Turkish J. Ineq., 7 (1) (2023), Pages 29 –39 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

NEW GENERALIZATIONS OF HERMITE-HADAMARD TYPE INEQUALITIES

MEHMET ZEKI SARIKAYA[1]

Abstract. In this study, we present a new generalization of the Hermite-Hadamard type inequalities for convex functions using a newly developed generalized an identity, which is rigorously proven. Moreover, we present new inequalities that are closely linked to both the left and right-hand side of the Hermite-Hadamard inequalities for Riemann and Riemann-Liouville fractional integrals. The results of this study build upon previous works and provide additional insights.

1. Introduction

Definition 1.1. The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

The theory of convex functions is a crucial area of mathematics that has applications in a wide range of fields, including optimization theory, control theory, operations research, geometry, functional analysis, and information theory. This theory is also highly relevant in other areas of science, such as economics, finance, engineering, and management sciences. One of the most well-known inequalities in the literature is the Hermite-Hadamard integral inequality (see, [4]), which is a fundamental tool for studying the properties of convex functions. This inequality has important implications in many areas of mathematics and has been extensively studied in recent years, leading to the development of new and powerful mathematical techniques for solving a broad range of problems.

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

Key words and phrases. Convex function, midpoint inequality, trapezoid inequality and Hermite-Hadamard inequality.

2010 *Mathematics Subject Classification.* Primary: 26A09. Secondary: 26D10, 26D15, 33E20.

Received: 02/06/2023 *Accepted*: 21/07/2023.

Cited this article as: M.Z. Sarıkaya, New Generalizations of Hermite-Hadamard type inequalities, Turkish Journal of Inequalities, 7(1) (2023), 29-39.

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

These inequalities were first introduced independently by Charles Hermite and Jacques Hadamard in the late 19th century and has since found numerous applications in various fields of mathematics, including analysis, geometry, and probability theory. The inequalities states that if a function is convex on a given interval, then the average value of the function over that interval is bounded from above by the midpoint value of the function, multiplied by the length of the interval. This inequalities provide a powerful tool for estimating integrals and has become a standard result in the theory of convex functions. The Hermite-Hadamard inequalities have numerous applications in mathematics. For example, they can be used to solve problems in integral calculus, probability theory, statistics, optimization, and number theory. The inequalities are also useful in solving physical and engineering problems that require the determination of function averages. In general, the Hermite-Hadamard inequalities provide a powerful tool for solving a wide range of mathematical problems. They are widely studied and used in various fields of mathematics, and their applications continue to grow as new problems are encountered. One of the most widely applied inequalities for convex functions is Hadamard's inequality, which has significant geometric implications. This inequality has been extensively studied in the literature, leading to numerous directions for extension and a rich mathematical literature (see [3–8, 11, 16]).

In [17], Zabandan gave the following important inequalities associated with the Hermite-Hadamard inequalities and he gave a few inequalities regarding the special cases of these inequalities.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ and $h : [0, 1] \to \mathbb{R}$ be a positive function such that $h \in L([0, 1])$. Then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2I_h\,(b-a)} \int_a^b \left[h\left(\frac{x-a}{b-a}\right) + h\left(\frac{b-x}{b-a}\right)\right] f(x)dx \leq \frac{f\,(a) + f\,(b)}{2} \tag{1.2}$$

where $I_h = \int_0^1 h\,(t)\,dt$.

The theory of fractional calculus has known an intensive development over the last few decades. It is shown that derivatives and integrals of fractional type provide an adequate mathematical modelling of real objects and processes see [10]. Therefore, the study of fractional differential equations need more developmental of inequalities of fractional type, for some of them, please see ([1, 2, 9, 12–15, 18–21]). Let us begin by introducing this type of inequality.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.2. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-t} u^{\alpha-1} du$. Here is $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Now, let's recall the basic expressions of Hermite-Hadamard inequality for fractional integrals is proved by Sarikaya et al. in [13] as follows:

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a function with $a < b$ and $f \in L_1([a, b])$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha > 0$.

In this paper, we introduce a novel extension of the Hermite-Hadamard inequalities for convex functions by utilizing a recently established generalized identities that are proven rigorously. Additionally, we derive new inequalities that have strong connections with both the left and right-hand sides of the Hermite-Hadamard inequalities for Riemann and Riemann-Liouville fractional integrals. Our findings not only expand upon previous research but also offer valuable insights and techniques for addressing a broad range of mathematical and scientific problems.

2. MAIN RESULTS

To prove our main results, we require the following lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$, $h : [0, 1] \to \mathbb{R}$ be a positive differentiable function, and $f' \in L[a, b]$. Then the following identities hold:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{2I_h(b-a)} \int_a^b \left[h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right)\right] f(x)\,dx \tag{2.1}$$

$$= \frac{b-a}{4I_h} \int_0^1 \left[\left(\int_t^1 H(s)ds\right) - \left(\int_0^t H(s)ds\right)\right] f'(ta + (1-t)b)\,dt$$

and

$$\frac{1}{2I_h(b-a)} \int_a^b \left[h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right)\right] f(x)\,dx - f\left(\frac{a+b}{2}\right) \tag{2.2}$$

$$= \frac{b-a}{2I_h} \left\{ \int_0^{\frac{1}{2}} \left(\int_0^t H(s)ds\right) f'(at + (1-t)b)\,dt \right.$$

$$-\int_{\frac{1}{2}}^{1}\left(\int_{t}^{1}H(s)ds\right)f'\left(at+(1-t)b\right)dt\Bigg\}$$

where $H(s)=h(s)+h(1-s)$.

Proof. By integration by parts, we have

$$\begin{aligned}
&\frac{b-a}{4I_h}\int_0^1\left[\left(\int_t^1 H(s)ds\right)-\left(\int_0^t H(s)ds\right)\right]f'\left(ta+(1-t)b\right)dt\\
&= -\left[\left(\int_t^1 H(s)ds\right)-\left(\int_0^t H(s)ds\right)\right]\left.\frac{f\left(at+(1-t)b\right)}{b-a}\right|_0^1\\
&\quad -\frac{2}{b-a}\int_0^1 H(t)f\left(ta+(1-t)b\right)dt\\
&= 2\left(\int_0^1 h(s)\,ds\right)\frac{f(a)+f(b)}{b-a}-\frac{2}{b-a}\int_0^1 H(t)f\left(ta+(1-t)b\right)dt.
\end{aligned}$$

Using the change of the variable and by multiplying the result by $\frac{b-a}{4I_h}$, we obtain desired equality (2.1).

Using a similar method, by integration by parts, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left(\int_0^t H(s)ds\right)f'\left(at+(1-t)b\right)dt-\int_{\frac{1}{2}}^1\left(\int_t^1 H(s)ds\right)f'\left(at+(1-t)b\right)dt\\
&= -\frac{1}{b-a}\left(\int_0^{\frac{1}{2}}H(s)ds\right)f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_0^{\frac{1}{2}}H(t)f\left(at+(1-t)b\right)dt\\
&\quad -\frac{1}{b-a}\left(\int_0^{\frac{1}{2}}H(s)ds\right)f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_{\frac{1}{2}}^1 H(t)f'\left(at+(1-t)b\right)dt\\
&= \frac{1}{b-a}\int_0^1 H(t)f\left(at+(1-t)b\right)dt-\frac{2}{b-a}\left(\int_0^1 h(s)\,ds\right)f\left(\frac{a+b}{2}\right).
\end{aligned}$$

Using the change of the variable and multiplying the result by $\frac{b-a}{2I_h}$, we obtain desired equality (2.2). □

Remark 2.1. In Lemma 2.1,

i) we choose $h(t) = t$ on $[0,1]$, then the equalities (2.1) and (2.2) become the following equalities, respectively,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{2}\int_0^1 (1-2t)\,f'(at+(1-t)b)\,dt$$

which is proved by Dragomir and Agarwal in [3], and

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) = (b-a)\left\{\int_0^{\frac{1}{2}} tf'(at+(1-t)b)\,dt - \int_{\frac{1}{2}}^1 (1-t)\,f'(at+(1-t)b)\,dt\right\}$$

which is proved by Kirmanci in [6].

ii) we choose $h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ on $[0,1]$ for $\alpha > 0$, then the equality (2.1) becomes the following equality

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] = \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(at+(1-t)b)\,dt$$

which is proved by Sarikaya et al. in [13].

Corollary 2.1. *With the assumptations in Lemma 2.1, if we take $h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ on $[0,1]$ for $\alpha \geq 1$, then the equality (2.2) reduse to*

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] - f\left(\frac{a+b}{2}\right) \\
= \;& \frac{b-a}{2}\left\{\int_0^{\frac{1}{2}} \left[t^{\alpha} - (1-t)^{\alpha} + 1\right] f'(at+(1-t)b)\,dt\right. \\
&\left. - \int_{\frac{1}{2}}^1 \left[(1-t)^{\alpha} - t^{\alpha} + 1\right] f'(at+(1-t)b)\,dt\right\}.
\end{aligned}$$

Theorem 2.1. *With the assumptations in Lemma 2.1. If $|f'|$ is convex on $[a,b]$, then we have the following inequalities*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{2I_h(b-a)}\int_a^b \left[h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right)\right] f(x)\,dx\right| \qquad (2.3)\\
\leq \;& \frac{b-a}{4I_h}\left(|f'(a)| + |f'(b)|\right) \\
&\times \left[I_h + \int_0^{\frac{1}{2}} H(s)\left(s^2 - \frac{1}{4}\right) ds + \int_{\frac{1}{2}}^1 H(s)\left(\frac{1}{4} - s^2\right) ds\right]
\end{aligned}$$

and

$$\left|\frac{1}{2I_h(b-a)}\int_a^b \left[h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right)\right] f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \qquad (2.4)$$

$$\leq \frac{b-a}{2I_h}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$
$$\times\left[\int_0^{\frac{1}{2}} H(s)\left(\frac{1}{8}-\frac{s^2}{2}\right)ds+\int_{\frac{1}{2}}^1 H(s)\left(\frac{s^2}{2}-\frac{1}{8}\right)ds\right]$$

where $I_h = \int_0^1 h(s)ds$.

Proof. We take absolute value of (2.1) and by using the convexity of $|f'|$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2I_h(b-a)}\int_a^b\left[h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right]f(x)\,dx\right|$$
$$\leq \frac{b-a}{4I_h}\int_0^1\left|\left(\int_t^1 H(s)ds\right)-\left(\int_0^t H(s)ds\right)\right|\left(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right)dt$$
$$=\frac{b-a}{4I_h}\left\{\int_0^{\frac{1}{2}}\left[\left(\int_t^1 H(s)ds\right)-\left(\int_0^t H(s)ds\right)\right]\left(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right)dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left[\left(\int_0^t H(s)ds\right)-\left(\int_t^1 H(s)ds\right)\right]\left(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right)dt\right\}$$
$$=\frac{b-a}{4I_h}\left\{\left(\frac{\left|f'(a)\right|+3\left|f'(b)\right|}{8}\right)\int_0^1 H(s)ds\right.$$
$$-\int_0^{\frac{1}{2}} H(s)\left(\left(\frac{1}{4}-s^2\right)\left|f'(a)\right|+\left((1-s)^2-\frac{1}{4}\right)\left|f'(b)\right|\right)ds$$
$$+\left(\frac{3\left|f'(a)\right|+\left|f'(b)\right|}{8}\right)\int_0^1 H(s)ds$$
$$\left.-\int_{\frac{1}{2}}^1 H(s)\left(\left(s^2-\frac{1}{4}\right)\left|f'(a)\right|+\left(\frac{1}{4}-(1-s)^2\right)\left|f'(b)\right|\right)ds\right\}$$
$$=\frac{b-a}{4I_h}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\left[\int_0^1 h(s)ds+\int_0^{\frac{1}{2}} H(s)\left(s^2-\frac{1}{4}\right)ds+\int_{\frac{1}{2}}^1 H(s)\left(\frac{1}{4}-s^2\right)ds\right].$$

Note that, by changing the order of the integrals

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \left[\left(\int_t^1 H(s)ds \right) - \left(\int_0^t H(s)ds \right) \right] \left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dt \\
= &\int_0^{\frac{1}{2}} \left(\int_0^1 H(s)ds - \int_0^t H(s)ds \right) \left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dt \\
&- \int_0^{\frac{1}{2}} \left(\int_0^t H(s)ds \right) \left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dt \\
= &\int_0^{\frac{1}{2}} \int_0^1 H(s)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dsdt - 2\int_0^{\frac{1}{2}} \int_0^t H(s)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dsdt \\
= &\int_0^1 \int_0^{\frac{1}{2}} H(s)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dtds - 2\int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} H(s)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dtds \\
= &\left(\frac{\left|f'(a)\right| + 3\left|f'(b)\right|}{8} \right) \int_0^1 H(s)ds - \int_0^{\frac{1}{2}} H(s) \left(\left(\frac{1}{4} - s^2 \right) \left|f'(a)\right| + \left((1-s)^2 - \frac{1}{4} \right) \left|f'(b)\right| \right) ds,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\frac{1}{2}}^1 \left[\left(\int_0^t H(s)ds \right) - \left(\int_t^1 H(s)ds \right) \right] \left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dt \\
= &\int_{\frac{1}{2}}^1 \int_0^1 H(s)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dsdt - 2\int_{\frac{1}{2}}^1 \int_t^1 H(s)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dsdt \\
= &\int_0^1 \int_{\frac{1}{2}}^1 H(s)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dtds - 2\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^s H(s)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right) dsdt \\
= &\left(\frac{3\left|f'(a)\right| + \left|f'(b)\right|}{8} \right) \int_0^1 H(s)ds - \int_{\frac{1}{2}}^1 H(s) \left(\left(s^2 - \frac{1}{4} \right) \left|f'(a)\right| + \left(\frac{1}{4} - (1-s)^2 \right) \left|f'(b)\right| \right) ds,
\end{aligned}
$$

$$
\int_{\frac{1}{2}}^1 H(s) \left(\frac{1}{4} - (1-s)^2 \right) ds = \int_0^{\frac{1}{2}} H(s) \left(\frac{1}{4} - s^2 \right) ds,
$$

and

$$
\int_0^{\frac{1}{2}} H(s) \left((1-s)^2 - \frac{1}{4} \right) ds = \int_{\frac{1}{2}}^1 H(s) \left(s^2 - \frac{1}{4} \right) ds.
$$

This proves the inequality (2.3).

Similarly, if we take absolute value of (2.2), by using the convexity of $|f'|$, and by changing the order of the integrals, we have

$$\begin{aligned}
&\left|\frac{1}{2I_h(b-a)}\int_a^b\left[h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right]f(x)\,dx-f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{2I_h}\left\{\int_0^{\frac{1}{2}}\left(\int_0^t H(s)ds\right)(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|)\,dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left(\int_t^1 H(s)ds\right)(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|)\,dt\right\} \\
=\ & \frac{b-a}{2I_h}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\left[\int_0^{\frac{1}{2}}H(s)\left(\frac{1}{8}-\frac{s^2}{2}\right)ds+\int_{\frac{1}{2}}^1 H(s)\left(\frac{s^2}{2}-\frac{1}{8}\right)ds\right]
\end{aligned}$$

which implies desired inequality (2.4).

Remark 2.2. In Theorem 2.1,

i) we choose $h(t)=t$ on $[0,1]$, then the inequalities (2.3) and (2.4) become the following inequalities, respectively,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{4}\left(\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right)$$

which is proved by Dragomir and Agarwal in [3], and

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{4}\left(\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right) \tag{2.5}$$

which is proved by Kirmaci in [6].

Corollary 2.2. *With the assumptations in Theorem 2.1, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\leq\frac{b-a}{2}\left(\frac{\left|f'(a)\right|+\left|f'(b)\right|}{\alpha+1}\right)\left(1-\frac{1}{2^{\alpha}}\right).$$

Proof. Let $h(s)=\frac{s^{\alpha-1}}{\Gamma(\alpha)}$ on $[0,1]$ for $\alpha\geq 1$, then the inequality (2.3) reduce to

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right| \qquad (2.6)\\
\leq\ & \frac{b-a}{4I_h}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\frac{1}{\Gamma(\alpha)} \\
&\times\left[\int_0^1 s^{\alpha-1}ds+\int_0^{\frac{1}{2}}\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\left(s^2-\frac{1}{4}\right)ds+\int_{\frac{1}{2}}^1\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\left(\frac{1}{4}-s^2\right)ds\right].
\end{aligned}$$

By calculating the above integrals

$$\int_0^{\frac{1}{2}}\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\left(s^2-\frac{1}{4}\right)ds \tag{2.7}$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{2}} \left[s^{\alpha+1} + s^2 (1-s)^{\alpha-1}\right] ds - \frac{1}{4}\int_0^{\frac{1}{2}} \left[s^{\alpha-1} + (1-s)^{\alpha-1}\right] ds \\
&= \int_0^{\frac{1}{2}} s^{\alpha+1} ds + \int_{\frac{1}{2}}^{1} \left(s^{\alpha-1} - 2s^{\alpha} + s^{\alpha+1}\right) ds - \frac{1}{4}\int_0^{\frac{1}{2}} \left[s^{\alpha-1} + (1-s)^{\alpha-1}\right] ds \\
&= \frac{1}{2^{\alpha+2}(\alpha+2)} + \left(\frac{1}{\alpha} - \frac{2}{\alpha+1} + \frac{1}{\alpha+2}\right) - \left(\frac{1}{2^{\alpha}\alpha} - \frac{2}{2^{\alpha+1}(\alpha+1)} + \frac{1}{2^{\alpha+2}(\alpha+2)}\right) \\
&\quad -\frac{1}{4}\int_0^{\frac{1}{2}} \left[s^{\alpha-1} + (1-s)^{\alpha-1}\right] ds \\
&= \left(\frac{1-\alpha}{\alpha(\alpha+1)} + \frac{1}{\alpha+2}\right) - \frac{1}{2^{\alpha}}\frac{1}{\alpha(\alpha+1)} - \frac{1}{4}\int_0^{\frac{1}{2}} \left[s^{\alpha-1} + (1-s)^{\alpha-1}\right] ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1} \left[s^{\alpha-1} + (1-s)^{\alpha-1}\right]\left(\frac{1}{4} - s^2\right) ds \qquad (2.8) \\
&= \frac{1}{4}\int_{\frac{1}{2}}^{1} \left[s^{\alpha-1} + (1-s)^{\alpha-1}\right] ds - \int_{\frac{1}{2}}^{1} \left[s^{\alpha+1} + s^2(1-s)^{\alpha-1}\right] ds \\
&= \frac{1}{4}\int_0^{\frac{1}{2}} \left[s^{\alpha-1} + (1-s)^{\alpha-1}\right] ds - \int_{\frac{1}{2}}^{1} s^{\alpha+1} ds - \int_0^{\frac{1}{2}} \left(s^{\alpha-1} - 2s^{\alpha} + s^{\alpha+1}\right) ds \\
&= \frac{1}{4}\int_0^{\frac{1}{2}} \left[s^{\alpha-1} + (1-s)^{\alpha-1}\right] ds - \left(\frac{1}{(\alpha+2)} - \frac{1}{2^{\alpha+2}(\alpha+2)}\right) \\
&\quad - \left(\frac{1}{2^{\alpha}\alpha} - \frac{1}{2^{\alpha}(\alpha+1)} + \frac{1}{2^{\alpha+2}(\alpha+2)}\right) \\
&= \frac{1}{4}\int_0^{\frac{1}{2}} \left[s^{\alpha-1} + (1-s)^{\alpha-1}\right] ds - \frac{1}{(\alpha+2)} - \frac{1}{2^{\alpha}}\frac{1}{\alpha(\alpha+1)}.
\end{aligned}
$$

If the integral values of (2.7) and (2.8) are written in (2.6), the desired result is achieved. This result is proved by Sarikaya et al. in [13]. □

We can obtain the midpoint inequality for Riemann-Liouville fractional integrals in a simpler way as follows.

Corollary 2.3. *With the assumptations in Theorem 2.1, we have*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \qquad (2.9) \\
&\leq \frac{(b-a)}{2}\left(|f'(a)| + |f'(b)|\right)\left[\frac{\alpha}{2(\alpha+2)} + \frac{1}{2^{\alpha}(\alpha+1)} - \frac{1}{(\alpha+1)(\alpha+2)}\right].
\end{aligned}
$$

Proof. Let $h(s)=\frac{s^{\alpha-1}}{\Gamma(\alpha)}$ on $[0,1]$ for $\alpha\geq 1$, then the inequality (2.4) reduce to

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{(b-a)\Gamma(\alpha+1)}{2\Gamma(\alpha)}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\left\{\int_0^{\frac{1}{2}}\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\left(\frac{1}{8}-\frac{s^2}{2}\right)ds\right.\\ &\left.+\int_{\frac{1}{2}}^{1}\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\left(\frac{s^2}{2}-\frac{1}{8}\right)ds\right\}.\end{aligned}$$

Note that

$$\int_0^{\frac{1}{2}}\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\left(\frac{1}{8}-\frac{s^2}{2}\right)ds=\frac{1}{8\alpha}+\frac{1}{2^{\alpha+1}}\frac{1}{\alpha(\alpha+1)}-\frac{1}{\alpha(\alpha+1)(\alpha+2)} \tag{2.10}$$

and

$$\int_{\frac{1}{2}}^{1}\left[s^{\alpha-1}+(1-s)^{\alpha-1}\right]\left(\frac{s^2}{2}-\frac{1}{8}\right)ds=\frac{1}{2(\alpha+2)}+\frac{1}{2^{\alpha+1}}\frac{1}{\alpha(\alpha+1)}-\frac{1}{8\alpha}. \tag{2.11}$$

Therefore, inequality (2.9) follows from (2.10) and (2.11). □

Remark 2.3. If we take $\alpha=1$ in Corollary 2.3, then the inequality (2.9) reduce to the inequality (2.5).

References

[1] H. Budak, E. Pehlivan, P. Kosem, *On new extensions of Hermite-Hadamard inequalities for generalized fractional integrals,* Sahand Communications in Mathematical Analysis, **18**(1) (2021), 73–88.

[2] H. Budak, C.C. Bilişik, M.Z. Sarikaya, *On some new extensions of inequalities of Hermite-Hadamard type for generalized fractional integrals,* Sahand Communications in Mathematical Analysis, **19**(2) (2022), 65–79.

[3] S.S. Dragomir, R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezodial formula,* Appl. Math. Lett., **11**(5) (1998), 91–95.

[4] S.S. Dragomir, C.E.M. Pearce, *Selected topics on Hermite–Hadamard inequalities and applications,* RGMIA Monographs, Victoria University, 2000.

[5] H. Kavurmaci, M. Avci, M.E. Özdemir, *New inequalities of Hermite-Hadamard type for convex functions with applications,* J. Inequal. Appl., **2011**(86) (2011), 1–11.

[6] U.S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula,* Appl. Math. Comput., **147** (2004), 137–146.

[7] U.S. Kirmaci, M.K. Bakula, M.E. Özdemir, J. Pečarić, *Hadamard-type inequalities for s-convex functions,* Applied Mathematics and Computation, **193**(1) (2007), 26–35.

[8] D.S. Mitrinovic, J.E. Pecaric, A. M. Fink, *Inequalities involving functions and their integrals and derivatives,* Kluwer Academic Publishers, Dordrecht, 1994.

[9] P.O. Mohammed, I. Brevik, *A new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals*, Symmetry, **12**(4) (2020), 610.

[10] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives Theory and Application*, Gordan and Breach Science, New York, 1993.

[11] H. Öğülmüş, M.Z. Sarıkaya, *Some Hermite–Hadamard type inequalities for h-convex functions and their applications*, Iranian Journal of Science and Technology, Transactions A: Science, **44** (2020), 813–819.

[12] M.Z. Sarıkaya, H. Yıldırım, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals,* Miskolc Mathematical Notes, **17**(2) (2016), 1049–1059.

[13] M.Z. Sarıkaya, E. Set, H. Yaldız, N. Başak, *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities,* Math. Comput. Model., **57** (2013), 2403–2407.

[14] M.Z. Sarıkaya, F. Ertuğral, *On the generalized Hermite-Hadamard inequalities,* Annals of the University of Craiova-Mathematics and Computer Science Series, **47**(1) (2020), 193–213.

[15] M.Z. Sarıkaya, H. Budak, *Generalized Hermite-Hadamard type integral inequalities for fractional integrals*, Filomat, **30**(5) (2016), 1315–1326.

[16] M.Z. Sarıkaya, N. Aktan, *On the generalization of some integral inequalities and their applications,* Mathematical and Computer Modelling, **54**(9-10) (2011), 2175–2182.

[17] G. Zabandan, *An extension and refinement of Hermite-Hadamard inequality and related results,* Int. J. Nonlinear Anal. Appl., **11**(2) (2020), 379–390.

[18] Y. Zhang, J. Wang, *On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals,* J. Inequal. Appl., **2013**(220) (2013), 1–27.

[19] J. Wang, X. Li, M. Feckan, Y. Zhou, *Hermite–Hadamard-type inequalities for Riemann–Liouville fractional integrals via two kinds of convexity,* Appl. Anal., **92**(11) (2012), 2241–2253.

[20] J. Wang, X. Li, C. Zhu, *Refinements of Hermite-Hadamard type inequalities involving fractional integrals,* Bull. Belg. Math. Soc. Simon Stevin, **20** (2013), 655–666.

[21] J. Wang, C. Zhu, Y. Zhou, *New generalized Hermite-Hadamard type inequalities and applications to special means,* Journal of Inequalities and Applications, **2013**(1) (2013), 1–15.

[1]Düzce University,
Department of Mathematics,
Faculty of Science and Arts, Düzce, Turkey
Email address: sarikayamz@gmail.com