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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR TWICE DIFFERENTIABLE FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS

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Abstract. In this paper we first obtain a new generalized identities for twice differentiable mappings involving newly defined generalized fractional integrals. Then by using this equality, we establish some Hermite-Hadamard type inequalities for functions whose second derivatives in absolute value are convex.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[13, 19], [32, p.137]). These inequalities state that if $\psi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$, then

$$\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\psi(\varkappa)d\varkappa \leq \frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if ψ is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The Hermite–Hadmard inequality, which is the first fundamental result for convex functions with natural geometric interpretation and multiple applications, has attracted much attention in elementary mathematics. Many mathematicians have devoted their efforts to generalizing, refining, counteracting, and using different classes of functions such as convex mapping.

The overall structure of the study takes the form of three sections including introduction. The remainder of this work is organized as follows: we first mention some works which

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focus on Hermite-Hadamard inequality. In Section 2, we introduce the generalized fractional integrals defined by Sarikaya and Ertuğral along with the very first results. In Section 3 we prove an identity for twice differentiable functions and using this equality we prove some trapezoid type inequalities for twice differentiable mappings.

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right hand side of the inequality (1.1). For some examples, please refer to ([3, 5, 9, 10, 14, 20, 29, 34–36]).

On the other hand, Sarikaya et al. obtain the Hermite-Hadamard inequality for the Riemann-Lioville fractional integrals in [40]. Whereupon Sarikaya et al. obtain the Hermite-Hadamard inequality for Riemann-Lioville fractional integrals, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality other fractional integrals such as k-fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, Conformable fractional integrals, etc. For some of them, please see ([4, 11, 16, 17, 21–27, 30, 31, 33, 37, 39, 41–48]). For more information about fractional calculus please refer to ([18, 28])

In this paper, we obtain the new generalized trapezoid type inequality for the generalized fractional integrals mentioned in next section.

2. Preliminaries

In this section we present the generalized fractional integrals which defined by Sarikaya and Ertuğral in [38].

Let's define a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions :

$$\int_0^1 \frac{\varphi(\xi)}{\xi} d\xi < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{\varkappa_1^+}I_\varphi \psi(x) = \int_{\varkappa_1}^{\varkappa} \frac{\varphi(x-\xi)}{x-\xi} \psi(\xi) d\xi, \quad \varkappa > \varkappa_1, \tag{2.1}$$

$$_{\varkappa_2^-}I_\varphi \psi(x) = \int_{\varkappa}^{\varkappa_2} \frac{\varphi(\xi - x)}{\xi - x} \psi(\xi) d\xi, \quad \varkappa_1 < \varkappa_2. \tag{2.2}$$

Remark 2.1. i) If we take $\varphi(\xi) = \xi$, the operator (2.1) and (2.2) reduce to the Riemann integral as follows:

$$I_{a^+}\psi(x) = \int_{\varkappa_1}^{\varkappa} \psi(\xi) d\xi, \quad \varkappa > \varkappa_1,$$

$$I_{b^-}\psi(x) = \int_{\varkappa}^{\varkappa_2} \psi(\xi) d\xi, \quad \varkappa_1 < \varkappa_2.$$

ii) If we take $\varphi(\xi) = \frac{\xi^\alpha}{\Gamma(\alpha)}$, the operator (2.1) and (2.2) reduce to the Riemann-Liouville fractional integral as follows:

$$I^\alpha_{\varkappa_1^+}\psi(x) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa_1}^{\varkappa} (x-\xi)^{\alpha-1} \psi(\xi) d\xi, \quad x > x_1,$$

$$I^\alpha_{x_2^-}\psi(x) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\varkappa_2} (\xi - x)^{\alpha-1} \psi(\xi) d\xi, \quad x < x_2.$$

iii) If we take $\varphi(\xi) = \frac{1}{k\Gamma_k(\alpha)}\xi^{\frac{\alpha}{k}}$, the operator (2.1) and (2.2) reduce to the k-Riemann-Liouville fractional integral as follows:

$$I^{\alpha}_{\varkappa_1^+,k}\psi(x) = \frac{1}{k\Gamma_k(\alpha)}\int_{\varkappa_1}^{\varkappa}(x-\xi)^{\frac{\alpha}{k}-1}\psi(\xi)d\xi, \quad x > x_1,$$

$$I^{\alpha}_{x_2^-,k}\psi(x) = \frac{1}{k\Gamma_k(\alpha)}\int_{\varkappa}^{\varkappa_2}(\xi-x)^{\frac{\alpha}{k}-1}\psi(\xi)d\xi, \quad x < x_2$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty}\xi^{\alpha-1}e^{-\frac{\xi^k}{k}}d\xi, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0$$

are given by Mubeen and Habibullah in [30].

Sarikaya and Ertuğral also give the following Hermite-Hadamard inequality for the generalized fractional integral operators:

Theorem 2.1. [38] *Let $\psi : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a convex function on $[\varkappa_1, \varkappa_2]$ with $\varkappa_1 < \varkappa_2$, then the following inequalities for fractional integral operators hold*

$$\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \le \frac{1}{2\Psi(1)}\left[_{\varkappa_1+}I_{\varphi}\psi(\varkappa_2) + _{\varkappa_2-}I_{\varphi}\psi(\varkappa_1)\right] \le \frac{\psi(\varkappa_1)+\psi(\varkappa_2)}{2} \tag{2.3}$$

where the mapping $\Psi : [0,1] \to \mathbb{R}$ is defined by

$$\Psi(\varkappa) = \int_0^{\varkappa}\frac{\varphi((\varkappa_2-\varkappa_1)\xi)}{\xi}d\xi.$$

For more recent results related to generalized fractional integral inequalities see, ([1, 2, 8, 38]).

3. Trapezoid Like Inequalities for newly defined Generalized Fractional Integral Operators

In this section, utilizing newly defined generalized fractional integrals, we establish some new trapezoid type inequalities for functions whose second derivatives in absolute value are convex.

Lemma 3.1. *Let $\psi : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be an absolutely continuous mapping on I° such that $\psi'' \in L([\varkappa_1, \varkappa_2])$, where $\varkappa_1, \varkappa_2 \in I^{\circ}$ with $\varkappa_1 < \varkappa_2$. Then the following equality for generalized fractional integrals holds:*

$$[(\varkappa-\varkappa_1)\Lambda_1(0) - (\varkappa_2-\varkappa)\Lambda_2(0)]\psi'(\varkappa)$$

$$+\Psi_1(1)\psi(\varkappa_1) + \Psi_2(1)\psi(\varkappa_2) - \left[_{\varkappa_1+}I_{\varphi}\psi(\varkappa) + _{\varkappa_2-}I_{\varphi}\psi(\varkappa)\right] \tag{3.1}$$

$$= \quad (\varkappa-\varkappa_1)^2 \int_0^1 \Lambda_1(\xi)\psi''\left(\xi\varkappa_1+(1-\xi)\varkappa\right)d\xi + (\varkappa_2-\varkappa)^2 \int_0^1 \Lambda_2(\xi)\psi''\left(\xi\varkappa_2+(1-\xi)\varkappa\right)d\xi$$

where

$$\begin{aligned}\Lambda_1(\xi) &= \int_\xi^1 \Psi_1(s)ds, \quad \Lambda_2(\xi)=\int_\xi^1 \Psi_2(s)ds,\\ \Psi_1(s) &= \int_0^s \frac{\varphi((x-a)u)}{u}du, \quad \Psi_2(s)=\int_0^s \frac{\varphi((b-x)u)}{u}du.\end{aligned}$$

Proof. First, we consider

$$\begin{aligned} I &= (\varkappa-\varkappa_1)^2 \int_0^1 \Lambda_1(\xi)\psi''\left(\xi\varkappa_1+(1-\xi)\varkappa\right)d\xi \\ &\quad + (\varkappa_2-\varkappa)^2 \int_0^1 \Lambda_2(\xi)\psi''\left(\xi\varkappa_2+(1-\xi)\varkappa\right)d\xi \qquad (3.2)\\ &= (\varkappa-\varkappa_1)^2 I_1 + (\varkappa_2-\varkappa)^2 I_2. \end{aligned}$$

Calculating I_1 and I_2 by integration by parts twice, we have

$$\begin{aligned} I_1 &= \int_0^1 \Lambda_1(\xi)\psi''\left(\xi\varkappa_1+(1-\xi)\varkappa\right)d\xi \\ &= -\frac{\Lambda_1(\xi)}{\varkappa-\varkappa_1}\psi'\left(\xi\varkappa_1+(1-\xi)\varkappa\right)\Big|_0^1 - \frac{1}{\varkappa-\varkappa_1}\int_0^1 \Psi_1(\xi)\psi'\left(\xi\varkappa_1+(1-\xi)\varkappa\right)d\xi \\ &= \frac{\Lambda_1(0)}{\varkappa-\varkappa_1}\psi'(\varkappa) - \frac{1}{\varkappa-\varkappa_1}\left[-\frac{\Psi_1(\xi)}{\varkappa-\varkappa_1}\psi\left(\xi\varkappa_1+(1-\xi)\varkappa\right)\Big|_0^1 \right.\\ &\quad \left. + \frac{1}{\varkappa-\varkappa_1}\int_0^1 \frac{\varphi((\varkappa-\varkappa_1)\xi)}{\xi}\psi\left(\xi\varkappa_1+(1-\xi)\varkappa\right)d\xi\right] \\ &= \frac{\Lambda_1(0)}{\varkappa-\varkappa_1}\psi'(\varkappa) + \frac{\Psi_1(1)}{(\varkappa-\varkappa_1)^2}\psi(\varkappa_1) - \frac{1}{(\varkappa-\varkappa_1)^2}\int_0^1 \frac{\varphi((\varkappa-\varkappa_1)\xi)}{\xi}\psi\left(\xi\varkappa_1+(1-\xi)\varkappa\right)d\xi \\ &= \frac{\Lambda_1(0)}{\varkappa-\varkappa_1}\psi'(\varkappa) + \frac{\Psi_1(1)}{(\varkappa-\varkappa_1)^2}\psi(\varkappa_1) - \frac{1}{(\varkappa-\varkappa_1)^2}\int_{\varkappa_1}^{\varkappa} \frac{\varphi(\varkappa-u)}{\varkappa-u}\psi(u)\,du \\ &= \frac{\Lambda_1(0)}{\varkappa-\varkappa_1}\psi'(\varkappa) + \frac{\Psi_1(1)}{(\varkappa-\varkappa_1)^2}\psi(\varkappa_1) - \frac{{}_{\varkappa_1+}I_\varphi\psi(\varkappa)}{(\varkappa-\varkappa_1)^2}, \end{aligned}$$

and similarly,

$$
\begin{aligned}
I_2 &= \int_0^1 \Lambda_2(\xi)\psi''\left(\xi\varkappa_2+(1-\xi)\varkappa\right)d\xi \\
&= -\frac{\Lambda_2(0)}{\varkappa_2-\varkappa}\psi'\left(\varkappa\right)+\frac{\Psi_2(1)}{\left(\varkappa_2-\varkappa\right)^2}\psi\left(\varkappa_2\right)-\frac{{}_{\varkappa_2-}I_{\varphi}\psi\left(\varkappa\right)}{\left(\varkappa_2-\varkappa\right)^2}.
\end{aligned}
$$

Substituting I_1 and I_2 in (3.2), then we get the desired result. □

Remark 3.1. If we choose $\varphi(\xi)=\xi$ in Lemma 3.1, then we have

$$
\begin{aligned}
&\frac{(\varkappa-\varkappa_1)^2-(\varkappa_2-\varkappa)^2}{2(\varkappa_2-\varkappa_1)}\psi'(\varkappa)+\frac{(\varkappa-\varkappa_1)\psi(\varkappa_1)+(\varkappa_2-\varkappa)\psi(\varkappa_2)}{(\varkappa_2-\varkappa_1)}-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\psi(\xi)d\xi \\
&= \frac{(\varkappa-\varkappa_1)^3}{2(\varkappa_2-\varkappa_1)}\int_0^1\left(1-\xi^2\right)\psi''\left(\xi\varkappa_1+(1-\xi)\varkappa\right)d\xi \\
&\quad+\frac{(\varkappa_2-\varkappa)^3}{2(\varkappa_2-\varkappa_1)}\int_0^1\left(1-\xi^2\right)\psi''\left(\xi\varkappa_2+(1-\xi)\varkappa\right)d\xi
\end{aligned}
$$

which is proved by Chu et al. in [12].

Remark 3.2. If we choose $\varphi(\xi)=\frac{\xi^{\alpha}}{\Gamma(\alpha)}$ in Lemma 3.1, then we get

$$
\begin{aligned}
&\frac{(\varkappa-\varkappa_1)^{\alpha+1}-(\varkappa_2-\varkappa)^{\alpha+1}}{(\alpha+1)(\varkappa_2-\varkappa_1)}\psi'(\varkappa)+\frac{(\varkappa-\varkappa_1)^{\alpha}\psi(\varkappa_1)+(\varkappa_2-\varkappa)^{\alpha}\psi(\varkappa_2)}{\varkappa_2-\varkappa_1} \\
&-\frac{\Gamma(\alpha+1)}{(\varkappa_2-\varkappa_1)}\left[I^{\alpha}_{\varkappa_1+}\psi(\varkappa)+I^{\alpha}_{\varkappa_2-}\psi(\varkappa)\right] \\
&= \frac{(\varkappa-\varkappa_1)^{\alpha+2}}{(\alpha+1)(\varkappa_2-\varkappa_1)}\int_0^1(1-\xi^{\alpha+1})\psi''\left(\xi\varkappa_1+(1-\xi)\varkappa\right)d\xi \\
&\quad+\frac{(\varkappa_2-\varkappa)^{\alpha+2}}{(\alpha+1)(\varkappa_2-\varkappa_1)}\int_0^1(1-\xi^{\alpha+1})\psi''\left(\xi\varkappa_2+(1-\xi)\varkappa\right)d\xi
\end{aligned}
$$

which is proved by Chu et al. in [12].

Corollary 3.1. *Under the assumptions of Lemma 3.1 with $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have following identity for the $k-$Riemann fractional integrals:*

$$
\frac{(\varkappa-\varkappa_1)^{\frac{\alpha}{k}+1}-(\varkappa_2-\varkappa)^{\frac{\alpha}{k}+1}}{(\alpha+k)(\varkappa_2-\varkappa_1)}k\psi'(\varkappa)+\frac{(\varkappa-\varkappa_1)^{\frac{\alpha}{k}}\psi(\varkappa_1)+(\varkappa_2-\varkappa)^{\frac{\alpha}{k}}\psi(\varkappa_2)}{(\alpha+k)(\varkappa_2-\varkappa_1)}
$$

$$-\frac{\Gamma_k(\alpha+k)}{(\varkappa_2-\varkappa_1)}\left[I^{\alpha}_{\varkappa_1+,k}\psi(\varkappa)+I^{\alpha}_{\varkappa_2-,k}\psi(\varkappa)\right]$$

$$= \frac{k\left(\varkappa-\varkappa_1\right)^{\frac{\alpha}{k}+2}}{(\alpha+k)(\varkappa_2-\varkappa_1)}\int_0^1(1-\xi^{\frac{\alpha}{k}+1})\psi''\left(\xi\varkappa_1+(1-\xi)\varkappa\right)d\xi$$

$$+\frac{k\left(\varkappa_2-\varkappa\right)^{\frac{\alpha}{k}+2}}{(\alpha+k)(\varkappa_2-\varkappa_1)}\int_0^1(1-\xi^{\frac{\alpha}{k}+1})\psi''\left(\xi\varkappa_2+(1-\xi)\varkappa\right)d\xi.$$

Theorem 3.1. *Let $\psi : I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be twice differentiable function on I° such that $\psi''\in L([\varkappa_1,\varkappa_2])$,where $\varkappa_1,\varkappa_2\in I^\circ$ with$\varkappa_1<\varkappa_2$. If the function $|\psi''|$ is convex on $[\varkappa_1,\varkappa_2]$, then we have the following inequality for generalized fractional integral operators*

$$\left|\left[\left(\varkappa-\varkappa_1\right)\Lambda_1(0)-\left(\varkappa_2-\varkappa\right)\Lambda_2(0)\right]\psi'\left(\varkappa\right)\right.$$

$$\left.+\Psi_1(1)\psi\left(\varkappa_1\right)+\Psi_2(1)\psi\left(\varkappa_2\right)-\left[{}_{\varkappa_1+}I_{\varphi}\psi\left(\varkappa\right)+{}_{\varkappa_2-}I_{\varphi}\psi\left(\varkappa\right)\right]\right|$$

$$\leq \left(\varkappa-\varkappa_1\right)^2\left|\psi''(\varkappa_1)\right|\int_0^1\xi\left|\Lambda_1(\xi)\right|d\xi+\left(\varkappa_2-\varkappa\right)^2\left|\psi''\left(\varkappa_2\right)\right|\int_0^1\xi\left|\Lambda_2(\xi)\right|d\xi$$

$$+\left[\left(\varkappa-\varkappa_1\right)^2\int_0^1\left(1-\xi\right)\left|\Lambda_1(\xi)\right|d\xi+\left(\varkappa_2-\varkappa\right)^2\int_0^1\left(1-\xi\right)\left|\Lambda_2(\xi)\right|d\xi\right]\left|\psi''(\varkappa)\right|.$$

Proof. Taking modulus in Lemma 3.1 and using the convexity of $|\psi''|$, we obtain

$$\left|\left[\left(\varkappa-\varkappa_1\right)\Lambda_1(0)-\left(\varkappa_2-\varkappa\right)\Lambda_2(0)\right]\psi'\left(\varkappa\right)\right.$$

$$\left.+\Psi_1(1)\psi\left(\varkappa_1\right)+\Psi_2(1)\psi\left(\varkappa_2\right)-\left[{}_{\varkappa_1+}I_{\varphi}\psi\left(\varkappa\right)+{}_{\varkappa_2-}I_{\varphi}\psi\left(\varkappa\right)\right]\right|$$

$$\leq \left(\varkappa-\varkappa_1\right)^2\left|\int_0^1\Lambda_1(\xi)\psi''\left(\xi\varkappa_1+(1-\xi)\varkappa\right)d\xi\right|$$

$$+\left(\varkappa_2-\varkappa\right)^2\left|\int_0^1\Lambda_2(\xi)\psi''\left(\xi\varkappa_2+(1-\xi)\varkappa\right)d\xi\right|$$

$$\leq \left(\varkappa-\varkappa_1\right)^2\int_0^1\left|\Lambda_1(\xi)\right|\left|\psi''\left(\xi\varkappa_1+(1-\xi)\varkappa\right)\right|d\xi$$

$$+\left(\varkappa_2-\varkappa\right)^2\int_0^1\left|\Lambda_2(\xi)\right|\left|\psi''\left(\xi\varkappa_2+(1-\xi)\varkappa\right)\right|d\xi$$

$$\begin{aligned}
&\leq (\varkappa-\varkappa_1)^2 \int_0^1 |\Lambda_1(\xi)| \left[\xi \left|\psi''(\varkappa_1)\right| + (1-\xi)\left|\psi''(\varkappa)\right|\right] d\xi \\
&\quad + (\varkappa_2-\varkappa)^2 \int_0^1 |\Lambda_2(\xi)| \left[\xi \left|\psi''(\varkappa_2)\right| + (1-\xi)\left|\psi''(\varkappa)\right|\right] d\xi \\
&= (\varkappa-\varkappa_1)^2 \left[\left|\psi''(\varkappa_1)\right| \int_0^1 \xi |\Lambda_1(\xi)| d\xi + \left|\psi''(\varkappa)\right| \int_0^1 (1-\xi)|\Lambda_1(\xi)| d\xi\right] \\
&\quad + (\varkappa_2-\varkappa)^2 \left[\left|\psi''(\varkappa_2)\right| \int_0^1 \xi |\Lambda_2(\xi)| d\xi + \left|\psi''(\varkappa)\right| \int_0^1 (1-\xi)|\Lambda_2(\xi)| d\xi\right] \\
&= (\varkappa-\varkappa_1)^2 \left|\psi''(\varkappa_1)\right| \int_0^1 \xi |\Lambda_1(\xi)| d\xi + (\varkappa-\varkappa_1)^2 \left|\psi''(\varkappa)\right| \int_0^1 (1-\xi)|\Lambda_1(\xi)| d\xi \\
&\quad + (\varkappa_2-\varkappa)^2 \left|\psi''(\varkappa_2)\right| \int_0^1 \xi |\Lambda_2(\xi)| d\xi + (\varkappa_2-\varkappa)^2 \left|\psi''(\varkappa)\right| \int_0^1 (1-\xi)|\Lambda_2(\xi)| d\xi \\
&= (\varkappa-\varkappa_1)^2 \left|\psi''(\varkappa_1)\right| \int_0^1 \xi |\Lambda_1(\xi)| d\xi + (\varkappa_2-\varkappa)^2 \left|\psi''(\varkappa_2)\right| \int_0^1 \xi |\Lambda_2(\xi)| d\xi \\
&\quad + \left[(\varkappa-\varkappa_1)^2 \int_0^1 (1-\xi)|\Lambda_1(\xi)| d\xi + (\varkappa_2-\varkappa)^2 \int_0^1 (1-\xi)|\Lambda_2(\xi)| d\xi\right] \left|\psi''(\varkappa)\right|.
\end{aligned}$$

The proof of Theorem 3.1 is completed. □

Remark 3.3. Let $\varkappa = \frac{\varkappa_1+\varkappa_2}{2}$, then Theorem 3.1 becomes

$$\begin{aligned}
&\left|\Psi_1^*(1)\left(\psi(\varkappa_1)+\psi(\varkappa_2)\right) - \left[{}_{\varkappa_1+}I_\varphi \psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) + {}_{\varkappa_2-}I_\varphi \psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right]\right| \\
&\leq \left(\frac{\varkappa_2-\varkappa_1}{2}\right)^2 \left(\left|\psi''(\varkappa_1)\right| + \left|\psi''(\varkappa_2)\right|\right) \int_0^1 |\Lambda_1^*(\xi)| d\xi,
\end{aligned}$$

where

$$\Lambda_1^*(\xi) = \int_\xi^1 \Psi_1^*(s) ds \text{ and } \Psi_1^*(s) = \int_0^s \frac{\varphi\left(\left(\frac{\varkappa_2-\varkappa_1}{2}\right)u\right)}{u} du.$$

Proof. If we take $\varkappa = \frac{\varkappa_1+\varkappa_2}{2}$ in Theorem 3.1, then using the convexity of $|\psi''(\varkappa)|$ we have

$$\left|\Psi_1^*(1)\left(\psi(\varkappa_1)+\psi(\varkappa_2)\right) - \left[{}_{\varkappa_1+}I_\varphi \psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right) + {}_{\varkappa_2-}I_\varphi \psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right]\right|$$

$$\leq \left(\frac{\varkappa_2-\varkappa_1}{2}\right)^2\left[\left(|\psi''(\varkappa_1)|+|\psi''(\varkappa_2)|\right)\int_0^1 \xi\,|\Lambda_1^*(\xi)|\,d\xi\right]$$

$$+\left(\frac{\varkappa_2-\varkappa_1}{2}\right)^2\left[2\int_0^1 (1-\xi)\,|\Lambda_1^*(\xi)|\,d\xi\right]\left|\psi''\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right|$$

$$\leq \left(\frac{\varkappa_2-\varkappa_1}{2}\right)^2\left[\left(|\psi''(\varkappa_1)|+|\psi''(\varkappa_2)|\right)\int_0^1 \xi\,|\Lambda_1^*(\xi)|\,d\xi\right]$$

$$+\left(\frac{\varkappa_2-\varkappa_1}{2}\right)^2\left[\int_0^1 (1-\xi)\,|\Lambda_1^*(\xi)|\,d\xi\right]\left(|\psi''(\varkappa_1)|+|\psi''(\varkappa_2)|\right)$$

$$= \left(\frac{\varkappa_2-\varkappa_1}{2}\right)^2\left(|\psi''(\varkappa_1)|+|\psi''(\varkappa_2)|\right)\int_0^1 |\Lambda_1^*(\xi)|\,d\xi.$$

This completes the proof. □

Remark 3.4. If we choose $\varphi(\xi)=\xi$ in Theorem 3.1, then we have the following inequality

$$\left|\frac{(\varkappa-\varkappa_1)^2-(\varkappa_2-\varkappa)^2}{2(\varkappa_2-\varkappa_1)}\psi'(\varkappa)+\frac{(\varkappa-\varkappa_1)\psi(\varkappa_1)+(\varkappa_2-\varkappa)\psi(\varkappa_2)}{(\varkappa_2-\varkappa_1)}-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\psi(\xi)d\xi\right|$$

$$\leq \frac{1}{8}\left[(\varkappa-\varkappa_1)^3|\psi''(\varkappa_1)|+(\varkappa_2-\varkappa)^3|\psi''(\varkappa_2)|\right]+\frac{5\left[(\varkappa-\varkappa_1)^3+(\varkappa_2-\varkappa)^3\right]}{24}|\psi''(\varkappa)|.$$

Corollary 3.2. *If we choose $\varphi(\xi)=\frac{\xi^\alpha}{\Gamma(\alpha)}$ in Theorem 3.1, then we have the following inequality*

$$\left|\frac{(\varkappa-\varkappa_1)^{\alpha+1}-(\varkappa_2-\varkappa)^{\alpha+1}}{(\alpha+1)(\varkappa_2-\varkappa_1)}\psi'(\varkappa)+\frac{(\varkappa-\varkappa_1)^{\alpha}\psi(\varkappa_1)+(\varkappa_2-\varkappa)^{\alpha}\psi(\varkappa_2)}{(\alpha+1)(\varkappa_2-\varkappa_1)}\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{(\varkappa_2-\varkappa_1)}\left[I_{\varkappa_1+}^{\alpha}\psi(\varkappa)+I_{\varkappa_2-}^{\alpha}\psi(\varkappa)\right]\right|$$

$$\leq \frac{1}{(\alpha+1)(\varkappa_2-\varkappa_1)}\left(\frac{1}{2}-\frac{1}{\alpha+3}\right)\left[(\varkappa-\varkappa_1)^{\alpha+2}|\psi''(\varkappa_1)|+(\varkappa_2-\varkappa)^{\alpha+2}|\psi''(\varkappa_2)|\right]$$

$$+\frac{1}{(\alpha+1)(\varkappa_2-\varkappa_1)}\left(\frac{1}{2}-\frac{1}{(\alpha+2)(\alpha+3)}\right)\left[(\varkappa-\varkappa_1)^{\alpha+2}+(\varkappa_2-\varkappa)^{\alpha+2}\right]|\psi''(\varkappa)|.$$

Corollary 3.3. *Under the assumptions of Theorem 3.1 with $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the following inequality for k-Riemann integrals holds:*

$$\left|\frac{(\varkappa-\varkappa_1)^{\frac{\alpha}{k}+1}-(\varkappa_2-\varkappa)^{\frac{\alpha}{k}+1}}{(\alpha+k)(\varkappa_2-\varkappa_1)}k\psi'(\varkappa)+\frac{(\varkappa-\varkappa_1)^{\frac{\alpha}{k}}\psi(\varkappa_1)+(\varkappa_2-\varkappa)^{\frac{\alpha}{k}}\psi(\varkappa_2)}{(\alpha+k)(\varkappa_2-\varkappa_1)}\right.$$

$$-\frac{\Gamma_k(\alpha+k)}{(\varkappa_2-\varkappa_1)}\left[I^{\alpha}_{\varkappa_1+,k}\psi(\varkappa)+I^{\alpha}_{\varkappa_2-,k}\psi(\varkappa)\right]\Big|$$

$$\leq \frac{k}{(\alpha+1)(\varkappa_2-\varkappa_1)}\left(\frac{1}{2}-\frac{k}{\alpha+3k}\right)\left[(\varkappa-\varkappa_1)^{\frac{\alpha}{k}+2}\left|\psi''(\varkappa_1)\right|+(\varkappa_2-\varkappa)^{\frac{\alpha}{k}+2}\left|\psi''(\varkappa_2)\right|\right]$$

$$+\frac{k}{(\alpha+k)(\varkappa_2-\varkappa_1)}\left(\frac{1}{2}-\frac{k^2}{(\alpha+2k)(\alpha+3k)}\right)\left[(\varkappa-\varkappa_1)^{\frac{\alpha}{k}+2}+(\varkappa_2-\varkappa)^{\frac{\alpha}{k}+2}\right]\left|\psi''(\varkappa)\right|.$$

Theorem 3.2. *Let $\psi : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable function on I° such that $\psi'' \in L([\varkappa_1,\varkappa_2])$, where $\varkappa_1,\varkappa_2 \in I^{\circ}$ with $\varkappa_1 < \varkappa_2$. If the function $|\psi''|^q, q>1$ is convex on $[\varkappa_1,\varkappa_2]$, then we have the following inequality for generalized fractional integral operators*

$$\left|\left[(\varkappa-\varkappa_1)\Lambda_1(0)-(\varkappa_2-\varkappa)\Lambda_2(0)\right]\psi'(\varkappa)\right.$$

$$\left.+\Psi_1(1)\psi(\varkappa_1)+\Psi_2(1)\psi(\varkappa_2)-\left[{}_{\varkappa_1+}I_{\varphi}\psi(\varkappa)+{}_{\varkappa_2-}I_{\varphi}\psi(\varkappa)\right]\right|$$

$$\leq (\varkappa-\varkappa_1)^2\left[\int_0^1|\Lambda_1(\xi)|^p d\xi\right]^{\frac{1}{p}}\left[\frac{|\psi''(\varkappa_1)|^q+|\psi''(\varkappa)|^q}{2}\right]^{\frac{1}{q}}$$

$$+(\varkappa_2-\varkappa)^2\left[\int_0^1|\Lambda_2(\xi)|^p d\xi\right]^{\frac{1}{p}}\left[\frac{|\psi''(\varkappa_2)|^q+|\psi''(\varkappa)|^q}{2}\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking modulus in Lemma 3.1 and using the convexity of $|f''|^q$, we obtain

$$\left|\left[(\varkappa-\varkappa_1)\Lambda_1(0)-(\varkappa_2-\varkappa)\Lambda_2(0)\right]\psi'(\varkappa)\right.$$

$$\left.+\Psi_1(1)\psi(\varkappa_1)+\Psi_2(1)\psi(\varkappa_2)-\left[{}_{\varkappa_1+}I_{\varphi}\psi(\varkappa)+{}_{\varkappa_2-}I_{\varphi}\psi(\varkappa)\right]\right|$$

$$\leq (\varkappa-\varkappa_1)^2\left[\int_0^1|\Lambda_1(\xi)|^p d\xi\right]^{\frac{1}{p}}\left[\int_0^1\left|\psi''(\xi\varkappa_1+(1-\xi)\varkappa)\right|^q d\xi\right]^{\frac{1}{q}}$$

$$+(\varkappa_2-\varkappa)^2\left[\int_0^1|\Lambda_2(\xi)|^p d\xi\right]^{\frac{1}{p}}\left[\int_0^1\left|\psi''(\xi\varkappa_2+(1-\xi)\varkappa)\right|^q d\xi\right]^{\frac{1}{q}}$$

$$\leq (\varkappa-\varkappa_1)^2\left[\int_0^1|\Lambda_1(\xi)|^p d\xi\right]^{\frac{1}{p}}\left[\int_0^1\left[\xi\left|\psi''(\varkappa_1)\right|^q+(1-\xi)\left|\psi''(\varkappa)\right|^q\right]d\xi\right]^{\frac{1}{q}}$$

$$+(\varkappa_2-\varkappa)^2\left[\int_0^1 |\Lambda_2(\xi)|^p\,d\xi\right]^{\frac{1}{p}}\left[\int_0^1 \xi\,|\psi''(\varkappa_2)|^q+(1-\xi)\,|\psi''(\varkappa)|^q\,d\xi\right]^{\frac{1}{q}}$$

$$\leq (\varkappa-\varkappa_1)^2\left[\int_0^1 |\Lambda_1(\xi)|^p\,d\xi\right]^{\frac{1}{p}}\left[\frac{|\psi''(\varkappa_1)|^q+|\psi''(\varkappa)|^q}{2}\right]^{\frac{1}{q}}$$

$$+(\varkappa_2-\varkappa)^2\left[\int_0^1 |\Lambda_2(\xi)|^p\,d\xi\right]^{\frac{1}{p}}\left[\frac{|\psi''(\varkappa_2)|^q+|\psi''(\varkappa)|^q}{2}\right]^{\frac{1}{q}}.$$

The proof of Theorem 3.2 is completed. □

Remark 3.5. Let $\varkappa=\frac{\varkappa_1+\varkappa_2}{2}$, then Theorem 3.2 leads

$$\left|\Psi_1^*(1)\left(\psi\left(\varkappa_1\right)+\psi\left(\varkappa_2\right)\right)-\left[{}_{\varkappa_1+}I_{\varphi}\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)+{}_{\varkappa_2-}I_{\varphi}\psi\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\right]\right|$$

$$\leq \left(\frac{\varkappa_2-\varkappa_1}{2}\right)^2\left(\int_0^1 |\Lambda_1^*(\xi)|^p\,d\xi\right)^{\frac{1}{p}}$$

$$\times\left(\left(\frac{3\,|\psi''(\varkappa_1)|^q+|\psi''(\varkappa_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\psi''(\varkappa_1)|^q+3\,|\psi''(\varkappa_2)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Corollary 3.4. *Under the assumptions of Theorem 3.2 with $\varphi(\xi)=\xi$, we have the following inequality:*

$$\left|\frac{(\varkappa-\varkappa_1)^2-(\varkappa_2-\varkappa)^2}{2(\varkappa_2-\varkappa_1)}\psi'(\varkappa)+\frac{(\varkappa-\varkappa_1)\psi(\varkappa_1)+(\varkappa_2-\varkappa)\psi(\varkappa_2)}{(\varkappa_2-\varkappa_1)}-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\psi(\xi)d\xi\right|$$

$$\leq \frac{2^{\frac{1}{p}}}{2(\varkappa_2-\varkappa_1)\left(\beta\left(\frac{1}{2},p+1\right)\right)^{\frac{1}{p}}}\left[(\varkappa-\varkappa_1)^3\left(\frac{|\psi''(\varkappa_1)|^q+|\psi''(\varkappa)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+(\varkappa_2-\varkappa)^3\left(\frac{|\psi''(\varkappa_2)|^q+|\psi''(\varkappa)|^q}{2}\right)^{\frac{1}{q}}\right],$$

where $\beta(\cdot,\cdot)$ is a Beta function.

Corollary 3.5. *Under the assumptions of Theorem 3.2 with $\varphi(\xi)=\frac{\xi^\alpha}{\Gamma(\alpha)}$, then the following inequality for the Riemann-Liouville fractional integral holds:*

$$\left|\frac{(\varkappa-\varkappa_1)^{\alpha+1}-(\varkappa_2-\varkappa)^{\alpha+1}}{(\alpha+1)(\varkappa_2-\varkappa_1)}\psi'(\varkappa)+\frac{(\varkappa-\varkappa_1)^{\alpha}\psi(\varkappa_1)+(\varkappa_2-\varkappa)^{\alpha}\psi(\varkappa_2)}{(\alpha+1)(\varkappa_2-\varkappa_1)}\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{(\varkappa_2-\varkappa_1)}\left[I_{\varkappa_1+}^{\alpha}\psi(\varkappa)+I_{\varkappa_2-}^{\alpha}\psi(\varkappa)\right]\right|$$

$$\leq \frac{(\alpha+1)^{\frac{1}{p}}}{(\alpha+1)(\varkappa_2-\varkappa_1)\left(\beta\left(\frac{1}{\alpha+1},p+1\right)\right)^{\frac{1}{p}}}\left[(\varkappa-\varkappa_1)^{\alpha+2}\left(\frac{\left|\psi''(\varkappa_1)\right|^q+|\psi''(\varkappa)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+(\varkappa_2-\varkappa)^{\alpha+2}\left(\frac{|\psi''(\varkappa_2)|^q+|\psi''(\varkappa)|^q}{2}\right)^{\frac{1}{q}}\right].$$

Corollary 3.6. *Under the assumptions of Theorem 3.2 with $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the following inequality for the k-Riemann-Liouville fractional integral holds:*

$$\left|\frac{(\varkappa-\varkappa_1)^{\frac{\alpha}{k}+1}-(\varkappa_2-\varkappa)^{\frac{\alpha}{k}+1}}{(\alpha+k)(\varkappa_2-\varkappa_1)}k\psi'(\varkappa)+\frac{(\varkappa-\varkappa_1)^{\frac{\alpha}{k}}\psi(\varkappa_1)+(\varkappa_2-\varkappa)^{\frac{\alpha}{k}}\psi(\varkappa_2)}{(\alpha+k)(\varkappa_2-\varkappa_1)}\right.$$
$$\left.-\frac{\Gamma_k(\alpha+k)}{(\varkappa_2-\varkappa_1)}\left[I^{\alpha}_{\varkappa_1+,k}\psi(\varkappa)+I^{\alpha}_{\varkappa_2-,k}\psi(\varkappa)\right]\right|$$
$$\leq \frac{k(\frac{\alpha}{k}+1)^{\frac{1}{p}}}{(\alpha+k)(\varkappa_2-\varkappa_1)\left(\beta\left(\frac{1}{\frac{\alpha}{k}+1},p+1\right)\right)^{\frac{1}{p}}}\left[(\varkappa-\varkappa_1)^{\frac{\alpha}{k}+2}\left(\frac{|\psi''(\varkappa_1)|^q+|\psi''(\varkappa)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+(\varkappa_2-\varkappa)^{\frac{\alpha}{k}+2}\left(\frac{|\psi''(\varkappa_2)|^q+|\psi''(\varkappa)|^q}{2}\right)^{\frac{1}{q}}\right].$$

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