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ON SOME BOUNDS FOR THE GENERALIZED GAMMA, DIGAMMA AND POLYGAMMA FUNCTIONS

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Abstract. We are interested in finding some strictly completely monotonic (CM) functions involving the generalized gamma, digamma and polygamma functions, where the generalized gamma function is $\Gamma_\mu(y) = \frac{\mu!\, \mu^y}{y(1+y)(2+y)\cdots(\mu+y)}$ for $y > 0$ and $\mu \in \mathbb{N}$ and the generalized digamma function is its logarithmic derivative. As a consequence, we establish some new upper and lower bounds for the generalized gamma, digamma and polygamma functions, which refine recent results.

1. Introduction

Special functions have many applications in pure mathematics and in applied areas such as electrical current, acoustics, heat conduction, fluid dynamics and solutions of wave equations. The gamma and the psi functions have great importance in the study of special functions. Euler [9] defined the gamma function as $\Gamma(y) = \lim_{\mu\to\infty} \Gamma_\mu(y)$, where the generalized gamma function

$$\Gamma_\mu(y) = \frac{\mu!\, \mu^y}{y(y+1)(y+2)\cdots(\mu+y)}, \qquad y \in \mathbb{R}^+, \qquad \mu \in \mathbb{N}$$

which satisfies

$$\Gamma_\mu(y+1) = \left(\frac{y\,\mu}{y+1+\mu}\right)\Gamma_\mu(y), \qquad y \in \mathbb{R}^+, \qquad \mu \in \mathbb{N}. \tag{1.1}$$

The digamma function is given by [1]

$$\psi(y) = \frac{\Gamma'(y)}{\Gamma(y)} = -\gamma + \sum_{l=0}^{\infty}\left(\frac{1}{1+l} - \frac{1}{l+y}\right), \qquad y \in (0,\infty)$$

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where the Euler-Mascheroni constant $\gamma := \lim_{s\to\infty}\left(\sum_{u=1}^{s}\frac{1}{u} - \log s\right) \approx 0.5772156649$. The generalized digamma and polygamma functions are given by [4]:

$$\psi_\mu(y) = \sum_{s=0}^{\mu}\frac{-1}{s+y} + \ln\mu = \int_0^\infty \left(\frac{e^{-\mu v} - e^{v}}{e^{v}-1}\right) e^{-yv}dv + \ln\mu, \qquad y\in(0,\infty), \quad \mu\in\mathbb{N} \quad (1.2)$$

and

$$\psi_\mu^{(u)}(y) = \sum_{s=0}^{\mu}\frac{(-1)^{u+1}\, u!}{(s+y)^{u+1}} = (-1)^u\int_0^\infty \left(\frac{e^{-\mu v} - e^{v}}{e^{v}-1}\right) v^u e^{-yv}dv, \quad y\in(0,\infty), \quad \mu; u\in\mathbb{N} \quad (1.3)$$

and they satisfy the following functional equation:

$$\psi_\mu^{(u)}(1+y) - \psi_\mu^{(u)}(y) = (-1)^{u+1}\, u!\Big((1+y+\mu)^{-1-u} - y^{-1-u}\Big), \qquad u\in\mathbb{N}\cup\{0\}. \quad (1.4)$$

$\psi_\mu^{(u)}(y)$ is strictly CM on $(0,\infty)$ for $u = 1, 3, 5, \cdots$. A function H defined on $J\subset\mathbb{R}$ is called CM if it has derivatives $H^{(u)}(y)$ for every $u\in\mathbb{N}\cup\{0\}$ such that

$$(-1)^u H^{(u)}(y) \geq 0 \qquad y\in J;\ u\in\mathbb{N}\cup\{0\}.$$

These functions occur in many areas such as elasticity, numerical analysis and probability theory. For more details about this topic, see [3,6–8]. The necessary and sufficient condition that the function $H(y)$ should be completely monotonic for $0<y<\infty$ is that

$$H(y) = \int_0^\infty e^{-yv}dv(y),$$

where $v(y)$ is non-decreasing and the integral converges for $0<y<\infty$ (see [11], Theorem 12b).

Batir [2] presented some bounds for $\Gamma(y)$ in terms of $\psi(y)$:

$$\exp\left[-\frac{1/6}{(y-\frac{1}{4})} - \frac{\psi(y)}{2}\right] < \frac{e^y\,\Gamma(y)}{\left(\sqrt{2\pi}\, y^y\right)} < \exp\left[-\frac{1/6}{y} - \frac{\psi(y)}{2}\right], \qquad y>0. \quad (1.5)$$

Şevli and Batir [10] presented some bounds for $\Gamma(y)$ in terms of $\psi'(y)$:

$$\exp\left(\frac{\psi'\left(\frac{1+2y}{2}\right)}{12}\right) < \frac{e^y\,\Gamma(y)}{\left(\sqrt{2\pi}\, y^{y-\frac{1}{2}}\right)} < \exp\left(\frac{\psi'(y)}{12}\right), \qquad y>0. \quad (1.6)$$

Recently, Mahmoud, Almuashi and Moustafa [5] presented some asymptotic expansions for $\Gamma_\mu(y)$, $\psi_\mu(y)$ and $\psi_\mu^{(u)}(y)$ functions:

$$\ln\Gamma_\mu(y) \sim \left(\frac{2y-1}{2}\right)\ln\left(\frac{\mu y}{1+y+\mu}\right) - (1+\mu)\ln(1+y+\mu) + \ln\left(\mu!\, e^{1+\mu}\sqrt{\mu}\right)$$
$$-\Big[\vartheta(1+y+\mu) - \vartheta(y)\Big], \quad (1.7)$$

$$\psi_\mu(y) \sim \frac{1}{2(1+y+\mu)} - \frac{1}{2y} + \ln\left(\frac{\mu y}{1+y+\mu}\right) - \Big[\vartheta'(1+y+\mu) - \vartheta'(y)\Big], \quad y\longrightarrow\infty \quad (1.8)$$

and

$$\psi_\mu^{(u)}(y) \sim (u-1)!\left(\frac{(-1)^u}{(1+y+\mu)^u} - \frac{(-1)^u}{y^u}\right) + \frac{(-1)^u}{2}\left(\frac{u!}{(1+y+\mu)^{u+1}} - \frac{u!}{y^{u+1}}\right)$$
$$+\vartheta^{(u+1)}(y) - \vartheta^{(u+1)}(1+y+\mu), \quad u \in \mathbb{N}, \quad y \longrightarrow \infty \tag{1.9}$$

where $\vartheta(y) = \frac{1}{12y} - \frac{1}{360y^3} - \frac{1}{120y^4}$ and $\vartheta^{(u)}(y) = (-1)^u\left[\frac{u!}{12y^{1+u}} - \frac{(2+u)!}{720y^{3+u}} - \frac{(3+u)!}{720y^{4+u}}\right]$ and they also deduced the following inequalities, for $\mu \in \mathbb{N}$, $y \in (0,\infty)$ and $u = 2, 3, \cdots$:

$$\sqrt{\frac{\mu y}{1+y+\mu}}\exp\left[-\frac{\psi_\mu\left(\frac{1+3y}{3}\right)}{2}\right] < \frac{\Gamma_\mu(y)}{\left(\frac{\mu^y\, y^{y-\frac{1}{2}}\, e^{1+\mu}\,\mu!}{(1+y+\mu)^{\frac{1}{2}+y+\mu}}\right)} < \sqrt{\frac{\mu y}{1+y+\mu}}\exp\left[-\frac{\psi_\mu(y)}{2}\right], \tag{1.10}$$

$$\frac{\psi_\mu'\left(\frac{1+3y}{3}\right)}{2} < \ln\left(\frac{\mu y}{y+\mu+1}\right) - \psi_\mu(y) < \frac{\psi_\mu'(y)}{2} \tag{1.11}$$

and

$$\frac{(-1)^{u+1}\psi_\mu^{(u)}\left(\frac{3y+1}{3}\right)}{2} < (-1)^u\psi_\mu^{(u-1)}(y) - (u-2)!\left(y^{1-u} - (1+y+\mu)^{1-u}\right) < \frac{(-1)^{1+u}\psi_\mu^{(u)}(y)}{2}. \tag{1.12}$$

Our purpose is to find some inequalities for $\Gamma_\mu(y), \psi_\mu(y)$ and $\psi_\mu^{(u)}(y)$ functions, which refine the results (1.10), (1.11) and (1.12).

The next corollary [8] will be needed in the next section:

Corollary 1.1. *Suppose that L is a function defined on $v > v_0$, $v_0 \in \mathbb{R}$ with $\lim_{v\to\infty} L(v) = 0$. Then for $\eta > 0$, $L(v) > 0$, if $L(v+\eta) - L(v) < 0$ for $v > v_0$ and $L(v) < 0$, if $L(v+\eta) - L(v) > 0$ for $v > v_0$.*

2. Auxiliary Results

Theorem 2.1. *Assume that $y > 0$ and $\mu \in \mathbb{N}$. Then the function*

$$u_\mu(y) = \frac{\psi_\mu'\left(\frac{1+2y}{2}\right)}{6} + \psi_\mu\left(\frac{1+3y}{3}\right) - \ln\left(\frac{y\mu}{1+y+\mu}\right)$$

is strictly CM on $(0,\infty)$.

Proof. Using (1.3) and the identity $\frac{1}{y^u} = \frac{1}{(u-1)!}\int_0^\infty v^{u-1}e^{-yv}dv$ for $y > 0$, (see [1]), yields

$$u_\mu'(y) = \frac{\psi_\mu''\big((1+2y)/2\big)}{6} + \psi_\mu'\left(\frac{1+3y}{3}\right) + (1+y+\mu)^{-1} - y^{-1} = \int_0^\infty \frac{e^{-(1+y+\mu)v}}{6(e^v-1)}\omega(v)dv,$$

where

$$\omega(v) = 6\Big[-1 + e^v + e^{(1+\mu)v} - e^{(\mu+2)v}\Big] - 6v\Big[e^{\frac{2v}{3}} - e^{(\frac{5}{3}+\mu)v}\Big] + v^2\Big[e^{\frac{v}{2}} - e^{(\frac{3}{2}+\mu)v}\Big]$$

and by using the series representation of the exponential function at $v = 0$, we obtain

$$\omega(v) = -\frac{(1+\mu)v^4}{6}\left[1 + \frac{(35+18\mu)v}{36} + \frac{(574+615\mu+180\mu^2)v^2}{1080}\right] + \sum_{u=5}^{\infty}\frac{f_\mu(u)}{(2+u)!}v^{2+u} < 0,$$

where

$$\begin{aligned}
f_\mu(u) &= 6\Big[(1+\mu)^{2+u} - (2+\mu)^{2+u} + 1\Big] - 6(2+u)\Big[(2/3)^{u+1} - (5/3+\mu)^{u+1}\Big] \\
&\quad +(2+u)(1+u)\left[2^{-u} - \left(\frac{3}{2}+\mu\right)^u\right] \\
&= -\frac{u(u+1)(u+2)\,(\mu+1)^{u-3}}{6480}\Big[30\mu\big(55+36\mu+17u\big) + u\big(93+139u\big)\Big] \\
&\quad -\sum_{s=1}^{u-4}(u+2)(u+1)\tbinom{u}{s}\left(\frac{1}{2}\right)^{u-s}(1+\mu)^s \\
&\quad -\sum_{s=1}^{u-4}\left[6\binom{u+2}{s} - 6(u+2)\binom{u+1}{s}(2/3)^{u+1-s}\right](1+\mu)^s \\
&< -\sum_{s=1}^{u-4}\left[6\binom{u+2}{s} - 6(u+2)\binom{u+1}{s}(2/3)^{u+1-s}\right](1+\mu)^s \\
&= -6\sum_{s=1}^{u-4}\frac{(2+u)\left(\frac{2}{3}\right)^{1+u-s}}{(2+u-s)}\binom{1+u}{s}\left[\frac{(u-s+4)(u-s-1)}{8} + \sum_{l=3}^{u-s+1}\binom{u-s+1}{l}2^{-l}\right](1+\mu)^s \\
&< 0.
\end{aligned}$$

From the Bernstein-Widder Theorem [11], we have $-u'_\mu(y)$ is strictly CM on $(0,\infty)$. It follows that $u_\mu(y)$ is strictly decreasing on $(0,\infty)$ and by using (1.8) and (1.9), we obtain $\lim_{y\to\infty} u_\mu(y) = 0$ and then $u_\mu(y) > 0$. Then $u_\mu(y)$ is strictly CM on $(0,\infty)$. ☐

Let us mention some important consequences of Theorem 2.1:

Corollary 2.1. *Assume that $y > 0$ and $\mu \in \mathbb{N}$. Then we have*

$$\frac{\psi'_\mu\left(\frac{1+2y}{2}\right)}{6} > \ln\left(\frac{y\mu}{1+y+\mu}\right) - \psi_\mu\left(\frac{1+3y}{3}\right) \tag{2.1}$$

and

$$\frac{(-1)^u}{6}\psi_\mu^{(1+u)}\left(\frac{2y+1}{2}\right) > \frac{(u-1)!}{(1+y+\mu)^u} - \frac{(u-1)!}{y^u} + (-1)^{1+u}\psi_\mu^{(u)}\left(\frac{1+3y}{3}\right), \qquad u \in \mathbb{N}. \tag{2.2}$$

Lemma 2.1. *For $\mu \in \mathbb{N}$, we have*

$$\frac{\psi'_\mu\left(\frac{1+2y}{2}\right)}{6} > \ln\left(\frac{y\mu}{y+1+\mu}\right) - \psi_\mu(y) - \frac{1}{3}\left(\frac{1}{\frac{4y-1}{4}} - \frac{1}{y+\frac{3}{4}+\mu}\right) \qquad \forall y > 0.25, \tag{2.3}$$

$$\frac{\psi''_\mu\left(\frac{2y+1}{2}\right)}{6} < y^{-1} - \frac{1}{1+y+\mu} - \psi'_\mu(y) + \frac{1}{3}\left(\frac{1}{(y-\frac{1}{4})^2} - \frac{1}{(\frac{3}{4}+y+\mu)^2}\right) \qquad \forall y > 0.25 \tag{2.4}$$

and

$$\frac{\psi_\mu'''\left(\frac{1+2y}{2}\right)}{6} > -\Big(\frac{1}{y^2}-\frac{1}{(1+y+\mu)^2}\Big)-\psi_\mu''(y)-\frac{2}{3}\left(\frac{1}{(y-\frac{1}{4})^3}-\frac{1}{(\frac{3}{4}+y+\mu)^3}\right) \qquad \forall y > 0.25. \tag{2.5}$$

Proof. Consider the function $v_\mu(y) = \frac{1}{6}\psi_\mu'\left(y+\frac{1}{2}\right)-\ln\left(\frac{y\mu}{y+\mu+1}\right)+\psi_\mu(y)+\frac{1}{3}\left(\frac{1}{y-\frac{1}{4}}-\frac{1}{y+\mu+\frac{3}{4}}\right)$ and by using (1.4) and Mathematica software, we have

$$v_\mu''(1+y)-v_\mu''(y) = \frac{\frac{-D_\mu\left(y-0.25\right)\,(1+\mu)}{3y^3(1+y)^2(1+\mu+y)^3(2+\mu+y)^2(1+2y)^4}}{(3+2\mu+2y)^4(-1+4y)^3(3+4y)^3(3+4\mu+4y)^3(7+4\mu+4y)^3},$$

where $D_\mu(y)$ is a polynomial of 23-th degree of y with positive coefficients of μ and then $v_\mu''(1+y)-v_\mu''(y) < 0$ for $y > 0.25$ and by using (1.9), we get $\lim_{y\to\infty} v_\mu''(y) = 0$. By using Corollary 1.1, we get that $v_\mu''(y) > 0$ for $y \in (0.25,\infty)$ and this proves (2.5). It follows that $v_\mu'(y)$ is strictly increasing on $y > 0.25$ with $\lim_{y\to\infty} v_\mu'(y) = 0$, then $v_\mu'(y) < 0$ for $y \in (0.25,\infty)$ and hence we obtain (2.4). Consequently, $v_\mu(y)$ is strictly decreasing on $y > 0.25$ with $\lim_{y\to\infty} v_\mu(y) = 0$, then $v_\mu(y)$ is positive for $y \in (0.25,\infty)$ and then we get (2.3). □

3. Some CM functions

Theorem 3.1. *Assume that $y > 0$ and $\mu \in \mathbb{N}$. Then the function*

$$N_{\delta,\mu}(y) = \frac{\psi_\mu(y)}{2}+\ln\Gamma_\mu(y)+(\mu+1)\ln(1+y+\mu)-y\ln\left(\frac{y\mu}{1+y+\mu}\right)-\ln\left(\mu!\,e^{1+\mu}\,\sqrt{\mu}\right)$$
$$+\frac{1}{6}\left(\frac{1}{y-\delta}-\frac{1}{\frac{3}{4}+y+\mu}\right)$$

is strictly CM on $(0,\infty)$ if and only if $\delta \geq \frac{1}{4}$. Also, $-N_{\delta,\mu}(y)$ is strictly CM on $(0,\infty)$ if and only if $\delta \leq 0$.

Proof. By using equations (1.2), (1.3) and the identity

$$\ln\left(\frac{h}{d}\right) = \int_0^\infty \frac{-e^{-hv}+e^{-dv}}{v}dv, \qquad h;d \in (0,\infty) \tag{3.1}$$

(see [1]), we have

$$\begin{aligned} N_{\delta,\mu}'(y) \;&=\; \psi_\mu(y)+\frac{\psi_\mu'(y)}{2}-\ln\left(\frac{y\mu}{y+1+\mu}\right) \\ &\quad -1/6\left((y-\delta)^{-2}-\frac{1}{(\frac{3}{4}+y+\mu)^2}\right) = \int_0^\infty \frac{e^{-(y+1+\mu)v}}{v(e^v-1)}\phi(v)dv, \end{aligned}$$

where

$$\begin{aligned}\phi(v) &= e^{(\mu+2)v}+1-e^{v}-e^{(1+\mu)v}-v\Big[e^{(2+\mu)v}-e^{v}\Big]-\frac{v^2}{2}\Big[e^{v}-e^{(2+\mu)v}\Big]\\ &- \frac{v^2}{6}\left[e^{\frac{v}{4}}-e^{\frac{5v}{4}}-\Big(e^{(1+\mu)v}-e^{(2+\mu)v}\Big)e^{\delta v}\right].\end{aligned}$$

Letting $\delta \geq \frac{1}{4}$ yields

$$\begin{aligned}\phi(v) &\leq e^{(\mu+2)v}+1-e^{v}-e^{(1+\mu)v}-v\Big[e^{(2+\mu)v}-e^{v}\Big]-\frac{v^2}{2}\Big[e^{v}-e^{(2+\mu)v}\Big]\\ &- \frac{v^2}{6}\Big[e^{\frac{v}{4}}-e^{\frac{5v}{4}}-e^{(\frac{5}{4}+\mu)v}+e^{(\frac{9}{4}+\mu)v}\Big]\end{aligned}$$

and hence

$$\phi(v) \leq -\frac{11(1+\mu)v^6}{2880}+\sum_{u=5}^{\infty}\frac{S_\mu(u)}{(2+u)!}v^{2+u} < 0,$$

where

$$\begin{aligned}S_\mu(u) &= -1-(1+\mu)^{u+2}+(2+\mu)^{u+2}+(u+2)\Big[1-(2+\mu)^{u+1}\Big]\\ &\quad -\frac{(u+1)(u+2)}{6}\left[4^{-u}-\left(\frac{5}{4}\right)^{u}+3-\left(\frac{5}{4}+\mu\right)^{u}+\left(\frac{9}{4}+\mu\right)^{u}-3(2+\mu)^{u}\right]\\ &= \sum_{s=1}^{u-4}\left[\binom{u+2}{s}-(u+2)\binom{u+1}{s}+\frac{(u+1)(u+2)}{6}\binom{u}{s}\left(3+\left(\frac{1}{4}\right)^{u-s}-\left(\frac{5}{4}\right)^{u-s}\right)\right](1+\mu)^{s}\\ &\quad -\frac{11\binom{2+u}{5}(1+\mu)^{u-3}}{24}\\ &< -\sum_{s=1}^{u-4}\frac{(1+u)(2+u)}{6(2+u-s)}\binom{u}{s}\left[(2+u-s)\Big(\left(\frac{5}{4}\right)^{u-s}-\left(\frac{1}{4}\right)^{u-s}\Big)-3(u-s)\right](1+\mu)^{s}\\ &= -\sum_{s=1}^{u-4}\frac{(1+u)(2+u)}{6(2+u-s)}\binom{u}{s}\left[(2+u-s)\sum_{l=2}^{u-s-1}\binom{u-s}{l}4^{-l}+\frac{(u-s-4)(u-s-2)}{4}\right](1+\mu)^{s}\\ &< 0.\end{aligned}$$

It follows that $-N'_{\delta,\mu}(y)$ is strictly CM on $(0,\infty)$ for $\delta \geq \frac{1}{4}$. Thus $N_{\delta,\mu}(y)$ is strictly decreasing on $y>0$ and by using (1.7) and (1.8), we get $\lim_{y\to\infty} N_{\delta,\mu}(y)=0$ and hence $N_{\delta,\mu}(y)>0$. Then, for $\delta \geq \frac{1}{4}$, the function $N_{\delta,\mu}(y)$ is strictly CM on $(0,\infty)$. On the other hand, if $N_{\delta,\mu}(y)$ is strictly CM on $(0,\infty)$, then by using again (1.7) and (1.8), we get

$$\begin{aligned}\lim_{y\to\infty}\frac{y^2}{(1+\mu)}N_{\delta,\mu}(y) &= \lim_{y\to\infty}\frac{y^2}{(1+\mu)}\Bigg[\frac{\psi_\mu(y)}{2}+\ln\Gamma_\mu(y)-y\ln\left(\frac{\mu y}{1+y+\mu}\right)-\ln\left(\mu!\,\sqrt{\mu}\,e^{1+\mu}\right)\\ &\quad +(\mu+1)\ln(y+1+\mu)\Bigg]+\lim_{y\to\infty}\frac{y^2}{6(1+\mu)}\left(\frac{\mu+\frac{3}{4}+\delta}{(y-\delta)(\frac{3}{4}+y+\mu)}\right)\\ &= -\frac{1}{6}+\frac{\mu+\frac{3}{4}+\delta}{6(1+\mu)}=\frac{\delta-\frac{1}{4}}{6(1+\mu)}\geq 0\end{aligned}$$

and then $\delta\geq\frac{1}{4}$. Now for $\delta\leq 0$, we get $N'_{\delta,\mu}(y)=\int_0^\infty\frac{e^{-(1+y+\mu)v}}{v(e^v-1)}\chi(v)dv$, where

$$\chi(v)\geq\sum_{u=2}^{\infty}\left[\sum_{s=1}^{u-1}\binom{u}{s}\left(\frac{2(u-s-1)(1+\mu)^s}{(2+u-s)}+\left(\frac{1}{4}\right)^s\right)\right]\frac{v^{2+u}}{6\,u!}$$

and consequently, for $\delta\leq 0$, the function $N'_{\delta,\mu}(y)$ is strictly CM on $(0,\infty)$. We thus get $N_{\delta,\mu}(y)$ is strictly increasing on $\mathbb{R}^+$ and also $\lim_{y\to\infty}N_{\delta,\mu}(y)=0$ and hence $N_{\delta,\mu}(y)<0$. Then, for $\delta\leq 0$, the function $-N_{\delta,\mu}(y)$ is strictly CM on $(0,\infty)$. On the contrary, if $-N_{\delta,\mu}(y)$ is strictly CM, then we get for $y\in(0,\infty)$, $\mu\in\mathbb{N}$:

$$\delta\leq\lim_{y\to 0}\left[\frac{1}{6\,H_\mu(y)-\left(y+\frac{3}{4}+\mu\right)^{-1}}+y\right]=\lim_{y\to 0}\left[\frac{1}{6\,H_\mu(y)-\left(y+\frac{3}{4}+\mu\right)^{-1}}\right],$$

where

$$H_\mu(y)=\frac{\psi_\mu(y)}{2}+\ln\Gamma_\mu(y)-y\ln\left(\frac{y\mu}{1+y+\mu}\right)+(1+\mu)\ln(1+y+\mu)-\ln\left(\mu!\,e^{1+\mu}\,\sqrt{\mu}\right)$$

Using the functional equations (1.1) and (1.4), we get $\delta\leq\lim_{y\to 0}\frac{y}{A_\mu(y)}=0$, where

$$\begin{aligned}A_\mu(y) &= 6\Bigg(\frac{y\,\psi_\mu(y+1)}{2}-y(y+1)\ln\left(\frac{y\mu}{y+1+\mu}\right)+(1+\mu)y\ln(y+1+\mu)\\ &\quad +y\ln\Gamma_\mu(1+y)-\frac{(1+\mu)}{2(1+y+\mu)}-y\ln\left(\mu!\,e^{1+\mu}\,\sqrt{\mu}\right)\Bigg)-\frac{4y}{3+4y+4\mu}\end{aligned}$$

□

Theorem 3.2. *Let $\mu\in\mathbb{N}$, $y>0$ and $\rho\geq 0$. Then the function*

$$T_{\rho,\mu}(y)=-(y-1/2)\ln\left(\frac{y\mu}{1+y+\mu}\right)+(1+\mu)\ln(y+\mu+1)+\ln\Gamma_\mu(y)-\ln\left(\mu!\,e^{1+\mu}\,\sqrt{\mu}\right)$$
$$-\frac{\psi'_\mu(\rho+y)}{12}$$

is strictly CM on $(0,\infty)$ if and only if $\rho\geq\frac{1}{2}$. The function $-T_{\rho,\mu}(y)$ is also strictly CM on $(0,\infty)$ if and only if $\rho=0$.

Proof. Using equations (1.2), (1.3) and the identity (3.1) lead us to

$$T'_{\rho,\mu}(y) = 1/2\Big(y^{-1} - (1+y+\mu)^{-1}\Big) + \ln\left(\frac{y+1+\mu}{y\ \mu}\right) - \frac{\psi''_\mu(\rho+y)}{12} + \psi_\mu(y)$$
$$= \int_0^\infty \frac{e^{-(1+y+\mu)v}}{v(e^v-1)}\Omega(v)dv,$$

where

$$\Omega(v) = 1 - e^{(1+\mu)v} - e^v + e^{(2+\mu)v} + \frac{v}{2}\Big[1 - e^{(1+\mu)v} + e^v - e^{(2+\mu)v}\Big] - \frac{v^3}{12}\Big[e^v - e^{(2+\mu)v}\Big]e^{-\rho v}$$

Let $\rho \geq \frac{1}{2}$, then we obtain

$$\Omega(v) \leq 1 - e^{(1+\mu)v} - e^v + e^{(2+\mu)v} + \frac{v}{2}\Big[1 - e^{(1+\mu)v} + e^v - e^{(2+\mu)v}\Big] - \frac{v^3}{12}\Big[e^{\frac{v}{2}} - e^{(\frac{3}{2}+\mu)v}\Big]$$

and hence

$$\Omega(v) \leq \frac{-(1+\mu)v^6}{480} + \sum_{u=4}^{\infty}\frac{h_\mu(u)}{(3+u)!}v^{u+3} < 0,$$

where

$$\begin{aligned}
h_\mu(u) &= (2+\mu)^{3+u} - 1 - (1+\mu)^{3+u} + \frac{(3+u)}{2}\Big[1 - (1+\mu)^{2+u} - (2+\mu)^{2+u}\Big] \\
&\quad - \frac{(1+u)(2+u)(3+u)}{12}\left[2^{-u} - \left(\frac{3}{2}+\mu\right)^u\right] \\
&= \sum_{s=1}^{u-3}\left[\binom{3+u}{s} - \frac{(3+u)}{2}\binom{u+2}{s} + \frac{(1+u)(2+u)(3+u)}{12}\binom{u}{s}2^{s-u}\right](1+\mu)^s \\
&\quad - \frac{\binom{3+u}{5}(1+\mu)^{u-2}}{4} \\
&< -\sum_{s=1}^{u-3}\frac{(1+u)(2+u)(3+u)}{12\,(2^{u-s})\,(3+u-s)(2+u-s)}\binom{u}{s}\left[2(u-s)(u-s-1) + 6\sum_{l=3}^{u-s}\binom{u-s}{l}\right](1+\mu)^s \\
&< 0.
\end{aligned}$$

It follows that $-T'_{\rho,\mu}(y)$ is strictly CM on $(0,\infty)$ for $\rho \geq \frac{1}{2}$ and hence $T_{\rho,\mu}(y)$ is strictly decreasing on $(0,\infty)$ and by using (1.7) and (1.9), we get $\lim_{y\to\infty} T_{\rho,\mu}(y) = 0$ and then $T_{\rho,\mu}(y) > 0$. Then $T_{\rho,\mu}(y)$ is strictly CM on $(0,\infty)$ for $\rho \geq \frac{1}{2}$. On the contrary, if $T_{\rho,\mu}(y)$ is strictly CM, then we get for $\mu \in \mathbb{N},\ y > 0:$

$$\frac{y^3}{(1+\mu)}T_{\rho,\mu}(y) = \frac{y^3}{(1+\mu)}\Big[Q_\mu(y) + R_\mu(y)\Big] > 0, \tag{3.2}$$

where

$$Q_\mu(y) = (1+\mu)\ln(1+y+\mu) + \left(\frac{1-2y}{2}\right)\ln\left(\frac{\mu y}{1+y+\mu}\right) + \ln\left(\frac{\Gamma_\mu(y)}{\sqrt{\mu}\ \mu!\ e^{1+\mu}}\right)$$
$$-\frac{(1+\mu)}{12y(1+y+\mu)}$$

and

$$R_\mu(y) = -\frac{1}{12}\left[\psi'_\mu(\rho+y) - \frac{(1+\mu)}{y(1+y+\mu)}\right]$$

From (1.7), we have $\lim\limits_{y\to\infty} \frac{y^3}{(1+\mu)} Q_\mu(y) = 0$ and by using (1.9), we have

$$\begin{aligned}\lim_{y\to\infty} \frac{y^3}{(1+\mu)} R_\mu(y) &= \lim_{y\to\infty} \frac{y^3}{12(1+\mu)}\left[\frac{\rho(1+\mu)(1+\rho+\mu+2y)}{y(\rho+y)(1+\mu+y)(1+\rho+\mu+y)}\right] \\ &\quad - \lim_{y\to\infty} \frac{y^3}{12(1+\mu)}\left[\frac{(1+\mu)(1+2\rho+\mu+2y)}{2(\rho+y)^2(1+\rho+\mu+y)^2}\right] = \frac{2\rho-1}{12}.\end{aligned}$$

From (3.2), we conclude that $0 + \frac{2\rho-1}{12} \geq 0$ and then $\rho \geq \frac{1}{2}$. Now for $\rho = 0$, we get

$$T'_{0,\mu}(y) = \int_0^\infty \left[\sum_{u=2}^{\infty} \frac{v^u}{u!}\left(\sum_{s=1}^{u-1} \frac{(u-s)(5+u-s)}{12(3+u-s)(2+u-s)}\binom{u}{s}(1+\mu)^s\right)\right] \frac{v^2\, e^{-(1+y+\mu)v}}{(e^v-1)} dv,$$

and therefore $T'_{0,\mu}(y)$ is strictly CM on $(0,\infty)$. We thus get $T_{0,\mu}(y)$ is strictly increasing on $(0,\infty)$ with $\lim\limits_{y\to\infty} T_{0,\mu}(y) = 0$ and then $T_{0,\mu}(y) < 0$. It follows that $-T_{0,\mu}(y)$ is strictly CM on $(0,\infty)$. On the other hand, we suppose that $-T_{\rho,\mu}(y)$ is strictly CM on $(0,\infty)$ with $\rho > 0$, then $T_{\rho,\mu}(y) < 0$ on $(0,\infty)$, which contradicts with $\lim\limits_{y\to 0} T_{\rho,\mu}(x) = \infty$ for $\rho > 0$, and hence $\rho = 0$. ☐

Theorem 3.3. *Let $\mu \in \mathbb{N}$, $y > 0$ and the function*

$$U_{\beta,\mu}(y) = T_{0,\mu}(y) + \frac{1}{24}\left[\frac{1}{(y-\beta)^2} - \frac{1}{(y+\mu+\frac{4}{5})^2}\right],$$

where $T_{0,\mu}(y)$ is given in Theorem 3.2 for $\rho = 0$. Then $U_{\beta,\mu}(y)$ is strictly CM on $(0,\infty)$ if and only if $\beta \geq \frac{1}{5}$ and $-U_{\alpha,\mu}(y)$ is strictly CM on $(0,\infty)$ if and only if $\beta \leq 0$.

Proof. Taking the derivative of the function $U_{\beta,\mu}(y)$ with respect to y and using equations (1.2), (1.3) and the identity (3.1) yields

$$U'_{\beta,\mu}(y) = \int_0^\infty \frac{e^{-(y+\mu+1)v}}{v(e^v-1)} \sigma(v) dv,$$

where

$$\begin{aligned}\sigma(v) &= 1 - e^{(1+\mu)v} - e^v + e^{(2+\mu)v} + \frac{v}{2}\left[1 - e^{(1+\mu)v} + e^v - e^{(2+\mu)v}\right] - \frac{v^3}{12}\left[e^v - e^{(2+\mu)v}\right] \\ &\quad - \frac{v^3}{24}\left[e^{\frac{v}{5}} - e^{\frac{6v}{5}} - \left(e^{(1+\mu)v} - e^{(2+\mu)v}\right)e^{\beta v}\right].\end{aligned}$$

Let $\beta \geq \frac{1}{5}$, then we obtain

$$\begin{aligned}\sigma(v) &\leq 1 - e^{(1+\mu)v} - e^v + e^{(2+\mu)v} + \frac{v}{2}\left[1 - e^{(1+\mu)v} + e^v - e^{(2+\mu)v}\right] - \frac{v^3}{12}\left[e^v - e^{(2+\mu)v}\right] \\ &\quad - \frac{v^3}{24}\left[e^{\frac{v}{5}} - e^{\frac{6v}{5}} - e^{(\frac{6}{5}+\mu)v} + e^{(\frac{11}{5}+\mu)v}\right].\end{aligned}$$

and hence

$$\sigma(v) \leq \sum_{u=4}^{\infty} \frac{S_\mu(u)}{(3+u)!} v^{3+u} < 0,$$

where

$$\begin{aligned} S_\mu(u) &= -\sum_{s=1}^{u-3} \frac{(1+u)(2+u)(3+u)}{24(3+u-s)(2+u-s)} \binom{u}{s} \left[\frac{(u-3-s)(u-2-s)(u+5-s)}{5} \right. \\ &\left. +(3+u-s)(2+u-s) \sum_{l=2}^{u-1-s} \binom{u-s}{l} 5^{-l} \right] (1+\mu)^s < 0 \end{aligned}$$

Consequently, $-U'_{\beta,\mu}(y)$ is strictly CM on $(0,\infty)$ for $\beta \geq \frac{1}{5}$. Thus $U_{\beta,\mu}(y)$ is strictly decreasing on $(0,\infty)$ and also $\lim_{y\to\infty} U_{\beta,\mu}(y) = 0$ and then $U_{\beta,\mu}(y) > 0$. Then $U_{\beta,\mu}(y)$ is strictly CM on $(0,\infty)$ for $\beta \geq \frac{1}{5}$. On the other hand, if $U_{\beta,\mu}(y)$ is CM, then we get for $y > 0,\ \ \mu \in \mathbb{N}$:

$$\frac{y^3}{(1+\mu)} U_{\beta,\mu}(y) = \frac{y^3}{(1+\mu)} T_{0,\mu}(y) + \frac{y^3}{24(1+\mu)} \left(\frac{1}{(y-\beta)^2} - \frac{1}{(y+\mu+\frac{4}{5})^2} \right) > 0. \tag{3.3}$$

By using the equations (1.7) and (1.9)and the fact that $\lim_{y\to\infty} \frac{y^3}{(1+\mu)} T_{0,\mu}(y) = -\frac{1}{12}$, we have

$$\lim_{y\to\infty} \frac{y^3}{(1+\mu)} U_{\beta,\mu}(y) = -\frac{1}{12} + \frac{\mu+\frac{4}{5}+\beta}{12(1+\mu)} = \frac{\beta-\frac{1}{5}}{12(1+\mu)} \geq 0.$$

Then from (3.3), we conclude that $\beta \geq \frac{1}{5}$. Now for $\beta \leq 0$, we get

$$U'_{\beta,\mu}(y) = \int_0^\infty \frac{e^{-(y+\mu+1)v}}{v(e^v-1)} \nu(v) dv,$$

where

$$\nu(v) \geq \sum_{u=2}^{\infty} \left[\sum_{s=1}^{u-1} \binom{u}{s} \left(\frac{(u-s-1)(6+u-s)(1+\mu)^s}{(3+u-s)(2+u-s)} + 5^{-s} \right) \right] \frac{v^{3+u}}{24\ u!}.$$

Consequently, for $\beta \leq 0$, the function $U'_{\beta,\mu}(y)$ is strictly CM on $(0,\infty)$. Thus $U_{\beta,\mu}(y)$ is strictly increasing on $(0,\infty)$ with $\lim_{y\to\infty} U_{\beta,\mu}(y) = 0$, since $\lim_{y\to\infty} T_{0,\mu}(y) = 0$ and hence $U_{\beta,\mu}(y) < 0$. Then, for $\beta \leq 0$, the function $-U_{\beta,\mu}(y)$ is strictly CM on $(0,\infty)$. On the contrary, if $-U_{\beta,\mu}(y)$ is strictly CM, then we get $U_{\beta,\mu}(y) < 0$ for $y > 0$ and $\mu \in \mathbb{N}$ and it follows that

$$\beta \leq -\lim_{y\to 0} \frac{1}{\sqrt{-24\ T_{0,\mu}(y) + \frac{1}{(y+\mu+\frac{4}{5})^2}}}.$$

Using (1.1) and (1.4), we get $\beta \leq -\lim_{y\to 0} \frac{y}{\sqrt{B_\mu(y)}} = 0$, where

$$\begin{aligned} B_\mu(y) &= -24 \Big[-y^2 (y+1/2) \ln \left(\frac{y\mu}{1+y+\mu} \right) + y^2(\mu+1) \ln (y+\mu+1) + y^2 \ln \Gamma_\mu(1+y) \\ &-y^2 \ln \left(\mu!\ e^{1+\mu}\ \sqrt{\mu} \right) - \frac{y^2}{12} \psi'_\mu(y+1) - \frac{(1+\mu)(1+\mu+2y)}{12(y+1+\mu)^2} \Big] + \frac{25y^2}{(5y+5\mu+4)^2}. \end{aligned}$$

□

From (1.2), we deduce the following lemma:

Lemma 3.1. *Assume that* $y > 0$ *and* $\mu \in \mathbb{N}$. *Then we have*

$$\lim_{y \to 0} y^{1+u}\, \psi_\mu^{(u)}(y) = (-1)^{1+u}\, u!, \qquad u \in \mathbb{N} \cup \{0\} \tag{3.4}$$

and

$$\lim_{y \to 0} y^{u}\, \psi_\mu^{(1+u)}(b+y) = 0, \qquad b \in (0,\infty), \quad u \in \mathbb{N}. \tag{3.5}$$

4. Some bounds for the generalized Γ_μ, ψ_μ and $\psi_\mu^{(u)}$ functions

Let us mention some important consequences of the Theorems 3.1, 3.2 and 3.3.

Corollary 4.1. *For* $y \in (0,\infty)$, $\mu \in \mathbb{N}$ *and* $a \in \mathbb{R}$,

$$\sqrt{\frac{y\mu}{y+\mu+1}}\exp\left[-\frac{\psi_\mu(y)}{2} - 1/6\left((y-a)^{-1} - \left(y+\mu+\frac{3}{4}\right)^{-1}\right)\right] < \frac{\Gamma_\mu(y)}{\left(\frac{\mu^y\, y^{y-\frac{1}{2}}\, e^{\mu+1}\, \mu!}{(y+\mu+1)^{y+\mu+\frac{1}{2}}}\right)}$$

$$< \sqrt{\frac{y\mu}{y+\mu+1}}\exp\Big[-\frac{\psi_\mu(y)}{2} - 1/6\Big(y^{-1} - \frac{1}{y+\mu+\frac{3}{4}}\Big)\Big], \tag{4.1}$$

where the constant $a = \frac{1}{4}$ *is the best.*

Proof. If the left-hand side (L.H.S) of (4.1) holds, then we get

$$\frac{y^2}{(1+\mu)}\, N_{a,\mu}(y) > 0, \qquad y \in (0,\infty), \quad \mu \in \mathbb{N},$$

which yields $a \geq \frac{1}{4}$ as shown in the proof of Theorem 3.1 and by using the decreasing property of the function $\frac{1}{y^s}$ on $(0,\infty)$ for $s \in \mathbb{N}$, we conclude that the constant $a = \frac{1}{4}$ is the best in (4.1). Next, the right- hand side (R.H.S) of (4.1) follows from $N_{0,\mu}(y) < 0$ in Theorem 3.1. □

Remark 4.1. Letting $\mu \to \infty$ in (4.1) yields (1.5).

Remark 4.2. For every $y \in (0,\infty)$ and $\mu \in \mathbb{N}$, the upper bound of (4.1) refines the upper bound of (1.10).

Corollary 4.2. *For* $a \in \mathbb{R}$, $y \in (0,\infty)$ *and* $\mu \in \mathbb{N}$,

$$\frac{1}{6}\left(\frac{1}{(y+\mu+\frac{3}{4})^2} - \frac{1}{(y-a)^2}\right) + \frac{\psi'_\mu(y)}{2} < \ln\left(\frac{y\mu}{y+\mu+1}\right) - \psi_\mu(y)$$

$$< \frac{1}{6}\left(\frac{1}{(y+\mu+\frac{3}{4})^2} - \frac{1}{y^2}\right) + \frac{\psi'_\mu(y)}{2}, \tag{4.2}$$

where the constant $a = \frac{1}{4}$ *being the best.*

Proof. If the L.H.S of (4.2) is correct, then we get $\frac{y^3}{(1+\mu)}\, N'_{a,\mu}(x) < 0$ and by using (1.8) and (1.9), we deduce that

$$\begin{aligned}\lim_{y\to\infty} \frac{y^3}{(1+\mu)}\, N'_{a,\mu}(y) &= \lim_{y\to\infty} \frac{y^3}{(1+\mu)}\Bigg[1/2\, \psi'_\mu(y) + \psi_\mu(y) - \ln\left(\frac{y\mu}{y+\mu+1}\right) \\ &\quad + \frac{1}{6}\Big(\frac{1}{(y+\mu+\frac{3}{4})^2} - (y-a)^{-2}\Big)\Bigg] \\ &= \frac{1}{3} - \frac{\mu+a+\frac{3}{4}}{3(1+\mu)} = \frac{\frac{1}{4}-a}{3(1+\mu)} \le 0\end{aligned}$$

which gives $a \ge \frac{1}{4}$ and by the same way as in the proof of Corollary 4.1, the constant $a = \frac{1}{4}$ is the best for (4.2). Finally, the R.H.S of (4.2) comes from $N'_{0,\mu}(y) > 0$ in Theorem 3.1. □

Remark 4.3. For every $y \in (0,\infty)$ and $\mu \in \mathbb{N}$, the upper bound of (4.2) refines the upper bound of (1.11).

Corollary 4.3. *For $y \in (0,\infty)$, $a \in \mathbb{R}$, $\mu \in \mathbb{N}$ and $u = 2, 3, \cdots$, we have*

$$\frac{(-1)^{u+1}}{2}\psi_\mu^{(u)}(y) - \frac{u!}{6}\left(\frac{1}{(y-a)^{1+u}} - \frac{1}{(y+\mu+\frac{3}{4})^{1+u}}\right) < (-1)^u\psi_\mu^{(u-1)}(y)$$

$$-(u-2)!\left(y^{1-u} - (y+\mu+1)^{1-u}\right) < \frac{(-1)^{1+u}\,\psi_\mu^{(u)}(y)}{2} - \frac{u!}{6}\left(y^{-1-u} - \left(y+\mu+\frac{3}{4}\right)^{-1-u}\right), \tag{4.3}$$

where the constant $a = \frac{1}{4}$ is the best.

Proof. Firstly, if the L.H.S of (4.3) is valid, then we get $\frac{y^{u+2}}{(1+\mu)}\,(-1)^u N_{a,\mu}^{(u)}(y) > 0$ for $u = 2, 3, \cdots$, and by using (1.9), we conclude that

$$\begin{aligned}\lim_{y\to\infty} \frac{y^{u+2}}{(1+\mu)}\,(-1)^u N_{a,\mu}^{(u)}(y) &= \lim_{y\to\infty} \frac{y^{2+u}}{(1+\mu)}\Bigg[\frac{(-1)^u\,\psi_\mu^{(u)}(y)}{2} + (-1)^u\psi_\mu^{(u-1)}(y) \\ &\quad - \frac{(u-2)!}{y^{u-1}} + \frac{(u-2)!}{(y+\mu+1)^{u-1}}\Bigg] \\ &\quad + \lim_{y\to\infty} \frac{y^{2+u}}{6(1+\mu)}\left(\frac{u!}{(y-a)^{1+u}} - \frac{u!}{(y+\mu+\frac{3}{4})^{1+u}}\right) \\ &= -\frac{(1+u)!}{6} + \frac{(1+u)!\,(\mu+a+\frac{3}{4})}{6(1+\mu)} = \frac{(1+u)!\,(a-\frac{1}{4})}{6(1+\mu)} \\ &\ge 0,\end{aligned}$$

which yields $a \ge \frac{1}{4}$ and similarly as before, the constant $a = \frac{1}{4}$ being the best in (4.3). Next, the R.H.S of (4.3) is deduced from $(-1)^u N_{0,\mu}^{(u)}(y) < 0$ for $u = 2, 3, \cdots$, in Theorem 3.1. □

Remark 4.4. For every $y \in (0,\infty)$, $\mu \in \mathbb{N}$ and $u = 2, 3, \cdots$, the upper bound of (4.3) refines the upper bound of (1.12).

Corollary 4.4. *For $y \in (0,\infty)$, $a;b \in \mathbb{R}^+ \cup \{0\}$ and $\mu \in \mathbb{N}$,*

$$\exp\Big[\frac{\psi'_\mu(a+y)}{12}\Big] < \frac{\Gamma_\mu(y)}{\left(\frac{\mu^y\, y^{\frac{2y-1}{2}}\, \mu!}{(y+1+\mu)^{\frac{1}{2}+y+\mu}\, e^{-1-\mu}}\right)} < \exp\Big[\frac{\psi'_\mu(b+y)}{12}\Big], \tag{4.4}$$

where the constants $a=\frac{1}{2}$ and $b=0$ are the best.

Proof. If the L.H.S of (4.4) is valid, then we obtain $\frac{y^3}{(1+\mu)}\, T_{a,\mu}(y) > 0$ and consequently, we have $a \geq \frac{1}{2}$ as noted in the proof of Theorem 3.2 and by using the decreasing property of the function $\psi'_\mu(y)$ on $(0,\infty)$ yields the constant $a=\frac{1}{2}$ is the best in (4.4). Next, the R.H.S of (4.4) for $b=0$ follows from $T_{0,\mu}(y) < 0$ in Theorem 3.2. Now, if the R.H.S of (4.4) is true for $y \in (0,\infty)$ and $b \in \mathbb{R}^+$, then we have

$$\lim_{y\to 0} \Gamma_\mu(1+y) < \left(\frac{\mu\, e^{1+\mu}\, \mu!}{(1+\mu)^{\mu+\frac{3}{2}}}\right) \exp\Big[\frac{\psi'_\mu(b)}{12}\Big] \lim_{y\to 0}\, y^{\frac{1+2y}{2}},$$

which yields $\Gamma_\mu(1) < 0$ and this contradicts with $\Gamma_\mu(1) = \frac{\mu}{1+\mu}$ for $\mu \in \mathbb{N}$. Hence the constant $b=0$ being the best in (4.4). ☐

Remark 4.5. Letting $\mu \to \infty$ in (4.4) yields (1.6).

Remark 4.6. By using (2.1), we deduce that, for every $y \in (0,\infty)$ and $\mu \in \mathbb{N}$, the lower bound of (4.4) refines the lower bound of (1.10).

Remark 4.7. By using (2.3), we deduce that, for every $y \in (0.25,\infty)$ and $\mu \in \mathbb{N}$, the lower bound of (4.4) refines the lower bound of (4.1).

Corollary 4.5. *For $y \in (0,\infty)$, $a;b \in \mathbb{R}^+ \cup \{0\}$ and $\mu \in \mathbb{N}$,*

$$\frac{\mu+1}{2y(y+1+\mu)} - \frac{\psi''_\mu(a+y)}{12} < \ln\left(\frac{y\mu}{y+1+\mu}\right) - \psi_\mu(y) < \frac{\mu+1}{2y(y+1+\mu)} - \frac{\psi''_\mu(y+b)}{12}, \tag{4.5}$$

where the constants $a=\frac{1}{2}$ and $b=0$ are the best.

Proof. If the L.H.S of (4.5) is correct, then we obtain

$$\frac{y^4}{(\mu+1)}\, T'_{a,\mu}(y) = \frac{y^4}{(\mu+1)}\, [M_\mu(y) + V_\mu(y)] < 0, \tag{4.6}$$

where

$$M_\mu(y) = \psi_\mu(y) - \ln\left(\frac{y\mu}{y+1+\mu}\right) + \frac{(\mu+1)}{2y(y+1+\mu)} + \frac{(\mu+1)(1+\mu+2y)}{12y^2(y+1+\mu)^2}.$$

and

$$V_\mu(y) = -\frac{(\mu+1)(1+\mu+2y)}{12y^2(y+1+\mu)^2} - \frac{\psi''_\mu(a+y)}{12}.$$

By using the asymptotic expansions (1.8) and (1.9), we have $\lim_{y\to\infty} \frac{y^4}{(1+\mu)} M_\mu(y) = 0$ and

$$\begin{aligned}
\lim_{y\to\infty} \frac{y^4}{(\mu+1)} V_\mu(y) &= \lim_{y\to\infty} \frac{y^4}{12(1+\mu)} \left[\frac{1+h(\mu)+(1+\mu)\Big(3y^2+3(1+2a+\mu)y\Big)}{(a+y)^3(1+a+\mu+y)^3} \right] \\
&- \lim_{y\to\infty} \frac{y^4}{12(1+\mu)} \left[\frac{a(\mu+1)\Big(s_\mu(y)+6y^4+12(1+a+\mu)y^3\Big)}{y^2(y+a)^2(1+\mu+y)^2(1+a+\mu+y)^2} \right] \\
&= \frac{1-2a}{4},
\end{aligned}$$

where $h(\mu) = 2\mu + \mu^2 + 3a(1+\mu+a)$ and

$$\begin{aligned}
s_\mu(y) = 2(4+9a+4a^2+8\mu+9ap+4\mu^2)y^2 + 2(1+a+\mu)(1+3a+a^2+2\mu+3a\mu+\mu^2)y \\
+a(1+\mu)(1+a+\mu)^2.
\end{aligned}$$

From (4.6), we conclude that $\frac{1-2a}{4} \le 0$ and then $a \ge \frac{1}{2}$ and by using the increasing property of the function $\psi''_\mu(x)$ on $(0,\infty)$, we deduce that the constant $a = \frac{1}{2}$ is the best in (4.5). Next, the R.H.S of (4.5) for $b = 0$ follows from $T'_{0,\mu}(y) > 0$ in Theorem 3.2. Finally, if the R.H.S of (4.4) holds for $y \in (0,\infty)$ and $b \in \mathbb{R}^+$, then we get

$$\lim_{y\to 0} -y \left[\psi_\mu(y) - \ln\left(\frac{y\mu}{1+y+\mu} \right) \right] < \frac{1}{2} - \frac{1}{12} \lim_{x\to 0} y\, \psi''_\mu(y+b)$$

and by using (3.4) and (3.5), we have $1 < \frac{1}{2}$, which is impossible. Then the constant $b = 0$ being the best in (4.5). ☐

Remark 4.8. Taking $u = 1$ in the inequality (2.2) leads us that for every $y > 0$ and $\mu \in \mathbb{N}$, the lower bound of (4.5) refines the lower bound of (1.11).

Remark 4.9. By using (2.4), we deduce that, for every $y > 0.25$ and $\mu \in \mathbb{N}$, the lower bound of (4.5) refines the lower bound of (4.2).

Corollary 4.6. *For $y \in (0,\infty)$, $a; b \in \mathbb{R}^+ \cup \{0\}$, $\mu \in \mathbb{N}$ and $u = 2, 3, \cdots$,*

$$\begin{aligned}
\frac{(-1)^u\, \psi_\mu^{(u+1)}(a+y)}{12} + \frac{(u-1)!}{2}\left[\frac{1}{y^u} - \frac{1}{(1+y+\mu)^u} \right] < (-1)^u \psi_\mu^{(u-1)}(y) \\
-\frac{(u-2)!}{y^{u-1}} + \frac{(u-2)!}{(1+y+\mu)^{u-1}} < \frac{(-1)^u}{12} \psi_\mu^{(1+u)}(b+y) + \frac{(u-1)!}{2}\left[\frac{1}{y^u} - \frac{1}{(1+y+\mu)^u} \right], \quad (4.7)
\end{aligned}$$

where constants $a = \frac{1}{2}$ and $b = 0$ are the best.

Proof. If the L.H.S of (4.7) is valid, then we get

$$\frac{y^{3+u}}{(1+\mu)} (-1)^u T_{a,\mu}^{(u)}(y) = \frac{y^{3+u}}{(1+\mu)} \Big[F_\mu(y) + L_\mu(y) \Big] > 0, \qquad u = 2, 3, \cdots \quad (4.8)$$

where

$$\begin{aligned}
F_\mu(y) &= (-1)^u \psi_\mu^{(u-1)}(y) - (u-2)!\Big(y^{1-u} - (y+1+\mu)^{1-u}\Big) - \frac{(u-1)!}{2}\Big(y^{-u} - (y+1+\mu)^{-u}\Big) \\
&\quad - \frac{u!}{12 y^{1+u}} + \frac{u!}{12(y+1+\mu)^{1+u}}
\end{aligned}$$

and

$$L_\mu(y) = \frac{u!}{12y^{1+u}} - \frac{u!}{12(y+1+\mu)^{1+u}} - \frac{(-1)^u}{12}\psi_\mu^{(1+u)}(y+a).$$

Using the asymptotic expansion (1.9), we have $\lim_{y\to\infty} \frac{y^{3+u}}{(1+\mu)} F_\mu(y) = 0$ and

$$\lim_{y\to\infty} \frac{y^{3+u}}{(1+\mu)} \left[\frac{u!}{12y^{1+u}} - \frac{u!}{12(y+1+\mu)^{1+u}} - \frac{(-1)^u}{12}\psi_\mu^{(1+u)}(y)\right] = -\frac{(u+2)!}{24}$$

and using (1.3), we obtain $\lim_{y\to\infty} \frac{(-1)^u y^{3+u}}{12(1+\mu)} \left[\psi_\mu^{(1+u)}(y) - \psi_\mu^{(1+u)}(a+y)\right] = \frac{(2+u)!\,a}{12}$ and hence, we obtain

$$\lim_{y\to\infty} \frac{y^{3+u}}{(1+\mu)} L_\mu(y) = \frac{(2+u)!\,(2a-1)}{24}.$$

From (4.8), we conclude that $\frac{(2+u)!\,(2a-1)}{24} \geq 0$ and then $a \geq \frac{1}{2}$ and by using the completely monotonic property of $-\psi_\mu''(y)$ on $(0,\infty)$, we deduce that $(-1)^{1+u}\,\psi_\mu^{(2+u)}(y) > 0$ on $(0,\infty)$ for $u \in \mathbb{N}\cup\{0\}$ and then $(-1)^u\,\psi_\mu^{(u+1)}(y)$ is decreasing on $(0,\infty)$, which proves that $a = \frac{1}{2}$ is the best in (4.7). Secondly, the R.H.S of (4.7) for $b = 0$ follows from $(-1)^u T_{0,\mu}^{(u)}(y) < 0$ in Theorem 3.2. Finally, if the R.H.S of (4.7) holds for $y \in (0,\infty)$, $b \in \mathbb{R}^+$ and $u = 2, 3, \cdots$, then we would have

$$\lim_{y\to 0} y^u \left[(-1)^u \psi_\mu^{(u-1)}(y) - \left(\frac{(u-2)!}{y^{u-1}} - \frac{(u-2)!}{(y+1+\mu)^{u-1}}\right)\right] < \frac{(u-1)!}{2} + \frac{(-1)^u}{12} \lim_{y\to 0} y^u \psi_\mu^{(1+u)}(b+y),$$

and by using (3.4) and (3.5), for $u = 2, 3, \cdots$, we would have $(u-1)! < \frac{(u-1)!}{2}$, which is impossible. Then the constant $b = 0$ is the best in (4.7). $\square$

Remark 4.10. By using (2.2), we conclude that for $u = 2, 3, \cdots$, $y > 0$ and $\mu \in \mathbb{N}$, the lower bound of (4.7) refines the lower bound of (1.12).

Remark 4.11. By using (2.5), we conclude that for $u = 2$, $y > 0.25$ and $\mu \in \mathbb{N}$, the lower bound of (4.7) refines the lower bound of (4.3).

Corollary 4.7. *For $y \in (0,\infty)$, $a \in \mathbb{R}$ and $\mu \in \mathbb{N}$,*

$$\exp\left[\frac{1}{12}\psi_\mu'(y) - \frac{1}{24}\Big(\frac{1}{(y-a)^2} - \frac{1}{(y+\mu+\frac{4}{5})^2}\Big)\right] < \frac{\Gamma_\mu(y)}{\left(\frac{\mu^y\, y^{y-\frac{1}{2}}\, e^{\mu+1}\, \mu!}{(x+\mu+1)^{y+\mu+\frac{1}{2}}}\right)} < \exp\left[\frac{1}{12}\psi_\mu'(y) - \frac{1}{24}\Big(\frac{1}{y^2} - \frac{1}{(y+\mu+\frac{4}{5})^2}\Big)\right], \tag{4.9}$$

where the constant $a = \frac{1}{5}$ being the best.

Proof. If the L.H.S of (4.9) holds, then we get $\frac{y^3}{(\mu+1)}\, U_{a,\mu}(y) > 0$ and this gives $a \geq \frac{1}{5}$ as shown in the proof of Theorem 3.3 and by the same way as in the proof of Corollary 4.1, the constant $a = \frac{1}{5}$ is the best in (4.9). Finally, the R.H.S of (4.9) follows from $U_{0,\mu}(y) < 0$ in Theorem 3.3. $\square$

Remark 4.12. Letting $\mu \to \infty$ in (4.9) yields

$$\exp\left(\frac{1}{12}\psi'(y) - \frac{1}{24(y-\frac{1}{5})^2}\right) < \frac{e^y\,\Gamma(y)}{\left(\sqrt{2\pi}\,y^{y-\frac{1}{2}}\right)} < \exp\left(\frac{1}{12}\psi'(y) - \frac{1}{24y^2}\right), \qquad y>0,$$

which refines the upper bound of the inequality (1.6).

Remark 4.13. For every $y \in (0,\infty)$ and $\mu \in \mathbb{N}$, the upper bound of (4.9) refines the upper bound of (4.4).

Corollary 4.8. *For $y \in (0,\infty)$, $a \in \mathbb{R}$ and $\mu \in \mathbb{N}$,*

$$\frac{1+\mu}{2y(y+\mu+1)} - \frac{\psi_\mu''(y)}{12} + \frac{1}{12}\left(\frac{1}{(y+\mu+\frac{4}{5})^3} - \frac{1}{(y-a)^3}\right) < \ln\left(\frac{\mu y}{y+\mu+1}\right) - \psi_\mu(y)$$
$$< \frac{1+\mu}{2y(y+\mu+1)} - \frac{\psi_\mu''(y)}{12} + \frac{1}{12}\left(\frac{1}{(y+\mu+\frac{4}{5})^3} - y^{-3}\right), \tag{4.10}$$

where the constant $a=\frac{1}{5}$ is the best.

Proof. If the L.H.S of (4.10) is correct, then we get $U'_{a,\mu}(y)<0$ and hence

$$\frac{y^4}{(\mu+1)}\,U'_{a,\mu}(y) = \frac{y^4}{(\mu+1)}\left[T'_{0,\mu}(y) - \frac{1}{12}\left(\frac{1}{(y-a)^3} - \frac{1}{(y+\mu+\frac{4}{5})^3}\right)\right] < 0.$$

Then

$$\lim_{x\to\infty}\frac{y^4}{(\mu+1)}\,U'_{a,\mu}(y) = \frac{1}{4} - \frac{\mu+a+\frac{4}{5}}{4(\mu+1)} = \frac{-a+\frac{1}{5}}{4(\mu+1)} \le 0$$

and consequently, we get $a \ge \frac{1}{5}$ and by the same way as before, the constant $a=\frac{1}{5}$ being the best in (4.10). Next, the R.H.S of (4.10) follows from $U'_{0,\mu}(y)>0$ in Theorem 3.3. □

Remark 4.14. For every $y \in (0,\infty)$ and $\mu \in \mathbb{N}$, the upper bound of (4.10) refines the upper bound of (4.5) .

Corollary 4.9. *For $y \in (0,\infty)$, $a \in \mathbb{R}$, $\mu \in \mathbb{N}$ and $u=2,3,\cdots$, we have*

$$\frac{(-1)^u}{12}\psi_\mu^{(u+1)}(y) - \frac{(u-1)!}{2(y+\mu+1)^u} + \frac{(u-1)!}{2y^u} - \frac{(u+1)!}{24(y-a)^{u+2}} + \frac{(u+1)!}{24(y+\mu+\frac{4}{5})^{u+2}}$$
$$< (-1)^u\psi_\mu^{(u-1)}(y) - \frac{(u-2)!}{y^{u-1}} + \frac{(u-2)!}{(y+\mu+1)^{u-1}} < \frac{(-1)^u\,\psi_\mu^{(u+1)}(y)}{12}$$
$$-\frac{(u-1)!}{2(y+\mu+1)^u} + \frac{(u-1)!}{2y^u} - \frac{(u+1)!}{24y^{u+2}} + \frac{(u+1)!}{24(y+\mu+\frac{4}{5})^{u+2}}, \tag{4.11}$$

where the constant $a=\frac{1}{5}$ is the best.

Proof. If the L.H.S of (4.11) is valid, then we get for $u=2,3,\cdots$:

$$\frac{y^{u+3}}{(1+\mu)}\,(-1)^u U_{a,\mu}^{(u)}(y) = \frac{y^{u+3}}{(\mu+1)}\left[(-1)^u T_{0,\mu}^{(u)}(y) + \frac{(u+1)!}{24(y-a)^{u+2}} - \frac{(u+1)!}{24(y+\mu+\frac{4}{5})^{u+2}}\right] > 0,$$

Then

$$\lim_{y\to\infty} \frac{y^{u+3}}{(1+\mu)} (-1)^u U_{a,\mu}^{(u)}(y) = \frac{-(u+2)!}{24} + \frac{(u+2)!\,(\mu+a+\frac{4}{5})}{24(\mu+1)} = \frac{(u+2)!\,(a-\frac{1}{5})}{24(\mu+1)} \geq 0$$

and consequently, we get $a \geq \frac{1}{5}$ and by the same manner as before, the constant $a = \frac{1}{5}$ is the best in (4.11). Finally, the R.H.S of (4.11) follows from $(-1)^u U_{0,\mu}^{(u)}(y) < 0$ for $u = 2, 3, \cdots$, in Theorem 3.3. □

Remark 4.15. For every $y \in (0, \infty)$, $\mu \in \mathbb{N}$ and $u = 2, 3, \cdots$, the upper bound of (4.11) refines the upper bound of (4.7).

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables,* Dover, New York, 1965.

[2] N. Batir, *On some properties of the gamma function,* Expo. Math., **26**(2)(2008), 187–196.

[3] T. Burić and N. Elezović, *Some completely monotonic functions related to the psi function,* Math. Inequal. Appl., **14**(3)(2011), 679–691.

[4] V. Krasniqi and F. Qi, *Complete monotonicity of a function involving the p-psi function and alternative proofs,* Glob. J. Math. Anal., **2**(3)(2014), 204–208.

[5] M. Mahmoud, H. Almuashi and H. Moustafa, *An Asymptotic Expansion for the Generalized Gamma Function,* Symmetry, **14**(7)(2022), 1412.

[6] F. Qi and W.-H. Lic, *Integral representations and properties of some functions involving the logarithmic function,* Filomat, **30**(7)(2016), 1659–1674.

[7] F. Qi and S.-H. Wang, *Complete monotonicity, completely monotonic degree, integral representations, and an inequality related to the exponential, trigamma, and modified Bessel functions,* Glob. J. Math. Anal., **2**(3)(2014), 91–97.

[8] F. Qi, S. Guo and B.-N. Guo, *Complete monotonicity of some functions involving polygamma functions,* J. Comput. Appl. Math., **233**(9)(2010), 2149–2160.

[9] J. Sandor, *Selected chapters of geometry, Analysis and number theory,* RGMIA Monographs, Victoria University, 2005.

[10] H. Şevli and N. Batir, *Complete monotonicity results for some functions involving the gamma and polygamma functions,* Math. Comput. Modelling, **53**(9-10)(2011), 1771–1775.

[11] D. V. Widder, *The Laplace transform,* Princeton University Press, Princeton, 1946.

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