}\mathfrak{B}_1(p_2 - q_2). \tag{2.18}$$

Then, by using (2.13) and (2.14) in (2.18), we have

$$a_3 = \frac{\mathfrak{B}_1^2(p_1^2+q_1^2)}{8(1+\xi)^2(1+\mu)^{2m}} + \frac{\mathfrak{B}_1(p_2-q_2)}{4(1+2\xi)(1+2\mu)^m},$$

and by the help of (1.11), we conclude that

$$|a_3| \leq \frac{(\cos\alpha - 1)^2}{(1+\mu)^{2m}(1+\xi)^2} + \frac{|\cos\alpha - 1|}{(1+2\mu)^m(1+2\xi)}.$$

□

For the special choices of parameters μ, ξ, and m in Theorem 2.1, we obtain the following:

Corollary 2.1. *Let* $u \in \mathcal{Q}^{\Sigma,1}(\xi, m; x) = \mathcal{Q}^{\Sigma}(\xi, m; x)$*. Then,*

$$|a_2| \leq \frac{\sqrt{2}|\cos\alpha - 1|\sqrt{|\cos\alpha - 1|}}{\sqrt{|\{2(3)^m(1+2\xi)(\cos\alpha-1)^2 - (2)^{2m}(1+\xi)^2(\cos\alpha-1)(1-3\cos\alpha)\}|}} \tag{2.19}$$

and

$$|a_3| \leq \frac{(\cos\alpha-1)^2}{(2)^{2m}(1+\xi)^2} + \frac{|\cos\alpha-1|}{(3)^m(1+2\xi)} \tag{2.20}$$

Corollary 2.2. *Let* $u \in \mathcal{Q}^{\Sigma,\mu}(\xi, 0; x) = \mathcal{Q}^{\Sigma,\mu}(\xi; x)$*. Then,*

$$|a_2| \leq \frac{\sqrt{2}|\cos\alpha - 1|\sqrt{|\cos\alpha - 1|}}{\sqrt{|\{2(1+2\xi)(\cos\alpha-1)^2 - (1+\xi)^2(\cos\alpha-1)(1-3\cos\alpha)\}|}} \tag{2.21}$$

and

$$|a_3| \leq \frac{(\cos\alpha-1)^2}{(1+\xi)^2} + \frac{|\cos\alpha-1|}{(1+2\xi)}. \tag{2.22}$$

Corollary 2.3. *Let* $u \in \mathcal{Q}^{\Sigma,\mu}(1, 0; x) = \mathcal{Q}^{\Sigma,\mu}(x)$*. Then,*

$$|a_2| \leq \frac{\sqrt{2}|\cos\alpha - 1|\sqrt{|\cos\alpha - 1|}}{\sqrt{|\{6(\cos\alpha-1)^2 - 4(\cos\alpha-1)(1-3\cos\alpha)\}|}} \tag{2.23}$$

and

$$|a_3| \leq \frac{(\cos\alpha-1)^2}{4} + \frac{|\cos\alpha-1|}{3}. \tag{2.24}$$

Theorem 2.2. *Let* u *given by (1.1) belongs to the class* $\mathcal{Q}^{\Sigma,\mu}(\xi, m; x)$*. Then,*

$$|a_3 - \varsigma a_2^2| \leq \begin{cases} \dfrac{|\cos\alpha-1|}{(1+2\xi)(1+2\mu)^m}, & 0 \leq |t(\varsigma;x)| < \frac{1}{4(1+2\xi)(1+2\mu)^m} \\ 4|\cos\alpha-1||t(\varsigma;x)|, & |t(\varsigma;x)| \geq \frac{1}{4(1+2\xi)(1+2\mu)^m} \end{cases} \tag{2.25}$$

where

$$t(\varsigma;x) = \frac{(1-\varsigma)(\cos\alpha-1)^2}{2[2(\cos\alpha-1)^2(1+2\xi)(1+2\mu)^m + (\cos\alpha-1)(1-3\cos\alpha)(1+\xi)^2(1+\mu)^{2m}]}.$$

Proof. From equations (2.17) and (2.18), we get

$$a_3 - \varsigma a_2^2 = \frac{(1-\varsigma)\mathfrak{B}_1^3(p_2+q_2)}{4[\mathfrak{B}_1^2(1+2\xi)(1+2\mu)^m + (\mathfrak{B}_1 - \mathfrak{B}_2)(1+\xi)^2(1+\mu)^{2m}]} + \frac{\mathfrak{B}_1(p_2-q_2)}{4(1+2\xi)(1+2\mu)^m}$$

$$= (\cos\alpha - 1)\left[\left(t(\varsigma;x) + \frac{1}{4(1+2\xi)(1+2\mu)^m}\right)p_2 + \left(t(\varsigma;x) - \frac{1}{4(1+2\xi)(1+2\mu)^m}\right)q_2\right]$$

where

$$t(\varsigma;x) = \frac{(1-\varsigma)(\cos\alpha-1)^2}{2[2(\cos\alpha-1)^2(1+2\xi)(1+2\mu)^m + (\cos\alpha-1)(1-3\cos\alpha)(1+\xi)^2(1+\mu)^{2m}]}.$$

□

Corollary 2.4. *Let* $u \in \mathcal{Q}^{\Sigma,1}(\xi, m; x) = \mathcal{Q}^{\Sigma}(\xi, m; x)$ *and* $\varsigma \in R$*. Then,*

$$|a_3 - \varsigma a_2^2| \leq \begin{cases} \dfrac{|\cos\alpha-1|}{(1+2\xi)(3)^m}, & 0 \leq |t(\varsigma;x)| < \frac{1}{4(1+2\xi)(3)^m} \\ 4|\cos\alpha-1||t(\varsigma;x)|, & |t(\varsigma;x)| \geq \frac{1}{4(1+2\xi)(3)^m} \end{cases} \tag{2.26}$$

where

$$t(\varsigma;x)=\frac{(1-\varsigma)(\cos\alpha-1)^2}{2[2(\cos\alpha-1)^2(1+2\xi)(3)^m+(\cos\alpha-1)(1-3\cos\alpha)(1+\xi)^2(2)^{2m}]}.$$

Corollary 2.5. *Let $u\in\mathcal{Q}^{\Sigma,\mu}(\xi,0;x)=\mathcal{Q}^{\Sigma,\mu}(\xi;x)$ and $\varsigma\in R$. Then,*

$$|a_3-\varsigma a_2^2|\leq\begin{cases}\dfrac{|\cos\alpha-1|}{(1+2\xi)}, & 0\leq|t(\varsigma;x)|<\frac{1}{4(1+2\xi)}\\ 4|\cos\alpha-1||t(\varsigma;x)|, & |t(\varsigma;x)|\geq\frac{1}{4(1+2\xi)}\end{cases}\tag{2.27}$$

where

$$t(\varsigma;x)=\frac{(1-\varsigma)(\cos\alpha-1)^2}{2[2(\cos\alpha-1)^2(1+2\xi)+(\cos\alpha-1)(1-3\cos\alpha)(1+\xi)^2]}.$$

Corollary 2.6. *Let $u\in\mathcal{Q}^{\Sigma,\mu}(1,0;x)=\mathcal{Q}^{\Sigma,\mu}(x)$ and $\varsigma\in R$. Then,*

$$|a_3-\varsigma a_2^2|\leq\begin{cases}\dfrac{|\cos\alpha-1|}{3}, & 0\leq|t(\varsigma;x)|<\frac{1}{12}\\ 4|\cos\alpha-1||t(\varsigma;x)|, & |t(\varsigma;x)|\geq\frac{1}{12}\end{cases}\tag{2.28}$$

where

$$t(\varsigma;x)=\frac{(1-\varsigma)(\cos\alpha-1)^2}{2[6(\cos\alpha-1)^2+4(\cos\alpha-1)(1-3\cos\alpha)]}.$$

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