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ČEBYŠEV'S INEQUALITIES WITH SYMMETRIC FUNCTIONS

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Abstract. The paper studies Čebyšev's inequalities for ordered n-tuples and ordered functions. The study offers the concept of proportionally symmetric functions as a generalization of the notion of midpoint symmetric functions. This is the concept that gives the opportunity for expansion of Čebyšev's integral inequality and related inequalities.

1. Introduction

We will briefly present Čebyšev's sum inequality, and show how the integral inequality can be derived from the sum inequality.

Čebyšev's sum inequality deals with two real n-tuples $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ that satisfy

$$x_1 \leq \ldots \leq x_n \quad \text{or} \quad x_1 \geq \ldots \geq x_n,$$

and likewise

$$y_1 \leq \ldots \leq y_n \quad \text{or} \quad y_1 \geq \ldots \geq y_n.$$

So, monotonic n-tuples of real numbers are in focus. It is also used the product n-tuple (obtained by multiplying the corresponding members of given n-tuples)

$$x_1 y_1, \ldots, x_n y_n.$$

The inequality itself actually compares the product of arithmetic means of given n-tuples, and the arithmetic mean of the product of n-tuples. All possible cases can be covered by two statements (Čebyšev 1882-1883) as follows.

Čebyšev's Sum Inequality. *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be monotonic real n-tuples.*

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If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are monotonic in the same direction, then

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} x_i y_i. \tag{1.1}$$

If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are monotonic in the opposite directions, then the reverse inequality symbol stands in formula (1.1).

The sum inequality in formula (1.1) and its reverse inequality have integral forms which employ monotonic (nondecreasing or nonincreasing) functions defined on the unit interval $[0, 1]$. If we have a monotonic function $f : [0, 1] \to \mathbb{R}$, we can take an even positive integer $2n$ including partition points $x_i = i/(2n)$ for $i = 0$ to $i = 2n$, and the corresponding left Riemann integral sum

$$S_{2n}(f) = \frac{1}{2n}\sum_{i=1}^{2n} f(x_i^*) = \sum_{i=1}^{2n} f(x_i^*)\Delta x_i$$

including argument points $x_i^* = x_{i-1}$ and lengths $\Delta x_i = x_i - x_{i-1} = 1/(2n)$. The function f and the sequence $(S_{2n}(f))_{n=1}^{\infty}$ are monotonic in the same direction. The sequence $(S_{2n}(f))_{n=1}^{\infty}$ is convergent because it is monotonic and bounded. Conveniently, the reflection moment (letting n tend to infinity) produces

$$\lim_{n\to\infty} \frac{1}{2n}\sum_{i=1}^{2n} f(x_i^*) = \lim_{n\to\infty} \sum_{i=1}^{2n} f(x_i^*)\Delta x_i = \int_0^1 f(x)\,dx.$$

Incidentally, the implementation of right integral sums has the same effect. In that case, the function f and the sequence of corresponding right Riemann integral sums are monotonic in the opposite directions.

If we have two monotonic functions $f, g : [0, 1] \to \mathbb{R}$, we can consider monotonic $2n$-tuples

$$f(x_1^*), \ldots, f(x_{2n}^*) \quad \text{and} \quad g(x_1^*), \ldots, g(x_{2n}^*)$$

through Chebyshev's sum inequality, and proceed with the reflection moment to expose

Čebyšev's Integral Inequality. *Let $f, g : [0, 1] \to \mathbb{R}$ be monotonic functions.*

If f and g are monotonic in the same direction, then

$$\left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(x)\,dx\right) \leq \int_0^1 f(x)g(x)\,dx. \tag{1.2}$$

If f and g are monotonic in the opposite directions, then the reverse inequality symbol stands in formula (1.2).

Some variants of Čebyšev's inequality with applications to other inequalities can be found in [2].

2. Discrete and integral variants of Čebyšev's inequality

As it is known, Čebyšev's sum and integral inequalities can be generalized by using ordered n-tuples and ordered functions respectively.

Definition A. *Two real n-tuples $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are said to be similarly ordered if*

$$(x_i - x_j)(y_i - y_j) \geq 0 \tag{2.1}$$

for each pair of indices $i, j \in \{1, \ldots, n\}$. The given n-tuples are said to be oppositely ordered if the reverse inequality symbol stands in formula (2.1). The given n-tuples are said to be ordered if they are similarly or oppositely ordered.

If two n-tuples are monotonic in the same direction (opposite directions), then they are similarly ordered (oppositely ordered).

Čebyšev's sum inequality can be extended to convex combinations of given n-tuples, and the convex combination of the product n-tuple.

Theorem A. *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be ordered real n-tuples, and let $p_1, \ldots, p_n$ be a nonnegative real n-tuple with the sum equals 1.*

If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are similarly ordered, then

$$\left(\sum_{i=1}^{n} p_i x_i\right)\left(\sum_{i=1}^{n} p_i y_i\right) \leq \sum_{i=1}^{n} p_i x_i y_i. \tag{2.2}$$

If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are oppositely ordered, then the reverse inequality symbol stands in formula (2.2).

Proof. If the given n-tuples are similarly ordered, then from formula (2.1) it follows that

$$\sum_{j=1}^{n} \sum_{i=1}^{n} p_i p_j (x_i - x_j)(y_i - y_j) \geq 0.$$

By multiplying the differences in brackets, summing with respect to indices i and j over the set $\{1, \ldots, n\}$, and arranging, we get

$$2\sum_{i=1}^{n} p_i x_i y_i - 2\left(\sum_{i=1}^{n} p_i x_i\right)\left(\sum_{i=1}^{n} p_i y_i\right) \geq 0,$$

which easily gives the inequality in formula (2.2). If the given n-tuples are oppositely ordered, then we employ the reverse inequality symbol. □

If $p_1 = \ldots = p_n = 1/n$, then formula (2.2) is reduced to formula (1.1).

If $q_1, \ldots, q_n$ is a nonnegative real n-tuple with a positive sum, then formula (2.2) can be modified by the coefficients

$$p_i = \frac{q_i}{\sum_{i=1}^{n} q_i},$$

and thus take the form

$$\left(\frac{\sum_{i=1}^{n} q_i x_i}{\sum_{i=1}^{n} q_i}\right)\left(\frac{\sum_{i=1}^{n} q_i y_i}{\sum_{i=1}^{n} q_i}\right) \leq \frac{\sum_{i=1}^{n} q_i x_i y_i}{\sum_{i=1}^{n} q_i}. \tag{2.3}$$

We aspire to the integral form of Theorem A.

Definition B. *Two functions $f, g : [a, b] \to \mathbb{R}$ are said to be similarly ordered if*

$$\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \geq 0 \tag{2.4}$$

for each pair of arguments $x, y \in [a, b]$. The given functions are said to be oppositely ordered if the reverse inequality symbol stands in formula (2.4). The given functions are said to be ordered if they are similarly or oppositely ordered.

If two functions are monotonic in the same direction (opposite directions), then they are similarly ordered (oppositely ordered).

Theorem A can be exposed in the integral form if we use appropriate functions instead of n-tuples. This certainly includes ordered functions.

Theorem B. *Let $f, g : [a, b] \to \mathbb{R}$ be ordered integrable functions, and let $p : [a, b] \to \mathbb{R}$ be a nonnegative integrable function with the integral equals 1.*

If f and g are similarly ordered, then

$$\left(\int_a^b p(x) f(x)\, dx\right)\left(\int_a^b p(x) g(x)\, dx\right) \le \int_a^b p(x) f(x) g(x)\, dx. \tag{2.5}$$

If f and g are oppositely ordered, then the reverse inequality symbol stands in formula (2.5).

Proof. If f and g are similarly ordered, then from formula (2.4) it follows that

$$\int_a^b \int_a^b p(x)p(y)\big(f(x) - f(y)\big)\big(g(x) - g(y)\big)\, dx\, dy \ge 0.$$

By multiplying the differences in brackets, integrating with respect to arguments x and y over the interval $[a, b]$, and arranging, we get

$$2\int_a^b p(x) f(x) g(x)\, dx - 2\left(\int_a^b p(x) f(x)\, dx\right)\left(\int_a^b p(x) g(x)\, dx\right) \ge 0,$$

which easily gives the inequality in formula (2.5). If f and g are oppositely ordered, then we employ the reverse inequality symbol. $\square$

If $[a, b] = [0, 1]$ and $p(x) = 1$ for all $x \in [0, 1]$, then formula (2.5) is reduced to formula (1.2).

If $q : [a, b] \to \mathbb{R}$ is a nonnegative integrable function with a positive integral, then formula (2.5) can be modified by the function

$$p(x) = \frac{q(x)}{\int_a^b q(x)\, dx},$$

and thus take the form

$$\left(\frac{\int_a^b q(x) f(x)\, dx}{\int_a^b q(x)\, dx}\right)\left(\frac{\int_a^b q(x) g(x)\, dx}{\int_a^b q(x)\, dx}\right) \le \frac{\int_a^b q(x) f(x) g(x)\, dx}{\int_a^b q(x)\, dx}. \tag{2.6}$$

If $q(x) = 1$ for all $x \in [a, b]$, then formula (2.6) is reduced to

$$\left(\frac{\int_a^b f(x)\, dx}{b - a}\right)\left(\frac{\int_a^b g(x)\, dx}{b - a}\right) \le \frac{\int_a^b f(x) g(x)\, dx}{b - a}. \tag{2.7}$$

Formula (2.7) says that the product of integral arithmetic means of f and g is less than or equal to the integral arithmetic mean of the product fg.

3. Main results

We will introduce the concept of proportionally symmetric functions, and realize some generalizations of the integral variant of Čebyšev's inequality.

Definition 3.1. A function $f:[a,b]\to\mathbb{R}$ is said to be a proportion function or proportionally symmetric function with the center at a point $c\in(a,b)$ if the equality

$$f(x)=f(y) \tag{3.1}$$

holds for each pair of points $x\in[a,c]$ and $y\in[c,b]$ satisfying the relation

$$\frac{x-a}{c-a}=\frac{b-y}{b-c}. \tag{3.2}$$

A proportion function can be represented by using the explicit correlations $y=u(x)$ and $x=v(y)$ expressed from formula (3.2).

Corollary 3.1. *A function $f:[a,b]\to\mathbb{R}$ is a proportion function with the center at a point $c\in(a,b)$ if and only if the equalities*

$$f(x)=\begin{cases} f(u(x)) & \text{with } x\in[a,c]\\ f(v(x)) & \text{with } x\in[c,b]\end{cases} \tag{3.3}$$

hold for the pair of mutually inverse affine functions

$$u(x)=\frac{b-c}{a-c}(x-c)+c=\frac{x-a}{c-a}c+\frac{c-x}{c-a}b \tag{3.4}$$

and

$$v(x)=\frac{a-c}{b-c}(x-c)+c=\frac{x-c}{b-c}a+\frac{b-x}{b-c}c. \tag{3.5}$$

In the coordinate plane, the equation $y=u(x)$ for $x\in[a,c]$ represents the line segment between points (a,b) and (c,c), and the equation $y=v(x)$ for $x\in[c,b]$ represents the line segment between points (c,c) and (b,a).

A collection of proportion functions $f:[a,b]\to\mathbb{R}$ with the center at a point $c\in(a,b)$ is a real linear space. Let $\mathcal{F}_c$ be this space.

One more thing, Definition 3.1 can be simplified by using the convex combinations $x=(1-t)a+tc$ and $y=tc+(1-t)b$ with $t\in[0,1]$.

Corollary 3.2. *A function $f:[a,b]\to\mathbb{R}$ is a proportion function with the center at a point $c\in(a,b)$ if and only if the equality*

$$f\big((1-t)a+tc\big)=f\big(tc+(1-t)b\big) \tag{3.6}$$

holds for each number $t\in[0,1]$.

Formula (3.6) very easily reveals that the integral arithmetic mean of an integrable proportion function has three representations.

Corollary 3.3. *Let $f:[a,b]\to\mathbb{R}$ be an integrable proportion function with the center at a point $c\in(a,b)$.*

Then

$$\frac{\int_a^c f(x)\,dx}{c-a}=\frac{\int_c^b f(x)\,dx}{b-c}=\frac{\int_a^b f(x)\,dx}{b-a}. \tag{3.7}$$

Definition 3.2. A proportion function $f:[a,b]\to\mathbb{R}$ with the center at a point $c\in(a,b)$ is said to be a cup (cap) function with the vertex at c if it is nondecreasing (nonincreasing) in directions $\overrightarrow{ca}$ and $\overrightarrow{cb}$.

A cup (cap) function attains the global minimum (maximum) at the vertex. If f is a cup (cap) function, then $-f$ is a cap (cup) function.

A collection of cup (cap) functions $f:[a,b]\to\mathbb{R}$ with the vertex at a point $c\in(a,b)$ is a convex cone in the space $\mathcal{F}_c$. Let $\mathcal{K}_c$ be this cone. If $f,g\in\mathcal{K}_c$ and $\kappa,\lambda\geq 0$, then $\kappa f+\lambda g\in\mathcal{K}_c$. The zero function is the vertex of the cone $\mathcal{K}_c$.

Lemma 3.1. *Let $f,g:[a,b]\to\mathbb{R}$ be cup or cap functions with the vertex at a point $c\in(a,b)$.*

If either f and g are cup functions or f and g are cap functions, then f and g are similarly ordered.

If either f is a cup function and g is a cap function or vice versa, then f and g are oppositely ordered.

Proof. Suppose that f and g are cup functions. Take arguments $x,y\in[a,b]$. If

$$x,y\in[a,c]\quad\textit{or}\quad x,y\in[c,b],$$

then

$$\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\geq 0 \tag{3.8}$$

because the restrictions $f\!\restriction_{[a,c]}$ and $g\!\restriction_{[a,c]}$ are nonincreasing, and the restrictions $f\!\restriction_{[c,b]}$ and $g\!\restriction_{[c,b]}$ are nondecreasing. If

$$x\in[a,c],\ y\in[c,b]\quad\textit{or}\quad x\in[c,b],\ y\in[a,c],$$

then

$$x,v(y)\in[a,c]\quad\textit{or}\quad x,u(y)\in[c,b],$$

and formula (3.8) implies

$$\big(f(x)-f(v(y))\big)\big(g(x)-g(v(y))\big)\geq 0\ \text{ or }\ \big(f(x)-f(u(y))\big)\big(g(x)-g(u(y))\big)\geq 0.$$

Each of the above inequalities coincides with formula (3.8) when we consider that $f(u(y))=f(v(y))=f(y)$ and $g(u(y))=g(v(y))=g(y)$. So, cup functions f and g are similarly ordered. Further, cap functions f and g are also similarly ordered because $-f$ and $-g$ are cup functions.

Suppose that f is a cup function and g is a cap function. Then f and $-g$ are cup functions, and so similarly ordered. By using formula (3.8) for f and $-g$, and multiplying by -1, we get formula for f and g with the reverse inequality symbol. So, f and g are oppositely ordered. The same is true for the reverse case. $\square$

A cup or cap function $f : [a,b] \to \mathbb{R}$ with the vertex at $c \in (a,b)$ is integrable because it is monotonic on the intervals $[a,c]$ and $[c,b]$. The synergy of Lemma 3.1 and Theorem B gives the following.

Theorem 3.1. *Let $f, g : [a,b] \to \mathbb{R}$ be cup or cap functions with the vertex at a point $c \in (a,b)$, and let $p : [a,b] \to \mathbb{R}$ be a nonnegative integrable function with the integral equals 1.*

If either f and g are cup functions or f and g are cap functions, then

$$\left(\int_a^b p(x)f(x)\,dx\right)\left(\int_a^b p(x)g(x)\,dx\right) \le \int_a^b p(x)f(x)g(x)\,dx. \tag{3.9}$$

If either f is a cup function and g is a cap function or vice versa, then the reverse inequality symbol stands in formula (3.9).

Convex functions can be included in Theorem 3.1. The idea is to transform a convex function to a cup function.

Any function can be used as an input to obtain the proportionally symmetric function. General formula is presented in the following lemma.

Lemma 3.2. *Let $f : [a,b] \to \mathbb{R}$ be a function, let $c \in (a,b)$ be a point, let κ and λ be real numbers with the sum equals 1, and let $\overline{f} : [a,b] \to \mathbb{R}$ be the function defined by*

$$\overline{f}(x) = \begin{cases} \kappa f(x) + \lambda f(u(x)) & \text{with } x \in [a,c] \\ \lambda f(x) + \kappa f(v(x)) & \text{with } x \in [c,b] \end{cases}. \tag{3.10}$$

Then $\overline{f}$ is a proportion function with the center at c.

Proof. We will prove that the function $\overline{f}$ satisfies the equations in formula (3.3), expressed in two lines respecting intervals $[a,c]$ and $[c,b]$.

If $x \in [a,c]$, then $u(x) \in [c,b]$, and combining the second and first lines of formula (3.10), we obtain

$$\overline{f}(u(x)) = \lambda f(u(x)) + \kappa f(v(u(x))) = \lambda f(u(x)) + \kappa f(x) = \overline{f}(x).$$

If $x \in [c,b]$, we similarly prove that $\overline{f}(v(x)) = \overline{f}(x)$. ☐

Note that $\overline{f}(c) = f(c)$. Formula (3.9) guarantees that $\overline{f} = f$ for each function f which is proportionally symmetric with respect to c.

Intuitively, it is clear that a convex (concave) proportion function is a cup (cap) function. We rely on the special case of the function f_1.

Lemma 3.3. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, let $c \in (a,b)$ be a point, and let $\widetilde{f} : [a,b] \to \mathbb{R}$ be the function defined by*

$$\widetilde{f}(x) = \begin{cases} \dfrac{b-c}{b-a} f(x) + \dfrac{c-a}{b-a} f(u(x)) & \text{with } x \in [a,c] \\ \dfrac{c-a}{b-a} f(x) + \dfrac{b-c}{b-a} f(v(x)) & \text{with } x \in [c,b] \end{cases}. \tag{3.11}$$

Then $\widetilde{f}$ is a convex cup function with the vertex at c.

Proof. Let us demonstrate that the function $\widetilde{f}$ attains the global minimum at the point c. If $x \in [a, c]$, then we have the convex combination

$$\begin{aligned} c &= \frac{u(x)-c}{u(x)-x}x + \frac{c-x}{u(x)-x}u(x) \\ &= \frac{b-c}{b-a}x + \frac{c-a}{b-a}u(x), \end{aligned}$$

and employing the convexity of f, we get

$$\widetilde{f}(c) = f(c) \le \frac{b-c}{b-a}f(x) + \frac{c-a}{b-a}f(u(x)) = \widetilde{f}(x).$$

If $x \in [c, b]$, then we represent c as the convex combination of $v(x)$ and x, and we get $\widetilde{f}(c) \le \widetilde{f}(x)$ again. Thus $\widetilde{f}(c)$ is the global minimum of $\widetilde{f}$.

Let us prove that the function $\widetilde{f}$ is convex. The restrictions $\widetilde{f}\restriction_{[a,c]}$ and $\widetilde{f}\restriction_{[c,b]}$ are convex. Take a pair of points $x \in [a, c)$ and $y \in (c, b]$ such that $c - x = y - c$. Since $\widetilde{f}(c)$ is the global minimum of $\widetilde{f}$, we have $\widetilde{f}(c) + \widetilde{f}(c) \le \widetilde{f}(x) + \widetilde{f}(y)$, and so $\widetilde{f}(c) - \widetilde{f}(x) \le \widetilde{f}(y) - \widetilde{f}(c)$. It follows that

$$\frac{\widetilde{f}(c) - \widetilde{f}(x)}{c-x} \le \frac{\widetilde{f}(y) - \widetilde{f}(c)}{y-c},$$

and the convexity of restrictions $\widetilde{f}\restriction_{[a,c]}$ and $\widetilde{f}\restriction_{[c,b]}$ ensures that this is true for all pairs of points $x \in [a, c)$ and $y \in (c, b]$. As a consequence, we have the double inequality

$$\frac{\widetilde{f}(c) - \widetilde{f}(x)}{c-x} \le \frac{\widetilde{f}(y) - \widetilde{f}(x)}{y-x} \le \frac{\widetilde{f}(y) - \widetilde{f}(c)}{y-c},$$

where the middle slope is obtained as the convex combination of left and right slopes. The above inequality provides the global convexity of $\widetilde{f}$.

So we have the function $\widetilde{f}$ which is proportionally symmetric and convex, and therefore $\widetilde{f}$ is a cup function. □

A convex function f can be included in Čebyšev's inequality via the function $\widetilde{f}$.

Corollary 3.4. *Let $f : [a, b] \to \mathbb{R}$ be a convex function, let $c \in (a, b)$ be a point, let $\widetilde{f}$ be the function defined by formula (3.11), let $g : [a, b] \to \mathbb{R}$ be a cup or cap function with the vertex at c, and let $p : [a, b] \to \mathbb{R}$ be a nonnegative integrable function with the integral equals 1.*

If g is a cup function, then

$$\left(\int_a^b p(x)\widetilde{f}(x)\,dx\right)\left(\int_a^b p(x)g(x)\,dx\right) \le \int_a^b p(x)\widetilde{f}(x)g(x)\,dx. \tag{3.12}$$

If g is a cap function, then the reverse inequality symbol stands in formula (3.12).

Proof. The function $\widetilde{f}$ is a cup function by Lemma 3.3. We just need to apply Theorem 3.1 to functions $\widetilde{f}$, g and p. □

4. Applications to other inequalities

We will employ the following lemma, which presents one of the fundamental inequalities for convex functions.

Lemma 4.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, let $g : [a,b] \to \mathbb{R}$ be an integrable function with nonzero integral, and let $\alpha a + \beta b$ be the convex combination representing the barycenter of g as the equality*

$$\frac{\int_a^b x\,g(x)\,dx}{\int_a^b g(x)\,dx} = \alpha a + \beta b. \tag{4.1}$$

Then

$$f(\alpha a + \beta b) \le \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \le \alpha f(a) + \beta f(b). \tag{4.2}$$

By relying on Lemma 4.1, we can modify and upgrade Corollary 3.4 as follows.

Corollary 4.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, let $c \in (a,b)$ be a point, let $\widetilde{f}$ be the function defined by formula (3.11), let $g : [a,b] \to \mathbb{R}$ be a cup or cap function with the vertex at c and positive integral, and let $\alpha a + \beta b$ be the convex combination representing the barycenter of g.*

If g is a cup function, then

$$\widetilde{f}\left(\frac{a+b}{2}\right) \le \frac{\int_a^b \widetilde{f}(x)\,dx}{b-a} \le \frac{\int_a^b \widetilde{f}(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \le \alpha \widetilde{f}(a) + \beta \widetilde{f}(b). \tag{4.3}$$

If g is a cap function, then

$$\widetilde{f}(\alpha a + \beta b) \le \frac{\int_a^b \widetilde{f}(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \le \frac{\int_a^b \widetilde{f}(x)\,dx}{b-a} \le \frac{\widetilde{f}(a) + \widetilde{f}(b)}{2}. \tag{4.4}$$

Proof. Suppose that g is a cup function. By using functions $\widetilde{f}$ and g, and the function p as the constant $1/(b-a)$, the inequality in formula (3.12) can be rearranged to take the form

$$\frac{\int_a^b \widetilde{f}(x)\,dx}{b-a} \le \frac{\int_a^b \widetilde{f}(x)\,g(x)\,dx}{\int_a^b g(x)\,dx}. \tag{4.5}$$

To extend the inequality in formula (4.5) to the left, we will use the special case of the inequality in formula (4.2) in which g is the constant equals 1. Since

$$\frac{\int_a^b x\,dx}{b-a} = \frac{a+b}{2},$$

the left-hand side of the inequality in formula (4.2) for this special case gives

$$\widetilde{f}\left(\frac{a+b}{2}\right) \le \frac{\int_a^b \widetilde{f}(x)\,dx}{b-a}.$$

To extend the inequality in formula (4.5) to the right, we will take the right-hand side of the inequality in formula (4.2) which states

$$\frac{\int_a^b \widetilde{f}(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \leq \alpha\widetilde{f}(a) + \beta\widetilde{f}(b).$$

By composing the above inequalities, we achieve formula (4.3).

If g is a cap function, then we start with the reverse inequality in formula (4.5). By doing similarly as in the previous case, we can attain formula (4.4). □

Let us consider the special case of Corollary 4.1 in which $c = (a+b)/2$. This is the case of the proper symmetry with respect to the interval midpoint.

Corollary 4.2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, and let $g : [a,b] \to \mathbb{R}$ be a cup or cap function with the vertex at the midpoint $c = (a+b)/2$ and positive integral.*

If g is a cup function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \leq \frac{f(a)+f(b)}{2}. \tag{4.6}$$

If g is a cap function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq \frac{f(a)+f(b)}{2}. \tag{4.7}$$

Proof. In this case $u(x) = v(x) = a+b-x$, and hence $g(x) = g(a+b-x)$ for all $x \in [a,b]$. By using the substitution $a+b-x = y$, we can obtain the representation

$$\frac{\int_a^b x\,g(x)\,dx}{\int_a^b g(x)\,dx} = \frac{a+b}{2}.$$

According to formula (4.1), the coefficients are $\alpha = \beta = 1/2$.

Further, formula (3.11) is reduced to

$$\widetilde{f}(x) = \frac{1}{2}f(x) + \frac{1}{2}f(a+b-x)$$

for all $x \in [a,b]$. Thus we have

$$\widetilde{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right), \quad \frac{\widetilde{f}(a)+\widetilde{f}(b)}{2} = \frac{f(a)+f(b)}{2}.$$

The same substitution $a+b-x = y$ also gives the integral equalities

$$\int_a^b \widetilde{f}(x)\,dx = \int_a^b f(x)\,dx \tag{4.8}$$

and

$$\int_a^b \widetilde{f}(x)\,g(x)\,dx = \int_a^b f(x)\,g(x)\,dx. \tag{4.9}$$

The above coefficients and coincidences turn formula (4.3) into formula (4.6), and formula (4.4) into formula (4.7). □

Let us analyze the multiple inequality in formula (4.6). The inequality of the first, second and fourth members represents the well known Hermite-Hadamard inequality. More details on the Hermite-Hadamard inequality can be found in [3]. The inequality of the first, third and fourth members constitutes the known Fejér inequality. Some generalizations of the Fejér inequality can be seen in [4]. By using the unit interval $[a,b]=[0,1]$, the inequality of the second and third members corresponds to the Levin-Stečkin inequality. More details on the Levin-Stečkin inequality can be found in [1].

It should be noted that the first simple proof of the Levin-Stečkin inequality was recently demonstrated in [5].

References

[1] V. I. Levin, S. B. Stečkin, *Inequalities*, Amer. Math. Soc. Transl., **14** (1960), 1-29.

[2] M. Klaričić Bakula, A. Matković, J. Pečrić, *Variants of Čebyšev's inequality with applications*, J. Inequal. Appl., **2006** (2006), Article ID 39692.

[3] Z. Pavić, *Improvements of the Hermite-Hadamard inequality*, J. Inequal. Appl., **2015** (2015), Article ID 222.

[4] Z. Pavić, *The Fejér inequality and its generalizations,* Filomat, **32** (2018), 5793-5802.

[5] A. Witkowski, *Inequalities of Levin-Stečkin, Clausing and Čebyšev revisited, arXiv:1612.05373 [math.CA]* (2016), 1-4.

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