Turkish J. Ineq., 7 (1) (2023), Pages 12 –21 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

COMPLETE MONOTONICITY OF FUNCTIONS INVOLVING k-TRIGAMMA AND k-TETRAGAMMA FUNCTIONS WITH RELATED INEQUALITIES

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Abstract. In this paper, by using the Bernstein-Widder theorem and properties on k-special function, we present several complete monotonicity properties on the function related to k-trigamma and k-tetragama functions. As an immediate consequence, we give the double-sided inequality on the function $[\psi_k'(x)]^2 + \frac{1}{k}\psi_k''(x)$. All the results obtained in this work are not just the k-generalizations of classical ones but also are the improvements of the bounds of recent results on the function $[\psi_k'(x)]^2 + \frac{1}{k}\psi_k''(x)$.

1. Introduction

The second kind of Eular integral, also known as the gamma function, is defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

for all positive real values of x. The logarithmic derivative $\psi(x) = \frac{d}{dx}\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ of the function is called the psi or digamma function and its derivatives are generally called polygamma functions. In particular the first and second derivatives of digamma functions are called trigamma and tetragamma functions, respectively. These functions play major roles in the theory of special functions and have applications in many other branches. Many researchers interest in these functions and obtain complete monotonicity properties, convexity and/or concavity and inequalities on the special functions or related to these functions (some researches related to this work can be found in [1–3, 5, 6, 8–10, 15] and references therein). Some of the researchers find several generalizations of these functions, such as; Díaz and Pariguan in [4] introduced k-generalized Pochhammer symbol as follows:

Key words and phrases. Complete monotonicity, Inequalities, k-Tetragamma function, k-Trigamma function.

2010 *Mathematics Subject Classification.* Primary: 26A48. Secondary: 33B99, 26D07.

Received: 07/05/2023 *Accepted*: 01/08/2023.

Cited this article as: E. Yıldırım, Complete Monotonicity of Functions Involving k-Trigamma and k-Tetragamma Functions with Related Inequalities, Turkish Journal of Inequalities, 7(1) (2023), 12-21.

Definition 1.1. [4] Let $x \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N}^+$, the Pochhammer k-symbol is given by

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k).$$

By using the Definition 1.1, they defined k-gamma function Γ_k as the following limit expression.

Definition 1.2. [4] For $k > 0$, the k-gamma function Γ_k is given by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{n}{k}-1}}{(x)_{n,k}}, \quad x \in \mathbb{C} \setminus k\mathbb{Z}^-.$$

Also in the paper [4], they obtained integral and infinite product representations of the function by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, \tag{1.1}$$

$$\frac{1}{\Gamma_k(x)} = x k^{-\frac{x}{k}} e^{\frac{x}{k}\gamma} \prod_{n=1}^{\infty} \left(\left(1 + \frac{x}{nk}\right) e^{-\frac{x}{nk}} \right) \tag{1.2}$$

for $x \in \mathbb{C}$, $Re(x) > 0$. They proved the k-generalization of Bohr-Mollerup Theorem, Stirling formula and found some properties on k-gamma function such as

$$\Gamma_k(x+k) = x\Gamma_k(x), \tag{1.3}$$

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right). \tag{1.4}$$

The k-special functions have also been used in many applications, for instance; combinatorics, fractional calculus, theory of inequality etc. In [12], authors gave several integral representations of k-digamma function, one of them is defined by

$$\psi_k(x) = \frac{\ln k}{k} - \frac{\gamma}{k} + \int_0^1 \frac{t^{k-1} - t^{x-1}}{1 - t^k} dt \tag{1.5}$$

for x, $k > 0$. Applying logarithmic derivative of the equation (1.4) leads us to the recurrence formula for k-digamma function by

$$\psi_k(x+k) = \frac{1}{x} + \psi_k(x) \tag{1.6}$$

and for the first and second derivatives of the equation (1.6), we get

$$\psi'_k(x+k) = \psi'_k(x) - \frac{1}{x^2}, \tag{1.7}$$

$$\psi''_k(x+k) = \psi''_k(x) + \frac{2}{x^3} \tag{1.8}$$

respectively for x, $k > 0$ that are called recurrence formulas on k-trigamma $\psi'_k(x)$ and k-tetragamma $\psi''_k(x)$ functions respectively.

Yıldırım in [13] used Binet's first formula for $\ln \Gamma_k(x)$ and complete monotonicity properties on k-digamma function and its derivatives to obtain following inequalities:

Corollary 1.1. [13] *The following inequalities*

$$\frac{\ln x}{k}-\frac{1}{2x}-\frac{k}{12x^2}<\psi_k(x)<\frac{\ln x}{k}-\frac{1}{2x}, \tag{1.9}$$

$$\frac{1}{kx}+\frac{1}{2x^2}+\frac{k}{6x^3}-\frac{k^3}{30x^5}<\psi'_k(x)<\frac{1}{kx}+\frac{1}{2x^2}+\frac{k}{6x^3} \tag{1.10}$$

and

$$-\frac{1}{kx^2}-\frac{1}{x^3}-\frac{k}{2x^4}<\psi''_k(x)<-\frac{1}{kx^2}-\frac{1}{x^3} \tag{1.11}$$

are valid for all $x, k > 0$.

By using previous inequalities (1.10) and (1.11) and the recurrence formula (1.7), the author in [14] mentioned that the following double-sided inequality

$$\frac{p_k(x)}{900x^4(x+k)^{10}}<[\psi'_k(x)]^2+\frac{1}{k}\psi''_k(x)<\frac{q_k(x)}{36x^4(x+k)^6} \tag{1.12}$$

is valid for all positive real values of x and k, where functions p_k and q_k are defined by

$$\begin{aligned} p_k(x) &= 75x^{10}+900kx^9+4840k^2x^8+15370k^3x^7+31865k^4x^6+45050k^5x^5 \quad (1.13)\\ &\quad +44101k^6x^4+29700k^7x^3+13290k^8x^2+3600k^9x+450k^{10},\\ q_k(x) &= 21x^6+132kx^5+352k^2x^4+504k^3x^3+408k^4x^2+180k^5x+36k^6. \quad (1.14) \end{aligned}$$

Also in the same paper, the following lemmas were obtained:

Lemma 1.1. [14] *For all positive real values of x, k and r, we have*

$$\frac{1}{x^{r/k}}=\frac{k^{r/k-1}}{\Gamma_k(r)}\int_0^\infty t^{r/k-1}e^{-xt}dt. \tag{1.15}$$

By taking $r = nk$ and using the equation $\Gamma_k(nk) = (n-1)!k^{n-1}$ for $n \in \mathbb{Z}^+$, the equation (1.15) *becomes*

$$\frac{1}{x^n}=\frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-xt}dt. \tag{1.16}$$

Lemma 1.2. [14] *For all positive real values of x and k and positive integer n, k-digamma and k-polygamma functions are defined by the following integrals:*

$$\begin{aligned} \psi_k(x) &= \frac{\ln k-\gamma}{k}+\int_0^\infty \frac{e^{kt}-e^{-xt}}{1-e^{-kt}}dt, \quad (1.17)\\ \psi_k^{(n)}(x) &= (-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-kt}}e^{-xt}dt. \quad (1.18) \end{aligned}$$

Author used the well-known Bernstein-Widder theorem:

Theorem 1.1. [11, Theorem 12b] *A necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that*

$$f(x)=\int_0^\infty e^{-xt}d\alpha(t)$$

where $\alpha(t)$ is a non-decreasing function and the integral converges for $0 < x < \infty$.

Then the author showed complete monotonicity on the function $[\psi'_k(x)]^2+\frac{1}{k}\psi''_k(x)-p_k(x)$ and $q_k(x)-[\psi'_k(x)]^2-\frac{1}{k}\psi''_k(x)$, where the functions p_k and q_k are defined by (1.13) and (1.14), respectively. Hence the author obtained simpler bounds for the inequality (1.12) as follows:

Theorem 1.2. *The functions*

$$P(x)=\left[\psi'_k(x)\right]^2+\frac{1}{k}\psi''_k(x)-\frac{x^2+12k^2}{12x^4(x+k)^2} \tag{1.19}$$

and

$$Q(x)=\frac{x+12k}{12x^4(x+k)}-\left[\psi'_k(x)\right]^2-\frac{1}{k}\psi''_k(x) \tag{1.20}$$

are completely monotonic. As an immediate consequence, the following double-sided inequality

$$\frac{x^2+12k^2}{12x^4(x+k)^2}<\left[\psi'_k(x)\right]^2+\frac{1}{k}\psi''_k(x)<\frac{x+12k}{12x^4(x+k)} \tag{1.21}$$

is valid for all positive real values of x and k.

It is worth to mention that the left side of inequality (1.21) is better than inequality (1.12) for $0<x<1.8157k$ and $k>0$. Also the upper bound in (1.21) is better than inequality (1.12) for $x>6.58818k$.
Motivated by above results and classical developments, our aim in this paper is to give some complete monotonicity of the functions related to k-trigamma and k-tetragamma functions and then to establish double-sided inequality for the function $[\psi'_k(x)]^2+\dfrac{1}{k}\psi''_k(x)$.

2. Main Results

Now, we give our main results.

Theorem 2.1. *The functions*

$$P(x)=\left[\psi'_k(x)\right]^2+\frac{1}{k}\psi''_k(x)-\frac{x^2+3kx+3k^2}{3x^4(2x+k)^2} \tag{2.1}$$

and

$$Q(x)=\frac{625x^2+2275kx+5043k^2}{3x^4(50x+41k)^2}-\left[\psi'_k(x)\right]^2-\frac{1}{k}\psi''_k(x) \tag{2.2}$$

are completely monotonic and the following inequalities

$$\frac{x^2+3kx+3k^2}{3x^4(2x+k)^2}<\left[\psi'_k(x)\right]^2+\frac{1}{k}\psi''_k(x)<\frac{625x^2+2275kx+5043k^2}{3x^4(50x+41k)^2} \tag{2.3}$$

hold for all x, $k>0$.

Proof. By using the recurrence formulas (1.7) and (1.8), we get

$$\begin{aligned}
P(x) \quad - \quad & P(x+k) = \left[\psi_k'(x) - \psi_k'(x+k)\right]\left[\psi_k'(x) + \psi_k'(x+k)\right] + \frac{1}{k}\left[\psi_k''(x) - \psi_k''(x+k)\right] \\
& -\frac{x^2+3kx+3k^2}{3x^4(2x+k)^2} + \frac{(x+k)^2+3k(x+k)+3k^2}{3(x+k)^4+(2x+3k)^2} \\
= \quad & \frac{1}{x^2}\left[2\psi_k'(x) - \frac{1}{x^2}\right] - \frac{2}{kx^3} - \left[\frac{x^2+3kx+3k^2}{3x^4(2x+k)^2} - \frac{(x+k)^2+3k(x+k)+3k^2}{3(x+k)^4+(2x+3k)^2}\right] \\
= \quad & \frac{2}{x^2}\left[\psi_k'(x) - \frac{1}{kx} - \frac{1}{2x^2} - \frac{x^2}{2}\left(\frac{x^2+3kx+3k^2}{3x^4(2x+k)^2} - \frac{(x+k)^2+3k(x+k)+3k^2}{3(x+k)^4+(2x+3k)^2}\right)\right] \\
= \quad & \frac{2}{x^2}F(x)
\end{aligned}$$

where

$$\begin{aligned}
F(x) \quad = \quad & \psi_k'(x) + \frac{1}{2kx} - \frac{1}{x^2} - \frac{41}{2k(x+k)} + \frac{23}{3(x+k)^2} - \frac{5k}{2(x+k)^3} + \frac{k^2}{2(k+x)^4} \\
& - \frac{3}{2k(x+k/2)} - \frac{7}{24(x+k/2)^2} + \frac{41}{2k(x+3k/2)} + \frac{21}{8(x+3k/2)^2}
\end{aligned}$$

for all x, $k > 0$. By using the equation (1.16) and integral representation of k-trigamma function (1.18), we get

$$F(x) \quad = \quad \int_0^\infty \frac{f(t)}{24k(e^{kt}-1)} e^{-(x+3k/2)t} dt$$

where

$$\begin{aligned}
f(t) \quad = \quad & 12e^{5kt/2} - (7kt+36)e^{2kt} + 2(k^3t^3 - 15k^2t^2 + 104kt - 252)e^{3kt/2} + (70kt+528)e^{kt} \\
& -2(k^3t^3 - 15k^2t^2 - 92kt - 246)e^{kt/2} - 63kt - 492.
\end{aligned}$$

Straightforward differentiating leads us to

$$\begin{aligned}
f'(t) \quad = \quad & k\Big[30e^{5kt/2} - (14kt+79)e^{2kt} + (3k^3t^3 - 39k^2t^2 + 252kt - 548)e^{3kt/2} + (70kt+598)e^{kt} \\
& -(k^3t^3 - 9k^2t^2 + 32kt - 62)e^{kt/2} - 63\Big], \\
f''(t) \quad = \quad & \frac{k^2}{2}e^{kt/2}\Big[150e^{2kt} - 8(7kt+43)e^{3kt/2} + 3(3k^3t^3 - 33k^2t^2 + 200kt - 380)e^{kt} \\
& +4(35kt+334)e^{kt/2} - k^3t^3 + 3k^2t^2 + 4kt - 2\Big] = \frac{k^2}{2}e^{kt/2}f_1(t), \\
f_1'(t) \quad = \quad & k\Big[300e^{2kt} - 4(21kt+143)e^{3kt/2} + (9k^3t^3 - 72k^2t^2 + 402kt - 540)e^{kt} \\
& +(70kt+808)e^{kt/2} - 3k^2t^2 + 6kt + 4\Big], \\
f_1''(t) \quad = \quad & k^2\Big[600e^{2kt} - 6(21kt+157)e^{3kt/2} + 3(3k^3t^3 - 15k^2t^2 + 86kt - 46)e^{kt} \\
& +(35kt+474)e^{kt/2} - 6kt + 6\Big],
\end{aligned}$$

$$\begin{aligned}
f_1^{(3)}(t) &= \frac{k^3}{2}\Big[2400e^{2kt} - 54(7kt+57)e^{3kt/2} + 6(3k^3t^3 - 6k^2t^2 + 56kt + 40)e^{kt} \\
&\quad +(35kt+544)e^{kt/2} - 12\Big], \\
f_1^{(4)}(t) &= \frac{k^4}{4}e^{kt/2}\Big[9600e^{3kt/2} - 54(21kt+185)e^{kt} + 12(3k^3t^3 + 3k^2t^2 + 44kt + 96)e^{kt/2} \\
&\quad +35kt + 614] = \frac{k^4}{4}e^{kt/2}f_2(t), \\
f_2'(t) &= k\Big[14400e^{3kt/2} - 54(21kt+206)e^{kt} + 6(3k^3t^3 + 21k^2t^2 + 56kt + 184)e^{kt/2} + 35\Big], \\
f_2''(t) &= 3k^2e^{kt/2}\Big[7200e^{kt} - 18(21kt+227)e^{kt/2} + 3k^3t^3 + 39k^2t^2 + 140kt + 296\Big] \\
&= 3k^2e^{kt/2}f_3(t), \\
f_3'(t) &= k\Big[7200e^{kt} - 9(21kt+269)e^{kt/2} + 9k^2t^2 + 78kt + 140\Big], \\
f_3''(t) &= \frac{3k^2}{2}\Big[4800e^{kt} - 3(21kt+311)e^{kt/2} + 12kt + 52\Big], \\
f_3^{(3)} &= \frac{9k^3}{4}\Big[3200e^{kt} - (21kt+353)e^{kt/2} + 8\Big], \\
f_3^{(4)} &= \frac{9k^4}{8}e^{kt/2}\Big[6400e^{kt/2} - 21kt - 395\Big] = \frac{9k^4}{8}e^{kt/2}f_4(t), \\
f_4'(t) &= k\Big[3200e^{kt/2} - 21\Big]
\end{aligned}$$

and

$$f_4''(t) = 1600k^2e^{kt/2}.$$

By the aid of the these results, it is easy to conclude that the functions $f_i^{(n)}$ are positive and non-decreasing for $0 \le i \le 4$ and $n \in \mathbb{Z}^+$. Therefore the function α' defined by $\alpha'(t) = \dfrac{f(t)e^{\frac{-3kt}{2}}}{24k(e^{kt}-1)}$ is positive. It means that the function α is non-decreasing. Hence due to the Bernstein-Widder theorem 1.1, we get that the function F is completely monotonic for all positive real values of x and k. Since two functions $2/x^2$ and $F(x)$ are completely monotonic and the product of two completely monotonic functions is also completely monotonic, then the function $P(x)$ is completely monotonic. Furthermore we have that the function $P(x) - P(x+k) > 0$, that is, the function P is decreasing and since $\lim_{x\to\infty} P(x) = 0$, the function $P(x)$ is positive. Hence we get the left side of the inequality (2.3).
For the second part, using the equations (1.7) and (1.8) leads us to

$$\begin{aligned}
Q(x) &- Q(x+k) = \left[\psi_k'(x+k) - \psi_k'(x)\right]\left[\psi_k'(x+k) + \psi_k'(x)\right] + \frac{1}{k}\left[\psi_k''(x+k) - \psi_k''(x)\right] \\
&\quad + \frac{625x^2 + 2275kx + 5043k^2}{3x^4(50x+41k)^2} - \frac{625(x+k)^2 + 2275k(x+k) + 5043k^2}{3(x+k)^4(50x+91k)^2} \\
&= -\frac{1}{x^2}\left[2\psi_k'(x) - \frac{1}{x^2}\right] + \frac{2}{kx^3} - \Bigg[\frac{625(x+k)^2 + 2275k(x+k) + 5043k^2}{3(x+k)^4(50x+91k)^2}
\end{aligned}$$

$$
\begin{aligned}
& \quad -\frac{625x^2+2275kx+5043k^2}{3x^4(50x+41k)^2}\Bigg] \\
&= \frac{2}{x^2}\left[\frac{x^2}{2}\left(\frac{625x^2+2275kx+5043k^2}{3x^4(50x+41k)^2}-\frac{625(x+k)^2+2275k(x+k)+5043k^2}{3(x+k)^4(50x+91k)^2}\right)\right. \\
& \quad \left.+\frac{1}{kx}+\frac{1}{2x^2}-\psi_k'(x)\right] \\
&= \frac{2}{x^2}G(x)
\end{aligned}
$$

where

$$
\begin{aligned}
G(x) &= \frac{61}{10086kx}+\frac{1}{x^2}+\frac{122943275}{16954566k(x+k)}-\frac{291573}{68921(x+k)^2}+\frac{20111k}{10086(x+k)^3}-\frac{k^2}{2(x+k)^4} \\
& \quad +\frac{10025}{10086k(x+41k/50)}+\frac{117}{328(x+41k/50)^2}-\frac{122943275}{16954566k(x+91k/50)} \\
& \quad -\frac{968877}{551368(x+91k/50)^2}-\psi_k'(x).
\end{aligned}
$$

By using the equation (1.16) and integral representation of k-trigamma function (1.18), we obtain

$$
G(x) = \int_0^\infty \frac{e^{-91kt/50}}{67818264k(e^{kt}-1)}\,g(t)\,e^{-xt}dt
$$

where

$$
\begin{aligned}
g(t) &= 410164e^{141kt/50}+(24191271kt+67408100)e^{2kt} \\
& \quad -(5651522k^3t^3-67613182k^2t^2+354726096kt-491362936)e^{91kt/50} \\
& \quad -(143363142kt+559181200)e^{kt} \\
& \quad +(5651522k^3t^3-67613182k^2t^2+286907832kt-491773100)e^{41kt/50} \\
& \quad +119171871kt+491773100.
\end{aligned}
$$

Similarly, differentiating the function g yields that

$$
\begin{aligned}
& g'(t) = \frac{k}{25}\Big[28916562e^{141kt/50}+42025(28782kt+94591)e^{2kt} \\
& -(257144251k^3t^3-2652535631k^2t^2+12759378268kt-13488861188)e^{91kt)/50} \\
& +41(2825761k^3t^3-23468441k^2t^2+60998816kt-70942750)e^{41kt/50} \\
& -50(71681571kt-207909029)e^{kt}+2979296775\Big], \\
& g''(t) = \frac{ke^{41kt/50}}{50}[4077235242e^{2kt}+8405000(14391kt+54491)e^{59kt/50} \\
& -(23400126841k^3t^3-202809104771k^2t^2+895849859288kt-589517454708)e^{kt} \\
& -(179203927500kt-340568645000)e^{9kt/50}+4750104241k^3t^3-22072019171k^2t^2 \\
& +6318401596kt+5792810050] = \frac{ke^{41kt/50}}{50}g_1(t),
\end{aligned}
$$

$$g_1'(t) = k[8154470484e^{2kt} + 168100(849069kt + 3934519)e^{59kt/50}$$
$$-(23400126841k^3t^3 - 132608724248k^2t^2 + 490231649746kt + 306332404580)e^{kt}$$
$$-450(71681571kt + 262003492)e^{9kt/50} + 2825761(5043k^2t^2 - 15622kt + 2236)],$$
$$g_1''(t) = k^2[16308940968e^{2kt} + 198358(849069kt + 4654069)e^{59kt/50}$$
$$-(23400126841k^3t^3 - 62408343725k^2t^2 + 225014201250kt + 796564054326)e^{kt}$$
$$-81(71681571kt + 660234442)e^{9kt/50} + 5651522(5043kt - 7811)],$$
$$g_1^{(3)}(t) = k^3[32617881936e^{2kt} + \frac{5851561}{25}(849069kt + 5373619)e^{59kt/50}$$
$$-(23400126841k^3t^3 + 7792036798k^2t^2 + 100197513800kt + 1021578255576)e^{kt}$$
$$-\frac{729}{50}(71681571kt + 1058465392)e^{9kt/50} + 28500625446],$$
$$g_1^{(4)}(t) = \frac{k^4e^{9kt/50}}{1250}[81544704840000e^{91kt/50} + 345242099(849069kt + 6093169)e^{kt}$$
$$-1250(23400126841k^3t^3 + 77992417321k^2t^2 + 115781587396kt + 1121775769376)e^{41kt/50}$$
$$-470302787331/2kt - 4778692349931] = \frac{k^4e^{9kt/50}}{1250}g_2(t),$$
$$g_2'(t) = \frac{41k}{2}[7239578673600e^{91kt/50} + 16841078(849069kt + 6942238)e^{kt} - 11470799691$$
$$-50(23400126841k^3t^3 + 163602637471k^2t^2 + 306006995496kt + 1262972827176)e^{41kt/50}],$$
$$g_2''(t) = \frac{1681k^2e^{41kt/50}}{2}[321366663072e^{kt} + 410758(849069kt + 7791307)e^{9kt/50}$$
$$-1681(13920361k^3t^3 + 148252741k^2t^2 + 419415716kt + 973320696)] = \frac{1681k^2e^{41kt/50}}{2}g_3(t),$$
$$g_3'(t) = k[321366663072e^{kt} + \frac{1848411}{25}(849069kt + 12508357)e^{9kt/50}$$
$$-1681(41761083k^2t^2 + 296505482kt + 419415716)],$$
$$g_3''(t) = k^2[321366663072e^{kt} + \frac{16635699}{1250}(849069kt + 17225407)e^{9kt/50}$$
$$-305942(458913kt + 1629151)],$$
$$g_3^{(3)}(t) = k^3[321366663072e^{kt} + \frac{149721291}{62500}(849069kt + 21942457)e^{9kt/50} - 140400761046]$$
and
$$g_3^{(4)}(t) = \frac{9k^4e^{9kt/50}}{3125000}[111585646900000000e^{41kt/50} + 127123706828079kt + 3991495805463537].$$
Hence one can get that the function g is positive and increasing for all $x, k > 0$, which implies that the function α is non-decreasing. So the function G is completely monotonic for all real values of x and k according to Bernstein-Widder theorem 1.1. Moreover we have that since the function g is positive, the functions G and Q are also positive. Thus we get the right hand side of the inequality (2.3). □

Remark 2.1. The inequality (2.3) is a refinement of the inequality (1.12) for all positive real values of x and k. Also the lower bound of the inequality (2.3) is somewhat better than the lower bound of the inequality (1.21) for $k > 0$ and $x > \frac{\left(\sqrt{849}+9\right)k}{32}$ and the upper bound of inequality (2.3) is better than the upper bound of the inequality (1.21) for all positive real values of x and k. The inequality (2.3) is also a k-generalization of the inequality obtained by Anis et.al. in [2, eq. (15)].

Since the function F and G in the proof of Theorem 2.1 are positive, as an immediate consequence, we get the following result:

Corollary 2.1. *The following double-sided inequality*

$$\frac{54k^9+477k^8x+1977k^7x^2+4962k^6x^3+8157k^5x^4+8968k^4x^5+6536k^3x^6+3040k^2x^7+816kx^8+96x^9}{6kx^2(k+x)^4(k+2x)^2(3k+2x)^2} < \psi_k'(x)$$
$$< \frac{\begin{array}{c}83522166k^9+630089005k^8x+2205760185k^7x^2+4676259010k^6x^3-3824275k^5x^5+6534723072k^5x^4-9327500k^4x^6\\ +6165903591k^4x^5-5687500k^3x^7+3894815200k^3x^6+1576227500k^2x^7+366750000kx^8+37500000x^9\end{array}}{6kx^2(k+x)^4(41k+50x)^2(91k+50x)^2} \tag{2.4}$$

is valid for all positive real values of x and k.

Remark 2.2. When investigating the behavior of the k-trigamma function in the neighborhood of $x = 0$, the inequality (2.4) is more advantageous than the inequality (1.10). Because the inequality

$$\frac{k}{6x^3} - \frac{k^3}{30x^5} < \frac{x^2}{2}\left[\frac{x^2++3kx+3k^2}{3x^4(2x+k)^2} - \frac{(x+k)^2++3k(x+k)+3k^2}{3(x+k)^4(2x+3k)^2}\right]$$

is valid for $0 < k$ and $0 < x \lessapprox 0.821017k$. So the lower bound of the inequality (2.4) is somewhat better than the lower bound of the inequality (1.10) at these intervals. Also the inequality

$$\frac{x^2}{2}\left[\frac{625x^2+2275kx+5043k^2}{3x^4(50x+41k)^2} - \frac{625(x+k)^2+2275k(x+k)+5043k^2}{3(x+k)^4(50x+91k)^2}\right] < \frac{k}{6x^3}$$

holds for $0 < k$ and $0 < x \lessapprox 1.54387k$. Therefore the upper bound of the inequality (2.4) is more useful than the upper bound of the inequality (1.10) for these intervals.

Acknowledgements. We would like to express our sincere gratitude to anonymous referees whose valuable suggestions improve the quality of this paper significantly.

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