Turkish J. Ineq., 6 (1) (2022), Pages 38 –47 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

TAUBERIAN THEOREMS FOR THE STATISTICALLY $\left(\overline{N},p\right)$ SUMMABLE INTEGRALS

CEMAL BELEN[1]

ABSTRACT. In this paper we consider the Tauberian conditions of slow decrease and slow oscillation with respect to P, where P is an indefinite Lebesgue integral of a locally integrable positive weight function. We prove that these are sufficient conditions to obtain ordinary limit at infinity of a real- or complex-valued measurable function from the existence of its statistical limit at infinity. Furthermore it is proved that ordinary limit of an integral function follows from the existence of statistical limit of its weighted mean at infinity.

1. Introduction and Preliminaries

Let $\mathbb{R}^+ := [0,\infty)$ and $f : \mathbb{R}^+ \to \mathbb{C}$ be a measurable (in Lebesgue's sense) function. Following Móricz (see, [8]) we say $f(t)$ has statistical limit at ∞ if there exists a number l such that for each $\varepsilon > 0$,

$$\lim_{a\to\infty} \frac{1}{a} \left|\{t \in [0,a) : |f(t) - l| > \varepsilon\}\right| = 0, \tag{1.1}$$

where by $|\{.\}|$, we denote the Lebesgue measure of the set $\{.\}$. If this is the case we write $st\text{-}\lim_{t\to\infty} f(t) = l$ or $f(t) \overset{st}{\to} l$. If the ordinary limit $f(t) \to l$ as $t \to \infty$ (in short, we always write $f(t) \to l$) exists then $f(t) \overset{st}{\to} l$ also exists. But the converse implication

$$f(t) \overset{st}{\to} l \Rightarrow f(t) \to l \tag{1.2}$$

is not true in general. For example, if we consider the measurable function defined by

$$f(t) = \begin{cases} k, & t \in \left(k^2, k^2+1\right) \\ 0, & \text{otherwise} \end{cases}, \quad k = 1,2,3,...,$$

Then $st\text{-}\lim_{t\to\infty} f(t) = 0$ but the limit $\lim_{t\to\infty} f(t)$ does not exist (*cf.* [11]).

Key words and phrases. Tauberian theorems, Statistical convergence, Weighted mean method of integrals, Slowly decreasing and oscillating functions.

2010 *Mathematics Subject Classification.* Primary: 40E05. Secondary:40C10, 40G15.

Received: 26/05/2022 *Accepted*: 14/06/2022.

Cited this article as: C. Belen, Tauberian theorems for the statistically $\left(\overline{N},p\right)$ summable integrals, Turkish Journal of Inequalities, 6(1) (2022), 38-47.

Let $p:\mathbb{R}^{+}\to\mathbb{R}^{+}$ be a function which is locally integrable (in Lebesgue's sense) on $\mathbb{R}^{+}$, in symbols: $p\in L^{1}_{loc}\left(\mathbb{R}^{+}\right)$. Suppose throughout that

$$p(x)>0 \qquad \text{for almost all } x\in\mathbb{R}^{+}, \tag{1.3}$$

$$P(t):=\int_0^t p(x)dx\to\infty \qquad \text{as } t\to\infty, \qquad P\left(0\right)=0, \tag{1.4}$$

and that

$$\frac{p\left(t\right)}{P\left(t\right)}\to 0, \qquad \text{as } t\to\infty. \tag{1.5}$$

Given a real- or complex-valued function $f\in L^{1}_{loc}\left(\mathbb{R}^{+}\right)$, we set

$$s\left(t\right)=\int_0^t f\left(x\right)dx \tag{1.6}$$

and

$$\sigma\left(t\right)=\frac{1}{P\left(t\right)}\int_0^t s\left(x\right)dP\left(x\right),$$

where the second integral exists in Riemann-Stieltjes sense. If the finite limit

$$\sigma\left(t\right)\to l$$

exists then we say that the function s is summable to l with respect to weight function p, or in short $\left(\overline{N},p\right)$ summable to l and we write $s\left(t\right)\to l\ \left(\overline{N},p\right)$.

Analogously to the discrete case, it is easy to check that if the condition (1.5) is satisfied and

$$s\left(t\right)\to l \tag{1.7}$$

then we also have

$$s\left(t\right)\to l\ \left(\overline{N},p\right) \tag{1.8}$$

(see, [12]). Moreover we say that the function s is statistically $\left(\overline{N},p\right)$ summable to l if $\sigma\left(t\right)\overset{st}{\to}l$. In this case we write

$$s\left(t\right)\overset{st}{\to}l\ \left(\overline{N},p\right). \tag{1.9}$$

On the other hand the existence of (1.7) implies that of (1.9), but the converse implication

$$s\left(t\right)\overset{st}{\to}l\ \left(\overline{N},p\right)\Rightarrow s\left(t\right)\to l \tag{1.10}$$

is not true in general.

Note that the assumption (1.3) implies that the function $P(t)$ defined by (1.4) is strictly increasing on $\mathbb{R}^{+}$. Since p is integrable over any bounded interval $[0,t]$, $0<t<\infty$, its indefinite Lebesgue integral $P\left(t\right)$ is absolutely continuous and so continuous on $\left[0,t\right]$. Hence its inverse function $P^{-1}\left(t\right)$ exists, it is continuous and strictly increasing on $\mathbb{R}^{+}$.

A function $s:\mathbb{R}^{+}\to\mathbb{R}$ is said to be slowly decreasing with respect to P (in the sense of Karamata [6]) if

$$\lim_{\lambda\to1^{+}}\liminf_{x\to\infty}\min_{x<t\leq X_{\lambda}}\left\{s\left(t\right)-s\left(x\right)\right\}\geq 0, \tag{1.11}$$

where

$$X_{\lambda}:=P^{-1}\left(\lambda P\left(x\right)\right), \quad x>0 \tag{1.12}$$

The condition (1.11) is satisfied if and only if for each $\varepsilon > 0$ there exist $x_0 = x_0(\varepsilon) > 0$ and $\lambda = \lambda(\varepsilon) > 1$, as close to 1 as we wish, such that

$$s(t) - s(x) \geq -\varepsilon \qquad \text{whenever} \qquad x_0 \leq x < t \leq X_\lambda. \tag{1.13}$$

We also say that a function $s : \mathbb{R}^+ \to \mathbb{C}$ is slowly oscillating with respect to P if

$$\lim_{\lambda \to 1^+} \limsup_{x \to \infty} \max_{x < t \leq X_\lambda} |s(t) - s(x)| = 0, \tag{1.14}$$

where X_λ is defined by (1.12). Condition (1.14) holds if and only if for each $\varepsilon > 0$ there exist $x_0 = x_0(\varepsilon) > 0$ and $\lambda = \lambda(\varepsilon) > 1$, as close to 1 as we wish, such that

$$|s(t) - s(x)| \leq \varepsilon \qquad \text{whenever} \qquad x_0 \leq x < t \leq X_\lambda. \tag{1.15}$$

Note that if $p(x) = 1$ for all $x > 0$ then $X_\lambda = \lambda x$ and in this case the conditions (1.11) and (1.14) are reduced to the discrete forms of slowly decreasing and slowly oscillating functions with respect to $(C, 1)$ summability due to Schmidt [6] and Hardy [5], respectively.

The aim of this paper is to verify the converse implications (1.2) and (1.10) under some conditions known as Tauberian conditions. The corresponding results are called Tauberian theorems. Such kinds of results for the ordinary and statistical weighted mean summable integrals have been obtained by various authors (see, e.g., [1–4, 9, 15, 17]). In particular, the following two classical Tauberian theorems were given in [2].

Theorem 1.1. *Let $p \in L^1_{loc}(\mathbb{R}^+)$ for which (1.3) and (1.4) are satisfied. If $f \in L^1_{loc}(\mathbb{R}^+)$ be a real-valued function such that its integral function $s(t)$ is slowly decreasing with respect to P, then the implication (1.8)⇒(1.7) holds true.*

Theorem 1.2. *Let $p \in L^1_{loc}(\mathbb{R}^+)$ for which (1.3) and (1.4) are satisfied. If $f \in L^1_{loc}(\mathbb{R}^+)$ be a complex-valued function such that its integral function $s(t)$ is slowly oscillating with respect to P, then the implication (1.8)⇒(1.7) holds true.*

In this paper we extend these results with the weaker assumption (1.9) (Theorem 2.3 and Theorem 2.4, below).

2. Main Results

First we state and prove some auxiliary results which will be useful in proofs of our main results.

Our first two lemmas below generalizes [10, Lemma 2-3] and [13, Lemma 1-2]. These results known as a Vijayaraghavan type lemma (see, [18, Lemma 6]) and they can be considered as a nondiscrete analogoues of [7, Lemma 2] and [14, Lemma 4.1], respectively, under less restrictive conditions.

Lemma 2.1. *Let $s : \mathbb{R}^+ \to \mathbb{R}$ be a function such that the condition (1.13) is satisfied only for $\varepsilon = 1$, where $x_0 > 0$ and $\lambda > 1$. Then there exists a positive constant B such that*

$$s(t) - s(x) \geq -B \log \frac{P(t)}{P(x)} \qquad \textit{whenever} \qquad x_0 \leq x \leq P^{-1}\left(\frac{1}{\lambda} P(t)\right). \tag{2.1}$$

Proof. Assume that $x_0 \leq x \leq P^{-1}\left(\frac{1}{\lambda}P(t)\right)$. Define

$$t_0 = t = P^{-1}(P(t)) \text{ and } t_p = P^{-1}\left(\frac{1}{\lambda}P(t_{p-1})\right), \quad p = 1, 2, ..., q+1 \tag{2.2}$$

where q is determined by the condition

$$t_{q+1} \leq x < t_q. \tag{2.3}$$

By (1.13) and (2.3) we have

$$s(t) - s(x) = \sum_{p=1}^{q} (s(t_{p-1}) - s(t_p)) + s(t_q) - s(x) \geq -q - 1. \tag{2.4}$$

By the assumption $x < t_q = P^{-1}\left(\frac{1}{\lambda^q}P(t)\right)$, we have

$$P(x) < \frac{1}{\lambda^q}P(t) \quad \text{ or equivalently } \quad q < \frac{1}{\log\lambda}\log\frac{P(t)}{P(x)}. \tag{2.5}$$

On the other hand, the assumption $x \leq P^{-1}\left(\frac{1}{\lambda}P(t)\right)$ implies that

$$\log\lambda < \log\frac{P(t)}{P(x)}. \tag{2.6}$$

Combining (2.4)-(2.6) we obtain that (2.1) with $B = 2/\log\lambda$. ☐

Lemma 2.2. *Let $s : \mathbb{R}^+ \to \mathbb{C}$ be a function such that the condition (1.15) is satisfied only for $\varepsilon = 1$, where $x_0 > 0$ and $\lambda > 1$. Then with $B = 2/\log\lambda$ we have*

$$|s(t) - s(x)| \leq B\log\frac{P(t)}{P(x)} \qquad \text{whenever} \quad x_0 \leq x \leq P^{-1}\left(\frac{1}{\lambda}P(t)\right). \tag{2.7}$$

Proof. Let $x_0 \leq x \leq P^{-1}\left(\frac{1}{\lambda}P(t)\right)$ and define $t_0, t_1, ..., t_{q+1}$ by (2.2) and (2.3). Using (1.15) and (2.4) we have

$$|s(t) - s(x)| \leq \sum_{p=1}^{q} |s(t_{p-1}) - s(t_p)| + |s(t_q) - s(x)| \geq q + 1. \tag{2.8}$$

Hence by (2.5) and (2.8), we get

$$|s(t) - s(x)| \leq 1 + \frac{1}{\log\lambda}\log\frac{P(t)}{P(x)}.$$

Now if we consider (2.6), we obtain (2.9) with $B = 2/\log\lambda$. ☐

Lemma 2.3. *Let $f \in L^1_{loc}(\mathbb{R}^+)$ be a real-valued function such that the assumptions of Lemma 2.1 are satisfied for its integral function $s(t)$. Then there exists a positive constant B_1 such that*

$$\frac{1}{P(t)}\int_{x_0}^{t} (s(t) - s(x))\, dP(x) \geq -B_1 \qquad \text{whenever} \;\; t > P^{-1}(\lambda P(x_0)). \tag{2.9}$$

Proof. By the assumption and (2.1), we have

$$\begin{aligned}\int_{x_0}^{t}(s(t)-s(x))\,dP(x) &= \int_{x_0}^{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}(s(t)-s(x))\,dP(x)\\ &\quad+\int_{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}^{t}(s(t)-s(x))\,dP(x)\\ &\geq -B\int_{x_0}^{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}\log\frac{P(t)}{P(x)}dP(x)-\int_{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}^{t}dP(x) \qquad (2.10)\\ &\geq -B\log P(t)\int_{x_0}^{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}dP(x)+B\int_{x_0}^{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}\log P(x)\,dP(x)\\ &\quad+\left(1-\frac{1}{\lambda}\right)P(t)\\ &\geq \frac{-B}{\lambda}(\log P(t))\,P(t)+B\int_{x_0}^{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}\log P(x)\,dP(x)\end{aligned}$$

By the condition (1.5) the function $\log P(x)$ has bounded derivative, hence it is absolutely continuous on any bounded interval in $\mathbb{R}^+$. Hence we can apply the integration by parts formula to the integral in the right hand side of (2.10). So we have

$$\begin{aligned}\int_{x_0}^{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}\log P(x)\,dP(x) &= \left[(\log P(x))\,P(x)\right]_{x_0}^{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}-\int_{x_0}^{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}dP(x)\\ &= \log\left(\frac{P(t)}{\lambda}\right)\frac{P(t)}{\lambda}-(\log P(x_0))\,P(x_0)-\frac{1}{\lambda}P(t)\\ &\quad+P(x_0) \qquad (2.11)\\ &= \frac{1}{\lambda}(\log P(t))\,P(t)-\frac{\log\lambda}{\lambda}P(t)-(\log P(x_0))\,P(x_0)\\ &\quad-\frac{1}{\lambda}P(t)+P(x_0)\,.\end{aligned}$$

On the other hand we have $\frac{P(x_0)}{P(t)}<\frac{1}{\lambda}$ whenever $t>P^{-1}(\lambda P(x_0))$. Now it follows from (2.10) that

$$\begin{aligned}\int_{x_0}^{t}(s(t)-s(x))\,dP(x) &\geq \frac{-B}{\lambda}(\log P(t))\,P(t)+\frac{B}{\lambda}(\log P(t))\,P(t)\\ &\quad-B\frac{\log\lambda}{\lambda}P(t)-B(\log P(x_0))\,P(x_0) \qquad (2.12)\\ &\quad-\frac{B}{\lambda}P(t)+BP(x_0)\\ &\geq -BP(t)\left(\frac{\log\lambda}{\lambda}+(\log P(x_0))\frac{P(x_0)}{P(t)}+\frac{1}{\lambda}\right)\\ &\geq -B_1P(t)\end{aligned}$$

where

$$B_1=\frac{B}{\lambda}(\log\lambda+\log P(x_0)+1)\,. \qquad (2.13)$$

This completes the proof. □

Lemma 2.4. *Let $f \in L^1_{loc}(\mathbb{R}^+)$ be a complex-valued function such that the assumptions of Lemma 2.2 are satisfied for its integral function $s(t)$. Then there exists a positive constant B_1 such that*

$$\frac{1}{P(t)}\int_{x_0}^{t}|s(t)-s(x)|\,dP(x)\le B_1 \qquad \textit{whenever} \quad t>P^{-1}(\lambda P(x_0)). \tag{2.14}$$

Proof. The proof goes along similar lines to the proof of Lemma 2.3. Assume (1.15) with $\varepsilon=1$, and (2.7). Then the estimation (2.11) turns into form

$$\int_{x_0}^{t}|s(t)-s(x)|\,dP(x)\le B\int_{x_0}^{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}\log\frac{P(t)}{P(x)}dP(x)+\int_{P^{-1}\left(\frac{1}{\lambda}P(t)\right)}^{t}dP(x). \tag{2.15}$$

Then (2.12) together with (2.15) yields that

$$\int_{x_0}^{t}|s(t)-s(x)|\,dP(x)\le B_1P(t)$$

where B_1 is the same constant defined by (2.13). □

The first main result below states that ordinary limit at infinity follows from statistical limit at infinity for the measurable real-valued functions that are slowly decreasing with repect to P.

Theorem 2.1. *Let $s:\mathbb{R}^+\to\mathbb{R}$ be a measurable function. If $s(t)\overset{st}{\to} l$ and $s(t)$ is slowly decreasing with respect to P then $s(t)\to l$.*

Proof. Let $\varepsilon>0$, $x_0>0$ and $\lambda>1$ be arbitrarily given. Also let $s(t)\overset{st}{\to} l$. Then from (1.1), there exists $a_1\ge x_0$ such that

$$|s(a_1)-l|\le\varepsilon.$$

There are two cases. There exists some $a_2\in\left(P^{-1}\left(\sqrt{\lambda}P(a_1)\right),P^{-1}(\lambda P(a_1))\right)$ such that

$$|s(a_2)-l|\le\varepsilon. \tag{2.16}$$

or there exists no such a_2, that is

$$|s(t)-l|>\varepsilon \qquad \text{for all} \quad t\in\left(P^{-1}\left(\sqrt{\lambda}P(a_1)\right),P^{-1}(\lambda P(a_1))\right).$$

If the last case holds, then by (1.1) we can choose any $a_2>P^{-1}(\lambda P(a_1))$ such that (2.16) is satisfied. Otherwise we would have

$$\lim_{a\to\infty}\frac{1}{a}\left|\left\{t\in\left(P^{-1}(\lambda P(a_1)),a\right):|s(t)-l|>\varepsilon\right\}\right|=1$$

and this contradicts with (1.1). Note that $a_2>P^{-1}(\lambda P(a_1))$ implies that $P(a_2)>\lambda P(a_1)>P(a_1)$, and so $a_2>a_1$, since P is strictly increasing function. Now we repeat the previous step by beginning with a_2 instead of a_1, and so on. Then we obtain an increasing sequence (a_n) of real numbers such that

$$|s(a_n)-l|\le\varepsilon \qquad \text{for } n=1,2,.... \tag{2.17}$$

We assert that the case

$$|s(t)-l|>\varepsilon \qquad \text{for all} \quad t\in\left(P^{-1}\left(\sqrt{\lambda}P(a_n)\right),P^{-1}(\lambda P(a_n))\right) \tag{2.18}$$

can not occur for infinitely many n. Otherwise there exists $\varepsilon > 0$ such that for infinitely many n we obtain

$$\begin{aligned} & \frac{1}{a_n}\left|\{t \in (0, a_n) : |s(t) - l| > \varepsilon\}\right| \\ \geq\ & \frac{1}{a_n}\left|\left\{t \in P^{-1}\left(\sqrt{\lambda}P(a_n)\right), P^{-1}(\lambda P(a_n)) : |s(t) - l| > \varepsilon\right\}\right| \\ =\ & P^{-1}(\lambda P(a_n)) - P^{-1}\left(\sqrt{\lambda}P(a_n)\right) > 0, \end{aligned}$$

but this contradicts with (1.1). Hence, (2.18) is satisfied only for finitely many values of n. Let n_0 be the largest value of n for which (2.18) holds. Thus we have

$$a_{n+1} < P^{-1}(\lambda P(a_n)) \qquad \text{for} \quad n > n_0. \tag{2.19}$$

On the other hand by construction we have

$$a_{n+1} > P^{-1}\left(\sqrt{\lambda}P(a_n)\right) \qquad \text{for} \quad n > n_0.$$

So it follows that

$$\lim_{n\to\infty} a_n = \infty.$$

Since $s(t)$ is slowly decreasing with respect to P, by condition (1.13), we have

$$s(t) - s(a_n) \geq -\varepsilon \qquad \text{whenever} \qquad x_0 \leq a_n < t \leq P^{-1}(\lambda P(a_n)), \quad n > n_0. \tag{2.20}$$

Let $a_n < t \leq a_{n+1}$ for some $n > n_0$. By (2.19) we have

$$a_n < t \leq a_{n+1} < P^{-1}(\lambda P(a_n)) < P^{-1}(\lambda P(t)). \tag{2.21}$$

On the other hand it follows from (2.17) and (2.20) that if $n > n_0$ then for each $t \in (a_n, a_{n+1}]$

$$s(t) - l = (s(t) - s(a_n)) + (s(a_n) - l) \geq -2\varepsilon. \tag{2.22}$$

Moreover, it follows from (2.17) and (2.19)-(2.21) that

$$s(t) - l = (s(t) - s(a_{n+1})) + (s(a_{n+1}) - l) \leq 2\varepsilon. \tag{2.23}$$

Combining (2.22) and (2.23) we have

$$|s(t) - l| \leq 2\varepsilon$$

for every $t \in \bigcup_{n=n_0+1}^{\infty}(a_n, a_{n+1}] = (a_{n_0+1}, \infty)$. This proves that $s(t) \to l$. □

The next result is counter part of Theorem 2.1 in the complex-valued case.

Theorem 2.2. *Let $s : \mathbb{R}^+ \to \mathbb{C}$ be a measurable function. If $s(t) \overset{st}{\to} l$ and $s(t)$ is slowly oscillating with respect to P then $s(t) \to l$.*

Proof. We will use the similar method as in the proof of Theorem 2.1. Let $\varepsilon > 0$ and $\lambda > 1$. Then there exists an increasing sequence (a_n) of positive numbers tending to infinity such that (2.17) and (2.19) hold. Since $s(t)$ is slowly oscillating with respect to P, by condition (1.15), we have

$$|s(t) - s(a_n)| \leq \varepsilon \qquad \text{whenever} \qquad x_0 \leq a_n < t \leq P^{-1}(\lambda P(a_n)), \quad n > n_0. \tag{2.24}$$

Then it follows from (2.17), (2.19) and (2.23) that

$$|s(t)-l| \leq |s(t)-s(a_n)| + |s(a_n)-l| \leq 2\varepsilon$$

for every $t \in \bigcup_{n=n_0+1}^{\infty}(a_n, a_{n+1}] = (a_{n_0+1}, \infty)$. This proves that $\lim_{t\to\infty} s(t) = l$. □

Theorem 2.3. *Let $f \in L^1_{loc}(\mathbb{R}^+)$ be a real-valued function such that its integral function $s(t)$ is slowly decreasing with respect to P. If $s(t) \xrightarrow{st} l \left(\overline{N},p\right)$ then $s(t) \to l$.*

Proof. We first prove that if $s(t)$ is slowly decreasing with respect to P, then so is the function $\sigma(t)$. Let $\varepsilon > 0$ be given and let $x_0 \leq x < t \leq P^{-1}(\lambda P(x))$, where $x_0 = x_0(\varepsilon) > 0$ and $\lambda = \lambda(\varepsilon) > 1$ that is so close to 1. Then

$$\begin{aligned}
\sigma(t)-\sigma(x) &= \frac{1}{P(t)}\int_0^t s(u)\,dP(u) - \frac{1}{P(x)}\int_0^x s(u)\,dP(u) \\
&= \frac{1}{P(t)}\left(\int_0^x + \int_x^t\right) s(u)\,dP(u) - \frac{1}{P(x)}\int_0^x s(u)\,dP(u) \qquad (2.25)\\
&= -\frac{P(t)-P(x)}{P(t)P(x)}\int_0^x s(u)\,dP(u) + \frac{1}{P(t)}\int_x^t s(u)\,dP(u) \\
&= \frac{P(t)-P(x)}{P(t)P(x)}\int_0^x [s(x)-s(u)]\,dP(u) \\
&\quad + \frac{1}{P(t)}\int_x^t [s(u)-s(x)]\,dP(u).
\end{aligned}$$

By Lemma 2.3 there exists a positive constant B_1 such that

$$\frac{1}{P(x)}\int_0^x (s(x)-s(u))\,dP(u) \geq -B_1. \qquad (2.26)$$

On the other hand it follows from $x < t \leq P^{-1}(\lambda P(x))$ that $P(x) < P(t) \leq \lambda P(x)$ and so

$$\frac{1}{\lambda} \leq \frac{P(x)}{P(t)}. \qquad (2.27)$$

By using inequalities (2.26) and (2.27), and the condition (1.13) of slow decrease, we have

$$\begin{aligned}
\sigma(t)-\sigma(x) &\geq -B_1\frac{P(t)-P(x)}{P(t)} - \varepsilon\frac{1}{P(t)}\int_x^t dP(u) \\
&= -\left(1-\frac{P(x)}{P(t)}\right)(B_1+\varepsilon) \\
&\geq -\left(1-\frac{1}{\lambda}\right)(B_1+\varepsilon) \\
&> -(\lambda-1)(B_1+\varepsilon).
\end{aligned}$$

Now it follows from this inequality that

$$\sigma(t)-\sigma(x) \geq -\varepsilon \qquad \text{whenever} \quad x_0 \leq x < t \leq P^{-1}(\lambda P(x))$$

provided $1 < \lambda \leq 1 + \frac{\varepsilon}{B_1+\varepsilon}$. This proves that $\sigma(t)$ is also slowly decreasing with respect to P. Since $\sigma(t) \xrightarrow{st} l$ by assumption, we obtain that $\sigma(t) \to l$ by Theorem 2.1. Finally by Theorem 1.1 we conclude that $s(t) \to l$. □

Theorem 2.4. *Let $f \in L^1_{loc}(\mathbb{R}^+)$ be a complex-valued function such that its integral function $s(t)$ is slowly oscillating with respect to P. If $s(t) \to l\ \left(\overline{N}, p\right)$ then $s(t) \to l$.*

Proof. The proof is analogous to the proof of Theorem 2.3. We first prove that if $s(t)$ is slowly oscillating with respect to P, then so is the function $\sigma(t)$. Let $\varepsilon > 0$ be given and let $x_0 \leq x < t \leq P^{-1}(\lambda P(x))$, where $x_0 = x_0(\varepsilon) > 0$ and $\lambda = \lambda(\varepsilon) > 1$ that is so close to 1. It follows from (2.25) that

$$\begin{aligned} |\sigma(t) - \sigma(x)| &\leq \frac{P(t) - P(x)}{P(t)P(x)} \int_0^x |s(x) - s(u)|\, dP(u) \\ &\quad + \frac{1}{P(t)} \int_x^t |s(u) - s(x)|\, dP(u). \end{aligned}$$

By Lemma 2.4 there exists a positive constant B_1 such that

$$\frac{1}{P(x)} \int_0^x |s(x) - s(u)|\, dP(u) \leq B_1. \tag{2.28}$$

By using inequalities (2.28) and (2.27), and the condition (1.15) of slow oscillation, we have

$$\begin{aligned} |\sigma(t) - \sigma(x)| &\leq B_1 \frac{P(t) - P(x)}{P(t)} + \varepsilon \frac{1}{P(t)} \int_x^t s(u)\, dP(u) \\ &= \left(1 - \frac{P(x)}{P(t)}\right)(B_1 + \varepsilon) \\ &\leq \left(1 - \frac{1}{\lambda}\right)(B_1 + \varepsilon) \\ &< (\lambda - 1)(B_1 + \varepsilon). \end{aligned}$$

Now it follows from this inequality that

$$\sigma(t) - \sigma(x) \geq -\varepsilon \qquad \text{whenever} \quad x_0 \leq x < t \leq P^{-1}(\lambda P(x))$$

provided $1 < \lambda \leq 1 + \frac{\varepsilon}{B_1 + \varepsilon}$. This proves that $\sigma(t)$ is also slowly oscillating with respect to P. Since $\sigma(t) \overset{st}{\to} l$ by assumption, we obtain from Theorem 2.1 that $\sigma(t) \to l$. Finally by Theorem 1.2, we conclude that $s(t) \to l$. □

Finally note that the special cases of $P(x) = x$ for all $x \in \mathbb{R}^+$ and

$$P(x) = \begin{cases} 0, & 0 \leq x < 1 \\ \log x, & x \geq 1 \end{cases},$$

our Theorems 2.1-2.4 have been given by Móricz [10, Theorem 1-4] and Móricz and Németh [13, Theorem 1-4].

References

[1] C. P. Chen and C. T. Chang, *Tauberian theorems for the weighted means of measurable functions of several variables*, Taiwanese J. Math., **15**(1) (2011), 181-199.

[2] Á. Fekete and F. Móricz, *Necessary and sufficient Tauberian conditions in the case of weighted mean summable integrals over* $\mathbb{R}^+$ *II*, Publ. Math. Debrecen, **67** (2005), 65-78.

[3] Á. Fekete, *Tauberian conditions under which the statistical limit of an integrable function follows from its statistical summability*, Studia Sci. Math. Hungar, **43**(1) (2006), 115-129.

[4] G. Fındık and İ. Çanak, *Some Tauberian theorems for weighted means of double integrals*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **68**(2) (2018), 1452-1461.

[5] G. H. Hardy, *Theorems relating to the summability and convergence of slowly oscillating series*, Proc. Lond. Math. Soc., **2**(1) (1910), 301-320.

[6] J. Karamata, *Sur les théorèmes inverses des procédés de sommabilité*, Act. Sci. Ind. 450, Hermann, Paris, 1937.

[7] G. A. Mikhalin, *Theorems of Tauberian type for* (J, p_n) *summation methods*, Ukrain. Mat. Zh., **29**(6) (1977), 763-770.

[8] F. Móricz, *Statistical limits of measurable functions*, Analysis, **24** (2004), 1-18.

[9] F. Móricz, *Necessary and sufficient Tauberian conditions in the case of weighted mean summable integrals over* $\mathbb{R}^+$, Math. Inequal. Appl. **7**(1) (2004), 87-93.

[10] F. Móricz, *Statistical extensions of some classical Tauberian theorems in nondiscrete setting*, Colloq. Math., **107**(1) (2007), 45-56.

[11] F. Móricz, *Statistical limit of Lebesgue measurable functions at* ∞ *with applications in Fourier Analysis and Summability*, Anal. Math., **40**(2) (2014), 147-159.

[12] F. Móricz and U. Stadtmüller, *Characterization of the convergence of weighted averages in a more general setting*, Studia Sci. Math. Hungar, **50**(1) (2013), 51-66.

[13] F. Móricz and Z. Németh, *Statistical extension of classical Tauberian theorems in the case of logarithmic summability*, Anal. Math., **3**(40) (2014), 231-242.

[14] İ. Çanak, and Z. Önder, *Tauberian Conditions Under Which Convergence Follows from Statistical Summability by Weighted Means*, In Advances in Summability and Approximation Theory (pp. 1-22). Springer, Singapore, 2018.

[15] F. Özsaraç and İ. Çanak, *Tauberian theorems for iterations of weighted mean summable integrals*, Positivity, **23**(1) (2019), 219-231.

[16] R. Schmidt, *Über divergente Folgen und lineare Mittelbildungen*, Mathematische Zeitschrift, **22**(1) (1925), 89-152.

[17] Ü. Totur and M. A. Okur, *Alternative proofs of some classical Tauberian theorems for the weighted mean method of integrals*, Filomat, **29**(10) (2015), 2281-2287.

[18] T. Vijayaraghavan, *A Tauberian theorem*, J. Lond. Math. Soc., **1**(2) (1926), 113-120.

[1]Department of Mathematics Education,
Ordu University,
Ordu, Turkey
E-mail address: cbelen52@gmail.com