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OPERATOR MONOTONICITY OF AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES WITH APPLICATIONS

SILVESTRU SEVER DRAGOMIR[1]

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if $B \geq A > 0$, then $\mathcal{D}(w, \mu)(B) \leq \mathcal{D}(w, \mu)(A)$, namely $\mathcal{D}(w, \mu)$ is operator monotone decreasing on $(0, \infty)$. From this we derive that, if $f : [0, \infty) \to \mathbb{R}$ is an operator monotone function on $[0, \infty)$, then the function $[f(0) - f(t)]t^{-1}$ is operator monotone on $(0, \infty)$. Also, if $f : [0, \infty) \to \mathbb{R}$ is an operator convex function on $[0, \infty)$, then the function $[f(0) + f'_+(0)t - f(t)]t^{-2}$ is operator monotone on $(0, \infty)$. Some examples for integral transforms $\mathcal{D}(\cdot, \cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [4], see for instance [1, p. 144-145]:

Theorem 1.1. *A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.1}$$

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where $b \geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.2}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 1.2. *A function $f : [0,\infty) \to \mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_+(0) \in \mathbb{R}$ if and only if it has the representation*

$$f(t) = f(0) + f'_+(0)\, t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda), \tag{1.3}$$

where $c \geq 0$ and a positive measure μ on $[0,\infty)$ such that (1.2) holds.

Let A and B be strictly positive operators on a Hilbert space H such that $B - A \geq m > 0$. In 2015, [2], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0,\infty)$

$$\begin{aligned} f(B) - f(A) &\geq f(\|A\| + m) - f(\|A\|) \\ &\geq f(\|B\|) - f(\|B\| - m) > 0. \end{aligned} \tag{1.4}$$

If $B > A > 0$, then

$$\begin{aligned} f(B) - f(A) &\geq f\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right) - f(\|A\|) \\ &\geq f(\|B\|) - f\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right) > 0. \end{aligned} \tag{1.5}$$

The inequality between the first and third term in (1.3) was obtained earlier by H. Zuo and G. Duan in [6].

By taking $f(t) = t^r$, $r \in (0,1]$ in (1.3) Furuta obtained the following refinement of the celebrated Löwner-Heinz inequality [3]

$$\begin{aligned} B^r - A^r &\geq \left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r - \|A\|^r \\ &\geq \|B\|^r - \left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r > 0 \end{aligned} \tag{1.6}$$

provided $B > A > 0$.

With the same assumptions for A and B, we have the logarithmic inequality [2]

$$\ln B-\ln A\geq\ln\left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)-\ln\left(\|A\|\right) \tag{1.7}$$
$$\geq\ln\left(\|B\|\right)-\ln\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)>0.$$

Notice that the inequalities between the first and third terms in (1.6) and (1.7) were obtained earlier by M. S. Moslehian and H. Najafi in [5].

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda),$$

where the integral is assumed to exist for all T a positive operator on a complex Hilbert space H.

We show among others that, if $B\geq A>0$, then $\mathcal{D}(w,\mu)(B)\leq\mathcal{D}(w,\mu)(A)$, namely $\mathcal{D}(w,\mu)$ is operator monotone decreasing on $(0,\infty)$. From this we derive that, if $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function on $[0,\infty)$, then the function $[f(0)-f(t)]t^{-1}$ is operator monotone on $(0,\infty)$. Also, if $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$, then the function $\left[f(0)+f'_+(0)t-f(t)\right]t^{-2}$ is operator monotone on $(0,\infty)$. Some examples for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Basic Identities

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^r=\frac{\sin(r\pi)}{\pi}t\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{2.1}$$

Observe that for $t>0$, $t\neq1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{2.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{2.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (2.3) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{2.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^r = \frac{\sin(r\pi)}{\pi} t\mathcal{D}(w_r)(t), \ t > 0. \tag{2.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t), \ t > 0. \tag{2.6}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{2.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{2.8}$$

for $T > 0$.

In the following, whenever we write $\mathcal{D}(w,\mu)$ we mean that the integral from (2.3) exists and is finite for all $t > 0$.

Theorem 2.1. *For all A, $B > 0$ we have the representation*

$$\begin{aligned} &\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) \\ &= -\int_0^\infty \left(\int_0^1 (\lambda + (1-t)B + tA)^{-1}(B-A)(\lambda + (1-t)B + tA)^{-1} dt \right) \\ &\times w(\lambda) d\mu(\lambda). \end{aligned} \tag{2.9}$$

Proof. Observe that, for all A, $B > 0$

$$\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) = \int_0^\infty w(\lambda)\left[(\lambda+B)^{-1} - (\lambda+A)^{-1}\right] d\mu(\lambda). \tag{2.10}$$

Let T, $S > 0$. The function $f(t) = -t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t\to 0}\left[\frac{f(T+tS) - f(T)}{t}\right] = T^{-1}ST^{-1} \tag{2.11}$$

for T, $S > 0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D] : \{(1-t)C + tD, \ t \in [0,1]\}$

for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) := f((1-t)C + tD), \ t \in [0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.12}$$

If we write this equality for the function $f(t) = -t^{-1}$ and C, $D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.13}$$

Now, if we take in (2.13) $C = \lambda + B$, $D = \lambda + A$, then

$$\begin{aligned} &(\lambda+B)^{-1} - (\lambda+A)^{-1} \\ &= \int_0^1 ((1-t)(\lambda+B) + t(\lambda+A))^{-1}(A-B) \\ &\times ((1-t)(\lambda+B)+t(\lambda+A))^{-1}\,dt \\ &= \int_0^1 (\lambda+(1-t)B+tA)^{-1}(A-B)(\lambda+(1-t)B+tA)^{-1}\,dt \end{aligned} \tag{2.14}$$

and by (2.10) we derive (2.9). □

Remark 2.1. We observe that if A, $B > 0$ and $r \in (0,1]$, then by (2.5) we get the identity

$$\begin{aligned} B^{r-1} - A^{r-1} = -\frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1}\Big(\int_0^1 (\lambda+(1-t)B+tA)^{-1} \\ \times (B-A)(\lambda+(1-t)B+tA)^{-1}\,dt\Big)\,d\lambda. \end{aligned} \tag{2.15}$$

If A, $B > 0$ with $A-1$ and $B-1$ invertible, then

$$\begin{aligned} &(B-1)^{-1}\ln B - (A-1)^{-1}\ln A \\ &= -\int_0^\infty (\lambda+1)^{-1} \\ &\times \Big(\int_0^1 (\lambda+(1-t)B+tA)^{-1}(B-A)(\lambda+(1-t)B+tA)^{-1}\,dt\Big)\,d\lambda. \end{aligned} \tag{2.16}$$

Corollary 2.1. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function that has the representation (1.1). Then for all A, $B > 0$ we have the equality*

$$\begin{aligned} &B^{-1}f(B) - A^{-1}f(A) - f(0)\left(B^{-1} - A^{-1}\right) \\ &= -\int_0^\infty \lambda\Big(\int_0^1 (\lambda+(1-t)B+tA)^{-1}(B-A) \\ &\times (\lambda+(1-t)B+tA)^{-1}\,dt\Big)\,d\mu(\lambda). \end{aligned} \tag{2.17}$$

If $f(0)=0$, *then we have the simpler equality*

$$B^{-1}f(B)-A^{-1}f(A)-f(0)\left(B^{-1}-A^{-1}\right) \tag{2.18}$$
$$=-\int_0^\infty \lambda\left(\int_0^1 (\lambda+(1-t)B+tA)^{-1}(B-A)\right.$$
$$\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda).$$

Proof. From (1.1) we have that

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t), \tag{2.19}$$

where $\ell(\lambda)=\lambda$, $\lambda>0$. Then for A, $B>0$,

$$\mathcal{D}(\ell,\mu)(B)-\mathcal{D}(\ell,\mu)(A)=[f(B)-f(0)]B^{-1}-[f(A)-f(0)]A^{-1}$$
$$=B^{-1}f(B)-A^{-1}f(A)-f(0)\left(B^{-1}-A^{-1}\right)$$

and by (2.9) we derive (2.17). □

Corollary 2.2. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator convex function that has the representation* (1.2). *Then for all* A, $B>0$ *we have the equality*

$$f(B)B^{-2}-f(A)A^{-2}-f'_+(0)\left(B^{-1}-A^{-1}\right)-f(0)\left(B^{-2}-A^{-2}\right) \tag{2.20}$$
$$=-\int_0^\infty \lambda\left(\int_0^1 (\lambda+(1-t)B+tA)^{-1}(B-A)\right.$$
$$\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda).$$

If $f(0)=0$, *then we have the simpler equality*

$$f(B)B^{-2}-f(A)A^{-2}-f'_+(0)\left(B^{-1}-A^{-1}\right) \tag{2.21}$$
$$=-\int_0^\infty \lambda\left(\int_0^1 (\lambda+(1-t)B+tA)^{-1}(B-A)\right.$$
$$\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda).$$

Proof. From (1.3) we have that

$$\frac{f(t)-f(0)-f'_+(0)t}{t^2}-c=\mathcal{D}(\ell,\mu)(t),$$

for $t>0$. Then for A, $B>0$,

$$\mathcal{D}(\ell,\mu)(B)-\mathcal{D}(\ell,\mu)(A)=f(B)B^{-2}-f'_+(0)B^{-1}-f(0)B^{-2}$$
$$-f(A)A^{-2}+f'_+(0)A^{-1}+f(0)A^{-2}$$
$$=f(B)B^{-2}-f(A)A^{-2}-f'_+(0)\left(B^{-1}-A^{-1}\right)$$
$$-f(0)\left(B^{-2}-A^{-2}\right)$$

and by (2.9) we derive (2.20). □

Remark 2.2. Let $a > 0$ and $f(t) = (t+a)^p$ with $p \in [-1, 0) \cup [1, 2]$. This function is operator convex and $f(0) = a^p$, $f'(0) = pa^{p-1}$. Then for all A, $B > 0$ we have the equality

$$(B+a)^p B^{-2} - (A+a)^p A^{-2} - pa^{p-1}\left(B^{-1} - A^{-1}\right) - a^p\left(B^{-2} - A^{-2}\right) \tag{2.22}$$
$$= -\int_0^\infty \lambda \left(\int_0^1 (\lambda + (1-t)B + tA)^{-1}(B-A) \times (\lambda + (1-t)B + tA)^{-1} dt\right) d\mu(\lambda),$$

for some positive measure μ on $(0, \infty)$.

3. Monotonicity Properties

In what follows, we assume that the integral transform defined by (2.3) is well defined for a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$.

Theorem 3.1. *If* $B \geq A > 0$, *then*

$$\mathcal{D}(w, \mu)(B) \leq \mathcal{D}(w, \mu)(A), \tag{3.1}$$

namely, the function $\mathcal{D}(w, \mu)(\cdot)$ *is operator monotone decreasing on* $(0, \infty)$.

Proof. From $B - A \geq 0$, by multiplying both sides with $(\lambda + (1-t)B + tA)^{-1}$ for $t \in [0, 1]$ and $\lambda > 0$, we get

$$(\lambda + (1-t)B + tA)^{-1}(B-A)(\lambda + (1-t)B + tA)^{-1} \geq 0,$$

which gives, by integration over $t \in [0, 1]$, that

$$\int_0^1 (\lambda + (1-t)B + tA)^{-1}(B-A)(\lambda + (1-t)B + tA)^{-1} dt \geq 0,$$

for all $\lambda > 0$.

Now, if we multiply this inequality by $w(\lambda) > 0$ and integrate over the positive measure $d\mu(\lambda)$, we get

$$\int_0^\infty w(\lambda)\left(\int_0^1 (\lambda + (1-t)B + tA)^{-1}(B-A)(\lambda + (1-t)B + tA)^{-1} dt\right) d\mu(\lambda) \geq 0,$$

and by representation (2.9), we deduce (3.1). □

Corollary 3.1. *Assume that* $f : [0, \infty) \to \mathbb{R}$ *is an operator monotone function on* $[0, \infty)$. *Then for all* $B \geq A > 0$ *we have*

$$A^{-1}f(A) - B^{-1}f(B) \geq f(0)\left(A^{-1} - B^{-1}\right), \tag{3.2}$$

namely the function $[f(0) - f(t)]t^{-1}$ *is operator monotone on* $(0, \infty)$.

In particular, if $f(0) = 0$, *then*

$$A^{-1}f(A) \geq B^{-1}f(B) \tag{3.3}$$

for all $B \geq A > 0$, *namely* $-f(t)t^{-1}$ *is operator monotone on* $(0, \infty)$.

Proof. It follows by Theorem 3.1 by observing that, if $f:[0,\infty)\to\mathbb{R}$ is operator monotone, then by (1.1)

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t),\ t>0$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda>0$. □

Corollary 3.2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$. Then for all $B\geq A>0$ we have*

$$f(A)A^{-2}-f(B)B^{-2}\geq f'_{+}(0)\left(A^{-1}-B^{-1}\right)+f(0)\left(A^{-2}-B^{-2}\right) \tag{3.4}$$

namely the function $\left[f(0)+f'_{+}(0)t-f(t)\right]t^{-2}$ is operator monotone on $(0,\infty)$.

In particular, if $f(0)=0$, then

$$f(A)A^{-2}-f(B)B^{-2}\geq f'_{+}(0)\left(A^{-1}-B^{-1}\right) \tag{3.5}$$

for all $B\geq A>0$, namely $\left[f'_{+}(0)t-f(t)\right]t^{-2}$ is operator monotone on $(0,\infty)$

Proof. It follows by Theorem 3.1 by observing that, if $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$, then by (1.3) we have that

$$\frac{f(t)-f(0)-f'_{+}(0)t}{t^2}-c=\mathcal{D}(\ell,\mu)(t),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda>0$. □

Remark 3.1. Let $a>0$ and $p\in[-1,0)\cup[1,2]$. Then for all $B\geq A>0$ we have the inequality

$$(A+a)^pA^{-2}-(B+a)^pB^{-2}\geq pa^{p-1}\left(A^{-1}-B^{-1}\right)+a^p\left(A^{-2}-B^{-2}\right). \tag{3.6}$$

4. Related Inequalities

We start with the following inequalities that can be derived from Furuta's inequalities (1.4).

Proposition 4.1. *Assume that $g:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$. If $A>0$ and there exists $m>0$ such that $B-A\geq m>0$, then*

$$\begin{aligned} &A^{-1}g(A)-B^{-1}g(B)-g(0)\left(A^{-1}-B^{-1}\right) \\ &\geq\frac{g(\|A\|)}{\|A\|}-\frac{g(\|A\|+m)}{\|A\|+m}-g(0)\frac{m}{(\|A\|+m)\|A\|} \\ &\geq\frac{g(\|B\|-m)}{\|B\|-m}-\frac{g(\|B\|)}{\|B\|}-g(0)\frac{m}{(\|B\|-m)\|B\|}\geq 0. \end{aligned} \tag{4.1}$$

If $g(0)=0$, then

$$\begin{aligned} A^{-1}g(A)-B^{-1}g(B)&\geq\frac{g(\|A\|)}{\|A\|}-\frac{g(\|A\|+m)}{\|A\|+m} \\ &\geq\frac{g(\|B\|-m)}{\|B\|-m}-\frac{g(\|B\|)}{\|B\|}\geq 0. \end{aligned} \tag{4.2}$$

Proof. If we write the inequality (1.4) for $f(t)=\frac{g(0)-g(t)}{t}$, $t>0$, which, by Corollary 3.1, is operator monotone, then we have

$$\begin{aligned} &B^{-1}[g(0)-g(B)]-A^{-1}[g(0)-g(A)] \\ &\geq \frac{g(0)-g(\|A\|+m)}{\|A\|+m}-\frac{g(0)-g(\|A\|)}{\|A\|} \\ &\geq \frac{g(0)-g(\|B\|)}{\|B\|}-\frac{g(0)-g(\|B\|-m)}{\|B\|-m}>0. \end{aligned} \tag{4.3}$$

Observe that

$$\begin{aligned} &B^{-1}[g(0)-g(B)]-A^{-1}[g(0)-g(A)] \\ &=A^{-1}g(A)-B^{-1}g(B)-g(0)\left(A^{-1}-B^{-1}\right), \end{aligned}$$

$$\begin{aligned} &\frac{g(0)-g(\|A\|+m)}{\|A\|+m}-\frac{g(0)-g(\|A\|)}{\|A\|} \\ &=\frac{g(\|A\|)}{\|A\|}-\frac{g(\|A\|+m)}{\|A\|+m}-g(0)\frac{m}{(\|A\|+m)\|A\|} \end{aligned}$$

and

$$\begin{aligned} &\frac{g(0)-g(\|B\|)}{\|B\|}-\frac{g(0)-g(\|B\|-m)}{\|B\|-m} \\ &=\frac{g(\|B\|-m)}{\|B\|-m}-\frac{g(\|B\|)}{\|B\|}-g(0)\frac{m}{(\|B\|-m)\|B\|} \end{aligned}$$

and by (4.3) we get (4.1). □

Remark 4.1. If we take $g(t)=t^r$, $r\in(0,1]$ in (4.2), then we get

$$A^{r-1}-B^{r-1}\geq\|A\|^{r-1}-(\|A\|+m)^{r-1}\geq(\|B\|-m)^{r-1}-\|B\|^{r-1}>0, \tag{4.4}$$

provided $A>0$ and $B-A\geq m>0$.

Its is well known that, if $P\geq 0$, then

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle$$

for all x, $y\in H$.

Therefore, if $T>0$, then

$$\begin{aligned} 0&\leq\langle x,x\rangle^2=\left\langle T^{-1}Tx,x\right\rangle^2=\left\langle Tx,T^{-1}x\right\rangle^2 \\ &\leq\langle Tx,x\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle \end{aligned}$$

for all $x\in H$.

If $x\in H$, $\|x\|=1$, then

$$1\leq\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\leq\langle Tx,x\rangle\sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1}1_H\leq T. \tag{4.5}$$

Corollary 4.1. *Assume that $g:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$. If $A>0$ and $B-A>0$, then*

$$\begin{aligned}
&A^{-1}g(A)-B^{-1}g(B)-g(0)\left(A^{-1}-B^{-1}\right) \\
&\geq \frac{g(\|A\|)}{\|A\|}-\frac{g\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|A\|+\left\|(B-A)^{-1}\right\|^{-1}} \\
&-g(0)\frac{\left\|(B-A)^{-1}\right\|^{-1}}{\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)\|A\|} \\
&\geq \frac{g\left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|B\|-\left\|(B-A)^{-1}\right\|^{-1}}-\frac{g(\|B\|)}{\|B\|} \\
&-g(0)\frac{\left\|(B-A)^{-1}\right\|^{-1}}{\left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)\|B\|} \\
&\geq 0.
\end{aligned} \tag{4.6}$$

If $g(0)=0$, then

$$\begin{aligned}
&A^{-1}g(A)-B^{-1}g(B) \\
&\geq \frac{g(\|A\|)}{\|A\|}-\frac{g\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|A\|+\left\|(B-A)^{-1}\right\|^{-1}} \\
&\geq \frac{g\left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)}{\|B\|-\left\|(B-A)^{-1}\right\|^{-1}}-\frac{g(\|B\|)}{\|B\|}\geq 0.
\end{aligned} \tag{4.7}$$

Remark 4.2. If we take $g(t)=t^r$, $r\in(0,1]$ in (4.7), then we get

$$\begin{aligned}
A^{r-1}-B^{r-1} &\geq \|A\|^{r-1}-\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)^{r-1} \\
&\geq \left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)^{r-1}-\|B\|^{r-1}>0,
\end{aligned} \tag{4.8}$$

where $A>0$ and $B-A>0$.

Proposition 4.2. *Assume that* $h:[0,\infty)\to\mathbb{R}$ *is operator convex on* $[0,\infty)$*. If* $A>0$ *and there exists* $m>0$ *such that* $B-A\geq m>0$*, then*

$$\begin{aligned}
&h(A)A^{-2}-h(B)B^{-2}-h(0)\left(A^{-2}-B^{-2}\right)+h'_{+}(0)(B-A) \quad (4.9)\\
&\geq h(\|A\|)\|A\|^{-2}-h(\|A\|+m)(\|A\|+m)^{-2}\\
&-h(0)\left(\|A\|^{-2}-(\|A\|+m)^{-2}\right)-h'_{+}(0)\left(\|A\|^{-1}-(\|A\|+m)^{-1}\right)\\
&\geq h(\|B\|-m)(\|B\|-m)^{-2}-h(\|B\|)\|B\|^{-2}\\
&-h(0)\left((\|B\|-m)^{-2}-\|B\|^{-2}\right)-h'_{+}(0)\left((\|B\|-m)^{-1}-\|B\|^{-1}\right)\\
&\geq 0
\end{aligned}$$

If $h(0)=0$*, then*

$$\begin{aligned}
&h(A)A^{-2}-h(B)B^{-2}+h'_{+}(0)(B-A) \quad (4.10)\\
&\geq h(\|A\|)\|A\|^{-2}-h(\|A\|+m)(\|A\|+m)^{-2}\\
&-h'_{+}(0)\left(\|A\|^{-1}-(\|A\|+m)^{-1}\right)\\
&\geq h(\|B\|-m)(\|B\|-m)^{-2}-h(\|B\|)\|B\|^{-2}\\
&-h'_{+}(0)\left((\|B\|-m)^{-1}-\|B\|^{-1}\right)\\
&\geq 0
\end{aligned}$$

Proof. If we write the inequality (1.4) for $f(t)=\left[h(0)+h'_{+}(0)t-h(t)\right]t^{-2}$, $t>0$, which, by Corollary 3.2, is operator monotone, then we have

$$\begin{aligned}
&\left[h(0)+h'_{+}(0)B-h(B)\right]B^{-2}-\left[h(0)+h'_{+}(0)A-h(A)\right]A^{-2} \quad (4.11)\\
&\geq\left[h(0)+h'_{+}(0)(\|A\|+m)-h(\|A\|+m)\right](\|A\|+m)^{-2}\\
&-\left[h(0)+h'_{+}(0)\|A\|-h(\|A\|)\right](\|A\|)^{-2}\\
&\geq\left[h(0)+h'_{+}(0)\|B\|-h(\|B\|)\right]\|B\|^{-2}\\
&-\left[h(0)+h'_{+}(0)(\|B\|-m)-h(\|B\|-m)\right](\|B\|-m)^{-2}\\
&>0.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\left[h(0)+h'_{+}(0)B-h(B)\right]B^{-2}-\left[h(0)+h'_{+}(0)A-h(A)\right]A^{-2}\\
&=h(A)A^{-2}-h(B)B^{-2}-h(0)\left(A^{-2}-B^{-2}\right)+h'_{+}(0)(B-A),
\end{aligned}$$

$$\begin{aligned}
&\left[h(0)+h'_{+}(0)(\|A\|+m)-h(\|A\|+m)\right](\|A\|+m)^{-2}\\
&-\left[h(0)+h'_{+}(0)\|A\|-h(\|A\|)\right]\|A\|^{-2}\\
&=h(\|A\|)\|A\|^{-2}-h(\|A\|+m)(\|A\|+m)^{-2}\\
&-h(0)\left(\|A\|^{-2}-(\|A\|+m)^{-2}\right)-h'_{+}(0)\left(\|A\|^{-1}-(\|A\|+m)^{-1}\right)
\end{aligned}$$

and

$$\begin{aligned}&\left[h(0)+h'_+(0)\|B\|-h(\|B\|)\right]\|B\|^{-2}\\&-\left[h(0)+h'_+(0)(\|B\|-m)-h(\|B\|-m)\right](\|B\|-m)^{-2}\\&=h(\|B\|-m)(\|B\|-m)^{-2}-h(\|B\|)\|B\|^{-2}\\&-h(0)\left((\|B\|-m)^{-2}-\|B\|^{-2}\right)-h'_+(0)\left((\|B\|-m)^{-1}-\|B\|^{-1}\right)\end{aligned}$$

and by (4.11) we derive the desired inequality (4.9). □

Remark 4.3. If $A>0$ and $B-A>0$, then we can take $m=\left\|(B-A)^{-1}\right\|^{-1}$ in Proposition 4.2 to obtain other norm inequalities. The details are omitted.

The function $h(t):=-\ln(t+1)$ is operator convex with $h(0)=0$ and $h'(0)=-1$. Then by (4.10) we get

$$\begin{aligned}&B^{-2}\ln(B+1)-A^{-2}\ln(A+1)-(B-A)\\&\geq(\|A\|+m)^{-2}\ln(\|A\|+m+1)-\|A\|^{-2}\ln(\|A\|+1)\\&+\|A\|^{-1}-(\|A\|+m)^{-1}\\&\geq\|B\|^{-2}\ln(\|B\|+1)-(\|B\|-m)^{-2}\ln(\|B\|-m+1)\\&+(\|B\|-m)^{-1}-\|B\|^{-1}\\&>0\end{aligned}\tag{4.12}$$

provided that $A>0$ and $B-A\geq m>0$.

5. More Examples of Interest

We define the *upper incomplete Gamma function* as [7]

$$\Gamma(a,z):=\int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a):=\int_0^\infty t^{a-1}e^{-t}dt \text{ for } Re\ a>0.$$

We have the integral representation [8]

$$\Gamma(a,z)=\frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty\frac{t^{-a}e^{-t}}{z+t}dt\tag{5.1}$$

for $Re\ a<1$ and $|ph\ z|<\pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda):=\lambda^{-a}e^{-\lambda}$ for $\lambda>0$. Then by (5.1) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t)=\int_0^\infty\frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1-a)t^{-a}e^t\Gamma(a,t)\tag{5.2}$$

for $a<1$ and $t>0$.

For $a=0$ in (5.2) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t)=\int_0^\infty\frac{e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1)e^t\Gamma(0,t)=e^tE_1(t)\tag{5.3}$$

for $t>0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du. \tag{5.4}$$

Let $a=1-n$, with n a natural number with $n \geq 0$, then by (5.2) we have

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1-n,t) \\ &= (n-1)!t^{n-1}e^t\Gamma(1-n,t).\end{aligned} \tag{5.5}$$

If we define the generalized exponential integral [9] by

$$E_p(z) := z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n(t)$$

for $n \geq 1$ and $t>0$.

Using the identity [9, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}E_1(z) + \frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\,(-z)^k,$$

we get

$$\begin{aligned}&\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) \\ &= (n-1)!e^tE_n(t) \\ &= (n-1)!e^t\left[\frac{(-t)^{n-1}}{(n-1)!}E_1(t) + \frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\,(-t)^k\right] \\ &= \sum_{k=0}^{n-2}(-1)^k(n-k-2)!t^k + (-1)^{n-1}t^{n-1}e^tE_1(t)\end{aligned} \tag{5.6}$$

for $n \geq 2$ and $t>0$.

If $T>0$, then we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T) = \int_0^\infty \lambda^{-a}e^{-\lambda}(t+\lambda)^{-1}d\lambda = \Gamma(1-a)T^{-a}\exp(T)\Gamma(a,T) \tag{5.7}$$

for $a<1$.

In particular,

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(T) = \int_0^\infty e^{-\lambda}(T+\lambda)^{-1}d\lambda = \exp(T)E_1(T) \tag{5.8}$$

and, for $n \geq 2$

$$\begin{aligned}&\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(T) \\ &= \int_0^\infty \lambda^{n-1}e^{-\lambda}(T+\lambda)^{-1}d\lambda \\ &= \sum_{k=0}^{n-2}(-1)^k(n-k-2)!T^k + (-1)^{n-1}T^{n-1}\exp(T)E_1(T),\end{aligned} \tag{5.9}$$

where $T > 0$.

For $n = 2$, we also get

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(T) = \int_0^\infty \lambda e^{-\lambda}(T+\lambda)^{-1} d\lambda = 1 - T\exp(T) E_1(T) \tag{5.10}$$

for $T > 0$.

We consider the weight $w_{(\cdot+a)^{-1}}(\lambda) := \frac{1}{\lambda+a}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)(\lambda+a)} d\lambda = \frac{\ln t - \ln a}{t-a} \tag{5.11}$$

for all $a > 0$ and $t > 0$ with $t \neq a$.

From this, we get

$$\ln t = \ln a + (t-a)\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t)$$

for all t, $a > 0$.

If $T > 0$, then

$$\begin{aligned}\ln T &= \ln a + (T-a)\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t) \\ &= \ln a + (T-a)\int_0^\infty \frac{1}{(\lambda+a)}(\lambda+T)^{-1} d\lambda.\end{aligned} \tag{5.12}$$

Let $a > 0$. Assume that either $0 < T < a$ or $T > a$, then by (5.13) we get

$$(\ln T - \ln a)(T-a)^{-1} = \int_0^\infty \frac{1}{(\lambda+a)}(\lambda+T)^{-1} d\lambda. \tag{5.13}$$

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\begin{aligned}\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) &:= \int_0^\infty \frac{1}{(\lambda+t)(\lambda^2+a^2)} d\lambda \\ &= \frac{\pi t}{2a(t^2+a^2)} - \frac{\ln t - \ln a}{t^2+a^2}\end{aligned}$$

for $t > 0$ and $a > 0$.

For $a = 1$ we also have

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)(\lambda^2+1)} d\lambda = \frac{\pi t}{2(t^2+1)} - \frac{\ln t}{t^2+1}$$

for $t > 0$.

If $T > 0$ and $a > 0$, then

$$\begin{aligned}&\frac{\pi}{2a} T\left(T^2+a^2\right)^{-1} - (\ln T - \ln a)\left(T^2+a^2\right)^{-1} \\ &= \int_0^\infty \frac{1}{(\lambda^2+a^2)}(\lambda+T)^{-1} d\lambda\end{aligned} \tag{5.14}$$

and, in particular,

$$\frac{\pi}{2a} T\left(T^2+1\right)^{-1} - \left(T^2+1\right)^{-1}\ln T = \int_0^\infty \frac{1}{(\lambda^2+1)}(\lambda+T)^{-1} d\lambda. \tag{5.15}$$

Proposition 5.1. *Let $B \geq A > 0$ and $a < 1$, then*

$$A^{-a} \exp(A) \Gamma(a, A) \geq B^{-a} \exp(B) \Gamma(a, B). \tag{5.16}$$

In particular,

$$\exp(A) E_1(A) \geq \exp(B) E_1(B) \tag{5.17}$$

and

$$B \exp(B) E_1(B) \geq A \exp(A) E_1(A). \tag{5.18}$$

The proof follows by Theorem 3.1 and the identity (5.7).

Proposition 5.2. *Let $B \geq A > a > 0$ or $a > B \geq A > 0$, then*

$$(\ln A - \ln a)(A - a)^{-1} \geq (\ln B - \ln a)(B - a)^{-1}. \tag{5.19}$$

If $B \geq A > 1 > 0$ or $1 > B \geq A > 0$, then

$$(A - 1)^{-1} \ln A \geq (B - 1)^{-1} \ln B. \tag{5.20}$$

The proof follows by Theorem 3.1 and the identity (5.13).

Proposition 5.3. *Let $B \geq A > 0$ and $a > 0$, then*

$$\begin{aligned} &(\ln B - \ln a)\left(B^2 + a^2\right)^{-1} - (\ln A - \ln a)\left(A^2 + a^2\right)^{-1} \\ &\geq \frac{\pi}{2a}\left[B\left(B^2 + a^2\right)^{-1} - A\left(A^2 + a^2\right)^{-1}\right]. \end{aligned} \tag{5.21}$$

In particular, for $a = 1$,

$$\left(B^2 + 1\right)^{-1} \ln B - \left(A^2 + 1\right)^{-1} \ln A \geq \frac{\pi}{2}\left[B\left(B^2 + 1\right)^{-1} - A\left(A^2 + 1\right)^{-1}\right]. \tag{5.22}$$

The proof follows by Theorem 3.1 and the identity (5.14).

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[1]Mathematics, College of Sport, Health & Engineering,
Victoria University,
Melbourne City, MC 8001, Australia
Email address: sever.dragomir@vu.edu.au