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ON THE HERMITE-HADAMARD INEQUALITY VIA GENERALIZED INTEGRALS

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ABSTRACT. In this paper, we use the k-generalized fractional Riemann-Liouville integral of order α to obtain new integral inequalities of the Hermite-Hadamard type, in the class of $P-$functions.

1. INTRODUCTION

Recently, the theory of convexity has had the attention of different researchers due to its numerous applications in different fields of pure and applied sciences.

A important inequality, due to Hermite and Hadamard is widely studied in the literature, this inequality is known as Hermite-Hadamard inequality for convex functions [16]; which establishes the following: let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $\zeta_1, \zeta_2 \in I$ with $\zeta_1 < \zeta_2$, so the following inequality

$$\phi(\frac{\zeta_1+\zeta_2}{2}) \leq \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\phi(u)du \leq \frac{\phi(\zeta_1)+\phi(\zeta_2)}{2}$$

holds.

Definition 1.1. We say that $\phi : I \to \mathbb{R}$ is a $P-$function, or that ϕ belongs to the class $P(I)$, if ϕ is a non-negative function and for all $u_1, u_2 \in I, \delta \in [0,1]$, we have $\phi(\delta u_1 + (1-\delta)u_2) \leq \phi(u_1) + \phi(u_2)$.

The fractional calculus is a generalization of classical calculus involved with derivatives and integrals of non-integer order. It's first appearance dates back to the correspondence of G. W. Leibniz and L'Hospital in 1695. Since the 19th century, the theory of fractional calculus has had a rapid growth, proof of this is that today we can find applications of fractional calculus in many areas and disciplines. These new applications have given rise to

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new operators which are natural generalizations of the classical Riemann-Liouville fractional integral. The authors (see [15]) give a generalized operator containing as particular cases, several of the those reported in the literature.

Today, many classes of convex functions are defined in the literature, and many different generalizations and extensions of the Hadamard inequality have been obtained for these classes. In [20] authors summarize the main definitions of convex functions and the relationships between them.

For more information and some other extensions of the integral inequality, including fractional integral operators, some results can be found in many articles [1–5,7,8,10,12,13,22] and the references therein.

For example, Alomari et al. in [1] obtained some inequalities of Hermite-Hadamard type for functions whose second derivatives are quasi-convex. In [3, 4] the authors obtained generalized inequalities for some convex functions via fractional integral operators on sub-intervals of the integration interval. Du et al. in [11] the concept of generalized semi$-(m,h)-$preinvex functions was introduced and some estimates for the trapezoid inequality are obtained. By using $k-$fractional integrals But et al. in [7] for the quasi-convex and $s-$Godunova-Levin functions, new integral inequalities were obtained. Bessenyei and Páles in [6], generalized convex functions of high order are studied, as a result, extensions of the classical Hermite-Hadamard inequality are obtained.

Nàpoles et al. in [19] obtained new integral inequalities of the Hermite-Hadamard type for convex and quasi-convex functions in a generalized context. The study [18] authors, these inequalities for $h-$ convex functions establish via of a certain generalized integral. Xi and Qi in [25] obtained new integral inequalities for extended $s-$convex functions. In [23], Qi et al. introduced generalized $k-$fractional conformable integrals and generalized some integral inequalities through them.

Definition 1.2. ([14]) The $k-$generalized fractional Riemann-Liouville integral of order α with $\alpha \in \mathbb{R}$, and $s \neq -1$ of an integrable function $\phi(u)$ on $[0,\infty)$, are given as follows (right and left, respectively):

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,\zeta_1^+}\phi(u) = \frac{1}{k\ \Gamma_k(\alpha)}\int_{\zeta_1}^{u}\frac{F(\delta,s)\phi(\delta)}{[\ \mathbb{F}(u,\delta)]^{1-\frac{\alpha}{k}}}d\delta \tag{1.1} $$

and

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,\zeta_2^-}\phi(u) = \frac{1}{k\ \Gamma_k(\alpha)}\int_{u}^{\zeta_2}\frac{F(\delta,s)\phi(\delta)}{[\ \mathbb{F}(\delta,u)]^{1-\frac{\alpha}{k}}}d\delta \tag{1.2} $$

with $F(\delta,.) \in L[\zeta_1,\zeta_2]$, $F(.,s) \in C^1[\zeta_1,\zeta_2]$, $F(\delta,0)=1$, $\mathbb{F}(\delta,u) = \int_u^\delta F(\theta,s)d\theta$, $\mathbb{F}(u,\delta) = \int_\delta^u F(\theta,s)d\theta$, and the Euler gamma functions defined by (see [9, 24]):

$$ \Gamma(z) = \int_0^\infty \delta^{z-1}e^{-\delta}\,\mathrm{d}\delta, \quad \Re(z)>0, $$
$$ \Gamma_k(z) = \int_0^\infty \delta^{z-1}e^{-\delta^k/k}\,\mathrm{d}\delta, k>0. $$

It is known that $\Gamma_k(z+k) = z\Gamma_k(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\lim_{k\to 1}\Gamma_k(z) = \Gamma(z)$.

In the theory of integral inequalities, Hölder's inequality and its other form, the power mean inequality, are often used.

Theorem 1.1. *(Hölder inequality [17]) Let $q_1 > 1$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$. If $\phi(\zeta)$ and $g(\zeta)$ are real functions defined on $[\zeta_1, \zeta_2]$ and if $|\phi|^{p_1}, |g|^{q_1} \in L[\zeta_1, \zeta_2]$, then*

$$\int_{\zeta_1}^{\zeta_2} |\phi(\zeta)\, g(\zeta)|\, d\zeta \leq \left(\int_{\zeta_1}^{\zeta_2} |\phi(\zeta)|^{p_1}\, d\zeta\right)^{\frac{1}{p_1}} \left(\int_{\zeta_1}^{\zeta_2} |g(\zeta)|^{q_1}\, d\zeta\right)^{\frac{1}{q_1}} \tag{1.3}$$

with equality holds, if and only if $A|\phi(\zeta)|^{p_1} = B|g(\zeta)|^{q_1}$ almost everywhere, where A and B are constants.

Theorem 1.2. *(Power mean inequality [17]) Let $q_1 \geq 1$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$. If $\phi(\zeta)$ and $g(\zeta)$ are real functions defined on $[\zeta_1, \zeta_2]$ and if $|\phi|^{p_1}, |g|^{q_1} \in L[\zeta_1, \zeta_2]$, then*

$$\int_{\zeta_1}^{\zeta_2} |\phi(\zeta)\, g(\zeta)|\, d\zeta \leq \left(\int_{\zeta_1}^{\zeta_2} |\phi(\zeta)|\, d\zeta\right)^{1-\frac{1}{q_1}} \left(\int_{\zeta_1}^{\zeta_2} |\phi(\zeta)|\, |g(\zeta)|^{q_1}\, d\zeta\right)^{\frac{1}{q_1}}. \tag{1.4}$$

The aim of the study is to obtain new Hermite-Hadamard type integral inequalities in terms of generalized integral operators for functions of p-convex class.

2. Main Results

Let $\phi : I \to \mathbb{R}$ be a given function, where $\zeta_1, \zeta_2 \in I$ with $0 < \zeta_1 < \zeta_2 < \infty$. We assume that $\phi \in L_\infty[\zeta_1, \zeta_2]$ such that ${}^sJ^{\frac{\alpha}{k}}_{F,\zeta_1^+}\phi(u)$ and ${}^sJ^{\frac{\alpha}{k}}_{F,\zeta_2^-}\phi(u)$ are well defined. We define

$$\tilde{\phi}(u) := \phi(\zeta_1 + \zeta_2 - u) \text{ and } G(u) := \phi(u) + \tilde{\phi}(u),\ u \in [\zeta_1, \zeta_2].$$

The following lemma will be useful hereafter (see [14])

Lemma 2.1. *For $\alpha, k > 0$ and $s \neq -1$. If $\phi \in C^1(I)$ and $\phi' \in L[\zeta_1, \zeta_2]$, then we have*

$$\begin{aligned}
&\frac{\phi(\zeta_1) + \phi(\zeta_2)}{2} - \frac{\Gamma_k(\alpha + k)}{4[\mathbb{F}(\zeta_2, \zeta_1)]^{\frac{\alpha}{k}}}\left[{}^sJ^{\frac{\alpha}{k}}_{F,\zeta_1^+}G(\zeta_2) + {}^sJ^{\frac{\alpha}{k}}_{F,\zeta_2^-}G(\zeta_1)\right] \\
&= \frac{\zeta_2 - \zeta_1}{4[\mathbb{F}(\zeta_2, \zeta_1)]^{\frac{\alpha}{k}}} \int_0^1 \Delta_{\alpha,s}(\delta)\phi'(\delta\zeta_1 + (1-\delta)\zeta_2)d\delta,
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
\Delta_{\alpha,s}(\delta) \;=\; & [\mathbb{F}(\delta\zeta_1 + (1-\delta)\zeta_2, \zeta_1)]^{\frac{\alpha}{k}} - [\mathbb{F}(\delta\zeta_2 + (1-\delta)\zeta_1, \zeta_1)]^{\frac{\alpha}{k}} \\
& + [\mathbb{F}(\zeta_2, \delta\zeta_2 + (1-\delta)\zeta_1)]^{\frac{\alpha}{k}} - [\mathbb{F}(\zeta_2, \delta\zeta_1 + (1-\delta)\zeta_2)]^{\frac{\alpha}{k}}.
\end{aligned}$$

Proof. Let us use the method of integration by parts and calculate the integral on the right side of identity (2.1)

$$\begin{aligned}
&\int_0^1 \Delta_{\alpha,s}(\delta)\phi'(\delta\zeta_1+(1-\delta)\zeta_2)d\delta \\
&= \frac{\Delta_{\alpha,s}(\delta)}{\zeta_1-\zeta_2}\phi(\delta\zeta_1+(1-\delta)\zeta_2)\Big|_0^1 - \frac{1}{\zeta_1-\zeta_2}\int_0^1 \left(\Delta_{\alpha,s}(\delta)\right)'\phi(\delta\zeta_1+(1-\delta)\zeta_2)d\delta \\
&= \frac{1}{\zeta_1-\zeta_2}\left[-2[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}\phi(\zeta_1) - 2[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}\phi(\zeta_2)\right] \\
&\qquad - \frac{1}{\zeta_1-\zeta_2}\int_0^1 \left(\Delta_{\alpha,s}(\delta)\right)'\phi(\delta\zeta_1+(1-\delta)\zeta_2)d\delta \\
&= \frac{2[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}{\zeta_2-\zeta_1}\left[\phi(\zeta_1)+\phi(\zeta_2)\right] + \frac{1}{\zeta_2-\zeta_1}\int_0^1 \left(\Delta_{\alpha,s}(\delta)\right)'\phi(\delta\zeta_1+(1-\delta)\zeta_2)d\delta.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{\zeta_2-\zeta_1}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\int_0^1 \Delta_{\alpha,s}(\delta)\phi'(\delta\zeta_1+(1-\delta)\zeta_2)d\delta \qquad (2.2)\\
&= \frac{\phi(\zeta_1)+\phi(\zeta_2)}{2} + \frac{1}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\int_0^1 \left(\Delta_{\alpha,s}(\delta)\right)'\phi(\delta\zeta_1+(1-\delta)\zeta_2)d\delta.
\end{aligned}$$

Here

$$\begin{aligned}
\left(\Delta_{\alpha,s}(\delta)\right)' &= \frac{\alpha}{k}[\mathbb{F}(\delta\zeta_1+(1-\delta)\zeta_2,\zeta_1)]^{\frac{\alpha}{k}-1}\mathbb{F}'(\delta\zeta_1+(1-\delta)\zeta_2,\zeta_1)\left(\zeta_1-\zeta_2\right) \\
&\quad - \frac{\alpha}{k}[\mathbb{F}(\delta\zeta_2+(1-\delta)\zeta_1,\zeta_1)]^{\frac{\alpha}{k}-1}\mathbb{F}'(\delta\zeta_2+(1-\delta)\zeta_1,\zeta_1)\left(\zeta_2-\zeta_1\right) \\
&\quad + \frac{\alpha}{k}[\mathbb{F}(\zeta_2,\delta\zeta_2+(1-\delta)\zeta_1)]^{\frac{\alpha}{k}-1}\mathbb{F}'(\zeta_2,\delta\zeta_2+(1-\delta)\zeta_1)\left(\zeta_2-\zeta_1\right) \\
&\quad - \frac{\alpha}{k}[\mathbb{F}(\zeta_2,\delta\zeta_1+(1-\delta)\zeta_2)]^{\frac{\alpha}{k}-1}\mathbb{F}'(\zeta_2,\delta\zeta_1+(1-\delta)\zeta_2)\left(\zeta_1-\zeta_2\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{F}'(\delta\zeta_1+(1-\delta)\zeta_2,\zeta_1) &= \left(\int_{\zeta_1}^{\delta\zeta_1+(1-\delta)\zeta_2} F(\theta,s)d\theta\right)' = F(\delta\zeta_1+(1-\delta)\zeta_2,s), \\
\mathbb{F}'(\delta\zeta_2+(1-\delta)\zeta_1,\zeta_1) &= \left(\int_{\zeta_1}^{\delta\zeta_2+(1-\delta)\zeta_1} F(\theta,s)d\theta\right)' = F(\delta\zeta_2+(1-\delta)\zeta_1,s), \\
\mathbb{F}'(\zeta_2,\delta\zeta_2+(1-\delta)\zeta_1) &= -\left(\int_{\zeta_1}^{\delta\zeta_2+(1-\delta)\zeta_1} F(\theta,s)d\theta\right)' = -F(\delta\zeta_2+(1-\delta)\zeta_1,s), \\
\mathbb{F}'(\zeta_2,\delta\zeta_1+(1-\delta)\zeta_2) &= -\left(\int_{\zeta_1}^{\delta\zeta_1+(1-\delta)\zeta_2} F(\theta,s)d\theta\right)' = -F(\delta\zeta_1+(1-\delta)\zeta_2,s).
\end{aligned}$$

Finally, for the derivative, we have

$$
\begin{aligned}
(\Delta_{\alpha,s}(\delta))' = &-\frac{\alpha}{k}[\mathbb{F}(\delta\zeta_1+(1-\delta)\zeta_2,\zeta_1)]^{\frac{\alpha}{k}-1}F(\delta\zeta_1+(1-\delta)\zeta_2,s)\,(\zeta_2-\zeta_1)\\
&-\frac{\alpha}{k}[\mathbb{F}(\delta\zeta_2+(1-\delta)\zeta_1,\zeta_1)]^{\frac{\alpha}{k}-1}F(\delta\zeta_2+(1-\delta)\zeta_1,s)\,(\zeta_2-\zeta_1)\\
&-\frac{\alpha}{k}[\mathbb{F}(\zeta_2,\delta\zeta_2+(1-\delta)\zeta_1)]^{\frac{\alpha}{k}-1}F(\delta\zeta_2+(1-\delta)\zeta_1,s)\,(\zeta_2-\zeta_1)\\
&-\frac{\alpha}{k}[\mathbb{F}(\zeta_2,\delta\zeta_1+(1-\delta)\zeta_2)]^{\frac{\alpha}{k}-1}F(\delta\zeta_1+(1-\delta)\zeta_2,s)\,(\zeta_2-\zeta_1)
\end{aligned}
$$

Or

$$
\begin{aligned}
(\Delta_{\alpha,s}(\delta))' = &-\frac{\alpha\,(\zeta_2-\zeta_1)}{k}\\
&\times\left[\frac{F(\delta\zeta_1+(1-\delta)\zeta_2,s)}{[\mathbb{F}(\delta\zeta_1+(1-\delta)\zeta_2,\zeta_1)]^{1-\frac{\alpha}{k}}}+\frac{F(\delta\zeta_2+(1-\delta)\zeta_1,s)}{[\mathbb{F}(\delta\zeta_2+(1-\delta)\zeta_1,\zeta_1)]^{1-\frac{\alpha}{k}}}\right.\\
&\left.+\frac{F(\delta\zeta_2+(1-\delta)\zeta_1,s)}{[\mathbb{F}(\zeta_2,\delta\zeta_2+(1-\delta)\zeta_1)]^{1-\frac{\alpha}{k}}}+\frac{F(\delta\zeta_1+(1-\delta)\zeta_2,s)}{[\mathbb{F}(\zeta_2,\delta\zeta_1+(1-\delta)\zeta_2)]^{1-\frac{\alpha}{k}}}\right].
\end{aligned}
\tag{2.3}
$$

Thus, taking into account (2.3) for the integral on the right side of (2.2), we obtain:

$$
\begin{aligned}
&\frac{1}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\int_0^1(\Delta_{\alpha,s}(\delta))'\,\phi(\delta\zeta_1+(1-\delta)\zeta_2)d\delta\\
&=-\frac{\alpha\,(\zeta_2-\zeta_1)}{4k[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\int_0^1\left[\frac{F(\delta\zeta_1+(1-\delta)\zeta_2,s)}{[\mathbb{F}(\delta\zeta_1+(1-\delta)\zeta_2,\zeta_1)]^{1-\frac{\alpha}{k}}}+\frac{F(\delta\zeta_2+(1-\delta)\zeta_1,s)}{\mathbb{F}(\delta\zeta_2+(1-\delta)\zeta_1,\zeta_1)]^{1-\frac{\alpha}{k}}}\right.\\
&\left.+\frac{F(\delta\zeta_2+(1-\delta)\zeta_1,s)}{[\mathbb{F}(\zeta_2,\delta\zeta_2+(1-\delta)\zeta_1)]^{1-\frac{\alpha}{k}}}+\frac{F(\delta\zeta_1+(1-\delta)\zeta_2,s)}{[\mathbb{F}(\zeta_2,\delta\zeta_1+(1-\delta)\zeta_2)]^{1-\frac{\alpha}{k}}}\right]\phi(\delta\zeta_1+(1-\delta)\zeta_2)d\delta.
\end{aligned}
$$

After the change of variables:

$\delta\zeta_1+(1-\delta)\zeta_2=u$ and $\delta\zeta_2+(1-\delta)\zeta_1=v$ and given that $\delta\zeta_1+(1-\delta)\zeta_2=\zeta_2+\zeta_1-v$, we get

$$\int_0^1 \frac{F(\delta\zeta_1+(1-\delta)\zeta_2,s)\phi(\delta\zeta_1+(1-\delta)\zeta_2)}{[\mathbb{F}(\delta\zeta_1+(1-\delta)\zeta_2,\zeta_1)]^{1-\frac{\alpha}{k}}}d\delta = \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} \frac{F(u,s)\phi(u)}{[\mathbb{F}(u,\zeta_1)]^{1-\frac{\alpha}{k}}}du,$$

$$\int_0^1 \frac{F(\delta\zeta_2+(1-\delta)\zeta_1,s)\phi(\delta\zeta_1+(1-\delta)\zeta_2)}{[\mathbb{F}(\delta\zeta_2+(1-\delta)\zeta_1,\zeta_1)]^{1-\frac{\alpha}{k}}}d\delta = \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} \frac{F(v,s)\phi(\zeta_2+\zeta_1-v)}{[\mathbb{F}(v,\zeta_1)]^{1-\frac{\alpha}{k}}}dv,$$

$$\int_0^1 \frac{F(\delta\zeta_2+(1-\delta)\zeta_1,s)\phi(\delta\zeta_1+(1-\delta)\zeta_2)d\delta}{[\mathbb{F}(\zeta_2,\delta\zeta_2+(1-\delta)\zeta_1)]^{1-\frac{\alpha}{k}}} = \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} \frac{F(v,s)\phi(\zeta_2+\zeta_1-v)}{[\mathbb{F}(\zeta_2,v)]^{1-\frac{\alpha}{k}}}dv,$$

$$\int_0^1 \frac{F(\delta\zeta_1+(1-\delta)\zeta_2,s)\phi(\delta\zeta_1+(1-\delta)\zeta_2)}{[\mathbb{F}(\zeta_2,\delta\zeta_1+(1-\delta)\zeta_2)]^{1-\frac{\alpha}{k}}}d\delta = \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} \frac{F(u,s)\phi(u)}{[\mathbb{F}(\zeta_2,u)]^{1-\frac{\alpha}{k}}}du.$$

Thus, taking into account the last equalities, we have

$$\frac{1}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\int_0^1 \left(\Delta_{\alpha,s}(\delta)\right)' \phi(\delta\zeta_1+(1-\delta)\zeta_2)d\delta$$

$$= -\frac{\alpha}{4k[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[\int_{\zeta_1}^{\zeta_2} \frac{F(u,s)\left[\phi(u)+\phi(\zeta_2+\zeta_1-u)\right]}{[\mathbb{F}(u,\zeta_1)]^{1-\frac{\alpha}{k}}}du \right.$$

$$\left. + \int_{\zeta_1}^{\zeta_2} \frac{F(v,s)\left[\phi(\zeta_2+\zeta_1-v)+\phi(v)\right]}{[\mathbb{F}(\zeta_2,v)]^{1-\frac{\alpha}{k}}}dv\right]$$

$$= -\frac{\alpha}{4k[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[\int_{\zeta_1}^{\zeta_2} \frac{F(u,s)G(u)}{\mathbb{F}(u,\zeta_1)]^{1-\frac{\alpha}{k}}}du + \int_{\zeta_1}^{\zeta_2} \frac{F(v,s)G(v)}{[\mathbb{F}(\zeta_2,v)]^{1-\frac{\alpha}{k}}}dv\right]$$

$$= -\frac{\alpha\Gamma_k(\alpha)}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[\, {}^sJ^{\frac{\alpha}{k}}_{F,\zeta_2^-}G(\zeta_1) + {}^sJ^{\frac{\alpha}{k}}_{F,\zeta_1^+}G(\zeta_2)\right].$$

The proof is complete. □

Now, for $\alpha, k > 0$, $s \neq -1$ and $v \in [\zeta_1, \zeta_2]$, we introduce the following operator:

$$\rho(s,v) := \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} \left|\mathbb{F}(v,w)\right|^{\frac{\alpha}{k}} dw - \int_{\frac{\zeta_1+\zeta_2}{2}}^{\zeta_2} \left|\mathbb{F}(v,w)\right|^{\frac{\alpha}{k}} dw.$$

By using Lemma 2.1, we can get the following result

Theorem 2.1. *For $\alpha, k > 0$ and $s \neq -1$. If $\phi \in C^1(\zeta_1, \zeta_2)$ such that $\phi' \in L[\zeta_1, \zeta_2]$ and $|\phi'| \in P([\zeta_1, \zeta_2])$, then*

$$\begin{aligned} &\left|\frac{\phi(\zeta_1)+\phi(\zeta_2)}{2} - \frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[{}^{s}J^{\frac{\alpha}{k}}_{F,\zeta_1^+}G(\zeta_2) + {}^{s}J^{\frac{\alpha}{k}}_{F,\zeta_2^-}G(\zeta_1)\right]\right| \\ &\leq \frac{(|\phi'(\zeta_1)|+|\phi'(\zeta_2)|)(\rho(s,\zeta_2)-\rho(s,\zeta_1))}{2[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}. \end{aligned} \tag{2.4}$$

Proof. By using the Lemma 2.1 and since $|\phi'|$ is $P-$function, we have

$$\begin{aligned} &\left|\frac{\phi(\zeta_1)+\phi(\zeta_2)}{2} - \frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[{}^{s}J^{\frac{\alpha}{k}}_{F,\zeta_1^+}G(\zeta_2) + {}^{s}J^{\frac{\alpha}{k}}_{F,\zeta_2^-}G(\zeta_1)\right]\right| \\ &\leq \frac{\zeta_2-\zeta_1}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left(|\phi'(\zeta_1)|+|\phi'(\zeta_2)|\right)\int_0^1 |\Delta_{\alpha,s}(\delta)|d\delta. \end{aligned} \tag{2.5}$$

Observe that

$$\int_0^1 |\Delta_{\alpha,s}(\delta)|d\delta = \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} |h(w)|dw,$$

where

$$h(w) = [\mathbb{F}(w,\zeta_1)]^{\frac{\alpha}{k}} - [\mathbb{F}(\zeta_2+\zeta_1-w,\zeta_1)]^{\frac{\alpha}{k}} + [\mathbb{F}(\zeta_2,\zeta_2+\zeta_1-w)]^{\frac{\alpha}{k}} - [\mathbb{F}(\zeta_2,w)]^{\frac{\alpha}{k}}.$$

Note that h is non-decreasing function on $[\zeta_1, \zeta_2]$. Moreover, we have $h(\zeta_1) < 0$ and $h(\frac{\zeta_1+\zeta_2}{2}) = 0$. Thus, we obtain

$$\begin{cases} \text{if } \ w \in \left[\zeta_1, \frac{\zeta_1+\zeta_2}{2}\right], \ \textit{then } h(w) \leq 0 \\ \text{if } \ w \in \left(\frac{\zeta_1+\zeta_2}{2}, \zeta_2\right], \ \textit{then } h(w) > 0. \end{cases}$$

Thus, we get $\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}|h(w)|dw = \frac{1}{\zeta_2-\zeta_1}(\lambda_1+\lambda_2+\lambda_3+\lambda_4)$, where

$$\lambda_1 = \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} [\mathbb{F}(\zeta_2, w)]^{\frac{\alpha}{k}} dw - \int_{\frac{\zeta_1+\zeta_2}{2}}^{\zeta_2} [\mathbb{F}(\zeta_2, w)]^{\frac{\alpha}{k}} dw,$$

$$\lambda_2 = -\int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} [\mathbb{F}(w, \zeta_1)]^{\frac{\alpha}{k}} dw + \int_{\frac{\zeta_1+\zeta_2}{2}}^{\zeta_2} [\mathbb{F}(w, \zeta_1)]^{\frac{\alpha}{k}} dw,$$

$$\lambda_3 = \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} [\mathbb{F}(\zeta_2+\zeta_1-w, \zeta_1)]^{\frac{\alpha}{k}} dw - \int_{\frac{\zeta_1+\zeta_2}{2}}^{\zeta_2} [\mathbb{F}(\zeta_2+\zeta_1-w, \zeta_1)]^{\frac{\alpha}{k}} dw,$$

$$\lambda_4 = -\int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} [\mathbb{F}(\zeta_2, \zeta_2+\zeta_1-w)]^{\frac{\alpha}{k}} dw + \int_{\frac{\zeta_1+\zeta_2}{2}}^{\zeta_2} [\mathbb{F}(\zeta_2, \zeta_2+\zeta_1-w)]^{\frac{\alpha}{k}} dw.$$

We note that $\lambda_1 = \rho(s,\zeta_2)$ and $\lambda_2 = -\rho(s,\zeta_1)$, and by using the change of variable $r = \zeta_2+\zeta_1-w$, we obtain that $\lambda_3 = -\rho(s,\zeta_1)$ and $\lambda_4 = \rho(s,\zeta_2)$. Therefore, we get

$$\int_0^1 |\Delta_{\alpha,s}(\delta)|d\delta = \frac{2(\rho(s,\zeta_2)-\rho(s,\zeta_1))}{\zeta_2-\zeta_1}. \tag{2.6}$$

Therefore, the inequality (2.4) follows from (2.5) and (2.6). □

Remark 2.1. In the Theorem 2.1 if we consider $F \equiv 1$, this result becomes Theorem 2.1 of [21]. If we additionally make $k = \alpha = 1$, Corollary 2.2 of said work follows.

If we impose additional conditions on the derivative of the function ϕ, we can obtain new refinements of the previous result, as the following theorems show.

Theorem 2.2. *For $\alpha, k > 0$ and $s \neq -1$. If $\phi \in C^1(\zeta_1,\zeta_2)$ such that $\phi' \in L[\zeta_1,\zeta_2]$ and $|\phi'|^{q_1} \in P([\zeta_1,\zeta_2])$, then we have*

$$\begin{aligned} &\left|\frac{\phi(\zeta_1)+\phi(\zeta_2)}{2} - \frac{\Gamma_k(\alpha+k)}{4[\,\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[\,{}^sJ^{\frac{\alpha}{k}}_{F,\zeta_1^+}G(\zeta_2) + {}^sJ^{\frac{\alpha}{k}}_{F,\zeta_2^-}G(\zeta_1)\right]\right| \\ &\leq \frac{(|\phi'(\zeta_1)|^{q_1}+|\phi'(\zeta_2)|^{q_1})^{\frac{1}{q_1}}(\rho(s,\zeta_2)-\rho(s,\zeta_1))}{2[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}, \end{aligned} \tag{2.7}$$

where $q_1 > 1$ with $\frac{1}{p_1}+\frac{1}{q_1} = 1$.

Proof. From Lemma 2.1 by using the power mean inequality (1.4) and the p convexity of $|\phi'|^q$, we get

$$\begin{aligned}
&\left|\frac{\phi(\zeta_1)+\phi(\zeta_2)}{2}-\frac{\Gamma_k(\alpha+k)}{4[\,\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[\,{}^{s}J^{\frac{\alpha}{k}}_{F,\zeta_1^+}G(\zeta_2)+\,{}^{s}J^{\frac{\alpha}{k}}_{F,\zeta_2^-}G(\zeta_1)\right]\right| \\
&\le \frac{(\zeta_2-\zeta_1)}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\int_0^1|\Delta_{\alpha,s}(\delta)||\phi'(\delta\zeta_1+(1-\delta)\zeta_2)|d\delta \\
&\le \frac{(\zeta_2-\zeta_1)}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left(\int_0^1|\Delta_{\alpha,s}(\delta)|d\delta\right)^{\frac{1}{p_1}}\left(\int_0^1|\Delta_{\alpha,s}(\delta)||\phi'(\delta\zeta_1+(1-\delta)\zeta_2)|^{q_1}d\delta\right)^{\frac{1}{q_1}} \\
&\le \frac{(\zeta_2-\zeta_1)}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left(\frac{2^{\frac{1}{p_1}}\left(\rho(s,\zeta_2)-\rho(s,\zeta_1)\right)^{\frac{1}{p_1}}}{(\zeta_2-\zeta_1)^{\frac{1}{p_1}}}\right) \\
&\qquad \left(\frac{2\left(|\phi'(\zeta_1)|^{q_1}+|\phi'(\zeta_2)|^{q_1}\right)}{\zeta_2-\zeta_1}(\rho(s,\zeta_2)-\rho(s,\zeta_1))\right)^{\frac{1}{q_1}} \\
&= \frac{\left(|\phi'(\zeta_1)|^{q_1}+|\phi'(\zeta_2)|^{q_1}\right)^{\frac{1}{q_1}}(\rho(s,\zeta_2)-\rho(s,\zeta_1))}{2[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}.
\end{aligned} \tag{2.8}$$

□

Remark 2.2. In the Theorem 2.2 if taking $F\equiv 1$, we obtain the Theorem 2.3 of [21].

Lemma 2.2. *Let $a_i\in\mathbb{R}$, where $i=1,2,3,\dots m$ and $m\in\mathbb{N}$, then*

$$\left(\left|\sum_{i=1}^m a_i\right|\right)^n \le m^n\sum_{i=1}^m|a_i|^n .$$

Proof. Notice that

$$\forall i,\ |a_i|\le\left(\sum_{i=1}^m|a_i|^n\right)^{\frac{1}{n}}\quad\text{thus}\quad\left|\sum_{i=1}^m a_i\right|\le\sum_{i=1}^m|a_i|\le m\left(\sum_{i=1}^m|a_i|^n\right)^{\frac{1}{n}}.$$

□

Theorem 2.3. *For $\alpha,k>0$ and $s\neq -1$. If $\phi\in C^1(\zeta_1,\zeta_2)$ such that $\phi'\in L[\zeta_1,\zeta_2]$ and $|\phi'|^{q_1}\in P([\zeta_1,\zeta_2])$, then we have*

$$\begin{aligned}
&\left|\frac{\phi(\zeta_1)+\phi(\zeta_2)}{2}-\frac{\Gamma_k(\alpha+k)}{4[\,\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[\,{}^{s}J^{\frac{\alpha}{k}}_{F,\zeta_1^+}G(\zeta_2)+\,{}^{s}J^{\frac{\alpha}{k}}_{F,\zeta_2^-}G(\zeta_1)\right]\right| \\
&\le \frac{\left[|\phi'(\zeta_1)|^{q_1}+|\phi'(\zeta_2)|^{q_1}\right]^{\frac{1}{q_1}}(\zeta_2-\zeta_1)}{[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[\int_{\zeta_1}^{\zeta_2}g(w)dw\right]^{\frac{1}{p_1}},
\end{aligned} \tag{2.9}$$

where
$g(w) = |\mathbb{F}(w,\zeta_1)|^{\frac{\alpha p_1}{k}} + |\mathbb{F}(\zeta_2+\zeta_1-w,\zeta_1)|^{\frac{\alpha p_1}{k}} + |\mathbb{F}(\zeta_2,\zeta_2+\zeta_1-w)|^{\frac{\alpha p_1}{k}} + |\mathbb{F}(\zeta_2,w)|^{\frac{\alpha p_1}{k}}$
and $q_1 > 1$ *with* $\frac{1}{p_1}+\frac{1}{q_1}=1$.

Proof. From Lemma 2.1 by using the Hölder's inequality (1.3) and Lemma 2.2, and the p convexity of $|\phi'|^{q_1}$, we get

$$\begin{aligned}
&\left|\frac{\phi(\zeta_1)+\phi(\zeta_2)}{2} - \frac{\Gamma_k(\alpha+k)}{4[\ \mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[\ {}^sJ^{\frac{\alpha}{k}}_{F,\zeta_1^+}G(\zeta_2) + \ {}^sJ^{\frac{\alpha}{k}}_{F,\zeta_2^-}G(\zeta_1)\right]\right| \\
&\leq \frac{(\zeta_2-\zeta_1)}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\int_0^1 |\Delta_{\alpha,s}(\delta)||\phi'(\delta\zeta_1+(1-\delta)\zeta_2)|d\delta \\
&\leq \frac{(\zeta_2-\zeta_1)}{4[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left(\int_0^1 |\Delta_{\alpha,s}(\delta)|^{p_1}d\delta\right)^{\frac{1}{p_1}}\left(\int_0^1 |\phi'(\delta\zeta_1+(1-\delta)\zeta_2)|^{q_1}d\delta\right)^{\frac{1}{q_1}} \\
&\leq \frac{\left[|\phi'(\zeta_1)|^{q_1}+|\phi'(\zeta_2)|^{q_1}\right]^{\frac{1}{q_1}}(\zeta_2-\zeta_1)}{[\mathbb{F}(\zeta_2,\zeta_1)]^{\frac{\alpha}{k}}}\left[\int_{\zeta_1}^{\zeta_2} g(w)dw\right]^{\frac{1}{p_1}}.
\end{aligned} \tag{2.10}$$

□

Remark 2.3. In the Theorem 2.3 with $F \equiv 1$ the Theorem 2.4 of [21] is derived. If in addition, $k=\alpha=1$, we obtain the Corollary 2.5 of same paper.

3. Conclusions

In this paper various new Hermite-Hadamard inequalities have been obtained. The paper utilized generalized integral operators, to produce results in the class of p-convex functions. We hope that this work will be an impetus to obtain results for other classes of convex functions by using generalized integral operators.

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