Turkish J. Ineq., 7 (1) (2023), Pages 22 –28 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

GENERALIZED INEQUALITIES BASED ON PARAMETRIC EXTENSIONS OF RELATIVE OPERATOR ENTROPY

WENSHI LIAO[1] AND PUJUN LONG[1]

Abstract. In this paper, we present some operator inequalities related to parametric relative operator entropy and Tallis relative operator entropy. Our results are some refinements and generalizations of some existing results.

1. Introduction

In this paper, all operators are bounded linear operators on a Hilbert space $\mathcal{H}$. $B(\mathcal{H})_{sa}$ is the set of all self-adjoint operators. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$ and an operator T is said to be strictly positive (denoted by $T > 0$) if $(Tx, x) \geq 0$ for all $x \neq 0$. A continuous real function f on $[0, +\infty)$ is said to be operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for $A, B \in B(\mathcal{H})_{sa}$.

For two strictly positive operators A and B, the weighted geometric mean is defined by $A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ for $0 \leq \alpha \leq 1$ and $A\sharp_0 B = A, A\sharp_1 B = B$. $A\sharp_\alpha B$ can be extended to $\alpha \in \mathbb{R}$, and we rewrite it as $A\natural_\alpha B$, which can be treated as a path from A to B.

A relative operator entropy of two strictly positive operators

$$S(A|B) = A^{\frac{1}{2}} \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$

was introduced by Fujii and Kamei [1] in the noncommutative information theory. When A is positive, one may set $S(A|B) := \lim_{\epsilon \to +0} S(A + \epsilon I|B)$ if the limit which is taken in the strong operator topology exists.

Tallis relative operator entropy was introduced by Yanagi-Kuriyama-Furuichi in [2]. For $A, B > 0$ and $0 < \lambda \leq 1$, Tallis relative operator entropy was defined as

$$T_\lambda(A|B) = \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}} - A}{\lambda} = \frac{A\sharp_\lambda B - A}{\lambda} = A^{\frac{1}{2}} \ln_\lambda(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}},$$

Key words and phrases. Relative operator entropy, Tallis relative operator entropy, operator inequalities.
2010 *Mathematics Subject Classification*: Primary: 15A45. Secondary: 47A63, 47A64.
Received: 30/05/2023 *Accepted*: 15/08/2023.
Cited this article as: W.S. Liao, P.J. Long, Generalized inequalities based on parametric extensions of relative operator entropy, Turkish Journal of Inequalities, 7(1) (2023), 22-28.

where $\ln_\lambda(X) = \frac{X^\lambda - 1}{\lambda}$ is λ-logarithmic function defined for $X > 0$ and $0 \neq \lambda \in \mathbb{R}$ and $\ln_{-\lambda}(X) = X^{-\lambda} \ln_\lambda(X)$. We remark that

$$T_0(A|B) = \lim_{\lambda \to +0} T_\lambda(A|B) = S(A|B),$$

since $\lim_{\lambda \to +0} \frac{x^\lambda - 1}{\lambda} = \log x$ for $x > 0$, and also the definition of $T_\lambda(A|B)$ can be extended for $\lambda \in \mathbb{R}$.

The generalized relative operator entropy for strictly positive operators A, B and $\alpha \in \mathbb{R}$ defined in [3] by setting

$$S_\alpha(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

In particular, we have $S_0(A|B) = S(A|B)$.

Nikoufar et al. [4] announced the notion of operator (α, β)-geometric mean ((α, β)-power mean) for two real numbers α, β as a generalization of operator weighted geometric mean of two strictly positive operators A, B as follows:

$$A \natural_{(\alpha,\beta)} B = A^{\frac{\beta}{2}}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^\alpha A^{\frac{\beta}{2}}.$$

Moreover $A \natural_{(\alpha,\beta)} B$ can be extended to nonnegative A and B as

$$A \natural_{(\alpha,\beta)} B = \lim_{\epsilon \to +0} (A + \epsilon I) \natural_{(\alpha,\beta)} (B + \epsilon I).$$

Note that $A \natural_\alpha B = A \natural_{(\alpha,1)} B$, $A \natural_{(-1,\beta)} B = A^\beta B^{-1} A^\beta$, $A \natural_{(0,\beta)} B = A^\beta$ and $A \natural_{(1,\beta)} B = B$.

Nikoufar [5] defined relative operator (α, β)-entropy (parametric extension of relative operator entropy) as

$$S_{(\alpha,\beta)}(A|B) = A^{\frac{\beta}{2}}(A^{-\frac{\beta}{2}}BA^{\frac{-\beta}{2}})^\alpha \log(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})A^{\frac{\beta}{2}}$$

for strictly positive operators A, B and real numbers α, β and Tsallis relative operator (λ, β)-entropy

$$T_{(\lambda,\beta)}(A|B) = A^{\frac{\beta}{2}} \ln_\lambda(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})A^{\frac{\beta}{2}}$$

for strictly positive operators A, B and real numbers $\beta, \lambda \neq 0$. $S_{(\alpha,\beta)}(A|B)$ can considered as a tangent at α of $A \natural_{(\alpha,\beta)} B$.

We collect these basic properties of relative operator (α, β)-entropy and Tsallis relative operator (λ, β)-entropy [5, 6]:

i) $S_{(\alpha,1)}(A|B) = S_\alpha(A|B), S_{(0,1)}(A|B) = S(A|B)$;

ii) $T_{(\lambda,1)}(A|B) = T_\lambda(A|B), \lim_{\lambda \to 0} T_{(\lambda,\beta)}(A|B) = S_{(0,\beta)}(A|B) = S(A^\beta|B)$.

In this paper, we discuss the relationships between $S_{(\alpha,\beta)}(A|B)$ and $T_{(\lambda,\beta)}(A|B)$, and the upper bounds and lower bounds of $S_{(\alpha,\beta)}(A|B)$ and $T_{(\lambda,\beta)}(A|B)$, we also obtain some refinements and generalizations of existing inequalities.

2. Main results

We begin this section with the inequalities between $S_{(0,\beta)}(A|B)$ and $T_{(\lambda,\beta)}(A|B)$.

Theorem 2.1. *Let $\lambda \in (0,1]$ and $\beta > 0$. For any invertible positive operators A and B, we have*

$$A\natural_{(0,\beta)}B - A\natural_{(-1,\beta)}B \leq T_{(-\lambda,\beta)}(A|B) \leq S_{(0,\beta)}(A|B) \leq T_{(\lambda,\beta)}(A|B) \leq A\natural_{(1,\beta)}B - A\natural_{(0,\beta)}B. \tag{2.1}$$

Moreover, $T_{(\lambda,\beta)}(A|B) = 0$ if and only if $A^{\beta} = B$.

Proof. Since

$$1 - x^{-1} \leq \ln_{\lambda} x \leq x - 1 \tag{2.2}$$

for any $x > 0$ and $0 < \lambda \leq 1$.

Let $\frac{1}{x}$ instead of x in the above inequality, we have

$$1 - x^{-1} \leq \ln_{-\lambda} x \leq x - 1. \tag{2.3}$$

It is known that for any $x > 0$ and $0 < \lambda \leq 1$,

$$\ln_{-\lambda} x \leq \log x \leq \ln_{\lambda} x, \tag{2.4}$$

then combining (2.2), (2.3) and (2.4), we get

$$1 - x^{-1} \leq \ln_{-\lambda} x \leq \log x \leq \ln_{\lambda} x \leq x - 1. \tag{2.5}$$

By the operator monotonicity and inequalities (2.5), we have

$$\begin{aligned} I - (A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{-1} &\leq \ln_{-\lambda}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}) \leq \log(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}) \\ &\leq \ln_{\lambda}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}) \leq A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}} - I. \end{aligned} \tag{2.6}$$

Multiply $A^{\frac{\beta}{2}}$ from the both sides of inequalities (2.6), we have

$$\begin{aligned} A^{\beta} - A^{\beta}B^{-1}A^{\beta} &\leq A^{\frac{\beta}{2}} \ln_{-\lambda}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})A^{\frac{\beta}{2}} \leq A^{\frac{\beta}{2}} \log(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})A^{\frac{\beta}{2}} \\ &\leq A^{\frac{\beta}{2}} \ln_{\lambda}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})A^{\frac{\beta}{2}} \leq B - A^{\beta}. \end{aligned}$$

This completes the proof. □

Remark 2.1. If we put $\beta = 1$ in (2.1), then we obtain the inequality (2.2) in [7] as follows

$$A - AB^{-1}A \leq T_{-\lambda}(A|B) \leq S(A|B) \leq T_{\lambda}(A|B) \leq B - A.$$

Theorem 2.2. *Let $a > 0$, $\lambda \in (0,1]$ and $\beta > 0$. For any invertible positive operators A and B, we have*

$$A\natural_{(0,\beta)}B - \frac{1}{a}A\natural_{(-1,\beta)}B + \frac{1-a^{\lambda}}{\lambda a^{\lambda}}A\natural_{(0,\beta)}B \leq a^{-\lambda}T_{(-\lambda,\beta)}(A|B). \tag{2.7}$$

Proof. For $a > 0$ and $\lambda \in (0,1]$, we have

$$1 - \frac{1}{ax} \leq a^{-\lambda}\frac{x^{-\lambda}-1}{-\lambda} - \frac{1-a^{\lambda}}{\lambda a^{\lambda}}, x > 0,$$

and so

$$I - \frac{1}{a}A^{\frac{\beta}{2}}B^{-1}A^{\frac{\beta}{2}} \le a^{-\lambda}\frac{(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{-\lambda} - I}{-\lambda} - \frac{1-a^{\lambda}}{\lambda a^{\lambda}}I.$$

Multiply both sides by $A^{\frac{\beta}{2}}$, we have

$$A^{\beta} - \frac{1}{a}A^{\beta}B^{-1}A^{\beta} \le a^{-\lambda}\frac{A^{\frac{\beta}{2}}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{-\lambda}A^{\frac{\beta}{2}} - A^{\beta}}{-\lambda} - \frac{1-a^{\lambda}}{\lambda a^{\lambda}}A^{\beta}.$$

This completes the proof. □

Remark 2.2. If we put $\beta = 1$, then we obtain the inequality (2.1) in [7]

$$A - \frac{1}{a}AB^{-1}A + \frac{1-a^{\lambda}}{\lambda a^{\lambda}}A \le a^{-\lambda}T_{-\lambda}(A|B).$$

If we put $a = 1$, then we have

$$A\natural_{(0,\beta)}B - A\natural_{(-1,\beta)}B \le T_{(-\lambda,\beta)}(A|B).$$

This is the first inequality of (2.1).

If we put $\beta = 1$ and $a = 1$, then we have

$$A - AB^{-1}A \le T_{-\lambda}(A|B).$$

Next, we discuss the relationships between $T_{(\lambda,\beta)}(A|B)$ and $S_{(\lambda,\beta)}(A|B)$.

Theorem 2.3. *Let $a > 0$, $\lambda \in (0,1]$ and $\beta > 0$. For any invertible positive operators A and B, we have*

$$\begin{aligned} A\natural_{(\lambda,\beta)}B - \frac{1}{a}A\natural_{(\lambda-1,\beta)}B + \ln_{\lambda}\frac{1}{a}A\natural_{(0,\beta)}B & \\ \le T_{(\lambda,\beta)}(A|B) & \\ \le S_{(\lambda,\beta)}(A|B) & \\ \le 2T_{(2\lambda,\beta)}(A|B) - T_{(\lambda,\beta)}(A|B) & \\ \le \frac{1}{a}A\natural_{(\lambda+1,\beta)}B - \ln_{\lambda}\frac{1}{a}A\natural_{(2\lambda,\beta)}B - A\natural_{(\lambda,\beta)}B. & \end{aligned} \tag{2.8}$$

Proof. It is known that for positive real number x,

$$\ln_{-\lambda}x = \frac{x^{-\lambda}-1}{-\lambda} \le \log x \le \frac{x^{\lambda}-1}{\lambda} = \ln_{\lambda}x,$$

So, we have

$$\frac{x^{\lambda}-1}{\lambda} = x^{\lambda}\frac{x^{-\lambda}-1}{-\lambda} \le x^{\lambda}\log x \le x^{\lambda}\frac{x^{\lambda}-1}{\lambda},$$

by Lemma 3.5 in [8], that is,

$$x^{\lambda} - \frac{1}{a}x^{\lambda-1} + \ln_{\lambda}\frac{1}{a} \le \ln_{\lambda}x \le x^{\lambda}\log x \le x^{\lambda}\ln_{\lambda}x = 2\ln_{2\lambda}x - \ln_{\lambda}x \le x^{\lambda}(\frac{1}{a}x - \ln_{\lambda}\frac{1}{a}x^{\lambda} - 1),$$

by the similar method in the proof of Theorem 2.1, we obtain the results. □

Remark 2.3. In the inequalities (2.8), if we put $a = 1$, then we get

$$\begin{aligned} A\natural_{(\lambda,\beta)}B - A\natural_{(\lambda-1,\beta)}B &\le T_{(\lambda,\beta)}(A|B) \\ &\le S_{(\lambda,\beta)}(A|B) \\ &\le 2T_{(2\lambda,\beta)}(A|B) - T_{(\lambda,\beta)}(A|B) \\ &\le A\natural_{(\lambda+1,\beta)}B - A\natural_{(\lambda,\beta)}B. \end{aligned} \tag{2.9}$$

Remark 2.4. It follows from inequalities (2.1) and (2.8) that

$$T_{(-\lambda,\beta)}(A|B) \le S_{(0,\beta)}(A|B) \le T_{(\lambda,\beta)}(A|B) \le S_{(\lambda,\beta)}(A|B).$$

If we put $\beta = 1$ in (2.9), then we have

$$T_\lambda(A|B) \le S_\lambda(A|B) \le 2T_{2\lambda}(A|B) - T_\lambda(A|B).$$

Theorem 2.4. *Let $a > 0$, $\lambda \in (0, 1]$ and $\beta > 0$. For any invertible positive operators A and B, we have*

$$S_{(\lambda,\beta)}(A|B) \ge T_{(\lambda,\beta)}(A|B) + (\ln_{-\lambda} a)A\natural_{(0,\beta)}B - (\log a)A\natural_{(\lambda,\beta)}B, \tag{2.10}$$

$$S_{(\lambda,\beta)}(A|B) \le 2T_{(2\lambda,\beta)}(A|B) - T_{(\lambda,\beta)}(A|B) + (\ln_\lambda a)A\natural_{(2\lambda,\beta)}B - (\log a)A\natural_{(\lambda,\beta)}B. \tag{2.11}$$

Proof.

$$\frac{x^\lambda - 1}{\lambda} \le x^\lambda \log x \le x^\lambda \frac{x^\lambda - 1}{\lambda},$$

so we have

$$\frac{(ax)^\lambda - 1}{\lambda} \le (ax)^\lambda \log(ax) \le (ax)^\lambda \frac{(ax)^\lambda - 1}{\lambda},$$

that is

$$\frac{x^\lambda - 1}{\lambda} + \ln_{-\lambda} a - x^\lambda \log a \le x^\lambda \log x \le 2\frac{x^{2\lambda} - 1}{\lambda} - \frac{x^\lambda - 1}{\lambda} + x^{2\lambda} \ln_\lambda a - x^\lambda \log a.$$

It follows that

$$T_{(\lambda,\beta)}(A|B) + (\ln_{-\lambda} a)A\natural_{(0,\beta)}B - (\log a)A\natural_{(\lambda,\beta)}B \le S_{(\lambda,\beta)}(A|B),$$

$$S_{(\lambda,\beta)}(A|B) \le 2T_{(2\lambda,\beta)}(A|B) - T_{(\lambda,\beta)}(A|B) + (\ln_\lambda a)A\natural_{(2\lambda,\beta)}B - (\log a)A\natural_{(\lambda,\beta)}B.$$

This completes the proof. □

Remark 2.5. It follows from inequalities (2.8), (2.10) and (2.11) that

$$(1 - \log a)A\natural_{(\lambda,\beta)}B - \frac{1}{a}A\natural_{(\lambda-1,\beta)}B \le S_{(\lambda,\beta)}(A|B),$$

$$S_{(\lambda,\beta)}(A|B) \le \frac{1}{a}A\natural_{(\lambda+1,\beta)}B + (\ln_\lambda a + \ln_{-\lambda} a)A\natural_{(2\lambda,\beta)}B - (1 + \log a)A\natural_{(\lambda,\beta)}B,$$

then putting $\lambda \to 0$ and $\beta = 1$, we get inequalities (1.4) in [7]

$$(1 - \log a)A - \frac{1}{a}AB^{-1}A \le S(A|B) \le (\log a - 1)A + \frac{1}{a}B.$$

Theorem 2.5. *Let $a > 0$, $\lambda \in (0, 1]$ and $\beta > 0$. For any invertible positive operators A and B, we have*

$$\begin{aligned} &a^{-\lambda}(l_1 T_{(\lambda,\beta)}(A|B) + l_2 A\natural_{(0,\beta)}B + l_3 A\natural_{(\lambda,\beta)}B) - (\log a)A\natural_{(\lambda,\beta)}B \\ &\leq S_{(\lambda,\beta)}(A|B) \\ &\leq l_1(2T_{(2\lambda,\beta)}(A|B) - T_{(\lambda,\beta)}(A|B)) + l_2 A\natural_{(\lambda,\beta)}B + l_3 A\natural_{(2\lambda,\beta)}B - (\log a)A\natural_{(\lambda,\beta)}B, \end{aligned} \tag{2.12}$$

where $l_1 = va^\lambda + (1-v)$, $l_2 = v\ln_\lambda a$ and $l_3 = (1-v)\ln_\lambda a$.

Proof. Note that

$$\frac{(ax)^\lambda - 1}{\lambda} = \frac{x^\lambda - 1}{\lambda}a^\lambda + \frac{a^\lambda - 1}{\lambda}, \tag{2.13}$$

$$\frac{(ax)^\lambda - 1}{\lambda} = \frac{a^\lambda - 1}{\lambda}x^\lambda + \frac{x^\lambda - 1}{\lambda}. \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$\begin{aligned} \frac{(ax)^\lambda - 1}{\lambda} &= v(\frac{x^\lambda - 1}{\lambda}a^\lambda + \frac{a^\lambda - 1}{\lambda}) + (1-v)(\frac{a^\lambda - 1}{\lambda}x^\lambda + \frac{x^\lambda - 1}{\lambda}) \\ &= [va^\lambda + (1-v)]\frac{x^\lambda - 1}{\lambda}a^\lambda + [v + (1-v)x^\lambda]\frac{a^\lambda - 1}{\lambda}, \end{aligned} \tag{2.15}$$

$$\frac{(ax)^\lambda - 1}{\lambda} \leq (ax)^\lambda \log(ax) \leq (ax)^\lambda \frac{(ax)^\lambda - 1}{\lambda}, \tag{2.16}$$

Combining (2.15) and (2.16), we have

$$a^{-\lambda}(l_1 \ln_\lambda x + l_2 + l_3 x^\lambda) - x^\lambda \log a \leq x^\lambda \log x \leq l_1(2\ln_{2\lambda} x - \ln_\lambda x) + l_2 x^\lambda + l_3 x^{2\lambda} - x^\lambda \log a.$$

Taking $x = A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}$ and multiply $A^{\frac{\beta}{2}}$ from both sides, we can obtain the results. $\square$

Remark 2.6. Putting $v = 1$ in (2.12), we get (2.10) and (2.11).

Acknowledgements. This research was supported by the Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No.KJQN201901546 and No.KJQN201901518) and the Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No.KJQN202101536). The authors wish to express their heartfelt thanks to the referees for their detailed and helpful suggestions for revising the manuscript.

References

[1] J.I. Fujii, E. Kamei, *Relative operator entropy in noncommutative information theory*, Math. Japonica, **34** (1989), 341–348.

[2] K. Yanagi, K. Kuriyama, S. Furuichi, *Generalized Shannon inequalities based on Tsallis relative operator entropy*, Linear Algebra Appl., **394** (2005), 109–118.

[3] T. Furuta, *Parametric extensions of Shannon inequality and its reverse one in Hilbert space operators*, Linear Algebra Appl., **381** (2004), 219–235.

[4] I. Nikoufar, A. Ebadian, M. Eshagi Gordji, *The simplest proof of Lieb concavity theorem*, Adv. Math., **248** (2013), 531–533.

[5] I. Nikoufar, *On operator inequalities of some relative operator entropies*, Advances in Mathematics, **259** (2014), 376–383.

[6] I. Nikoufar, *Convexity of parameter extensions of some relative operator entropies with a perspective approach*, Glasgow Math. J., **62** (2020), 737–744.
[7] L. Zou, *Operator inequalities associated with Tsallis relative operator entropy*, Math. Inequal. Appl., **18**(2) (2015), 401–406.
[8] S. Furuichi, K. Yanagi, K. Kuriyama, *A note on operator inequalities of Tsallis relative operator entropy*, Linear Algebra Appl., **407**(1) (2002), 19–31.

[1]College of Mathematics, Physics and Data science,
Chongqing University of Science and Technology,
Chongqing, 401331, P. R.China
Email address: liaowenshi@gmail.com
Email address: longpujun@gmail.com