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INTEGRAL INEQUALITIES FOR SOME CONVEXITY CLASSES VIA ATANGANA-BALEANU INTEGRAL OPERATORS

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Abstract. In this paper, firstly, definitions of different classes of convexity, Riemann-Liouville fractional integral and Atangana-Baleanu fractional integral operator are given. In the second part, which constitutes the main results, by using the identity given by Set et al. in [20], some new integral inequalities for quasi-convex and P-function via Atangana-Baleanu fractional integral operators are obtained.

1. Introduction

First of all, let us recall the concept of convex function which is the basic concept of convex analysis.

Definition 1.1. [17] The function $\kappa : [\mu, \nu] \subseteq \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$\kappa(\omega x + (1-\omega)y) \leq \omega\kappa(x) + (1-\omega)\kappa(y)$$

for all $x, y \in [\mu, \nu]$ and $\omega \in [0,1]$. We say that κ is concave if $(-\kappa)$ is convex.

There are many types of convexity in the literature. The two types of convexity that will be used in this article are as follows.

Definition 1.2. [9] Let $\kappa : I \to \mathbb{R}$ for all $\omega \in [0,1]$ and all $\mu, \nu \in I$, if the following inequality

$$\kappa(\omega\mu + (1-\omega)\nu) \leq \max\{\kappa(\mu), \kappa(\nu)\}$$

holds, then κ is called a quasi-convex function on I.

Definition 1.3. [18] A function $\kappa : I \to \mathbb{R}$ is P-function or that κ belongs to the class of $P(I)$, if it is nonnegative and, for all $\mu, \nu \in I$ and $\omega \in [0,1]$, satisfies the following inequality;

$$\kappa(\omega x + (1-\omega)y) \leq \kappa(x) + \kappa(y). \tag{1.1}$$

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There are many inequalities in the literature for convex functions. But among these inequalities the most take attention of researchers is the Hermite-Hadamard inequality on which hundreds of studies have been conducted. The classical Hermite-Hadamard integral inequalities are as the following.

Theorem 1.1. *Assume that $\kappa : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function defined on the interval I of $\mathbb{R}$ where $\mu < \nu$. The following statement;*

$$\kappa\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{\nu-\mu}\int_{\mu}^{\nu} \kappa(x)dx \leq \frac{\kappa(\mu)+\kappa(\nu)}{2} \tag{1.2}$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if κ is concave.

The inequality (1.2) was introduced by C. Hermite [7] and investigated by J. Hadamard [8] in 1893. Many mathematicians have paid great attention to the inequality of Hermite-Hadamard due to its quality and validity in mathematical inequalities. For significant developments, modifications, and consequences regarding the Hermite-Hadamard uniqueness property and general convex function definitions, for essential details, the interested reader would like to refer to the works in [2, 5, 14] and references therein.

Several new results have been proved related different kinds of convex functions and associated integral inequalities. In [10], Bakula et al. gave some new integ-ral inequalities of Hadamard type for $m-$convex and (α, m)-convex functions. A similar paper has been written by Kirmaci et al. for $s-$convex functions in [11]. Besides, in [13], Kavurmaci et al. proved some new inequalities for convex functions. In [15], the authors have given several new results for co-ordinated convexity which is a modification of convexity on the co-ordinates. In [16], Özdemir et al. have defined a generalization of convexity and proved some Hadamard type inequalities. On all of these, in [19], Sarikaya et al. gave a different perspective to the inequality (1.2) by using the Riemann-Liouville fractional integral operators.

As of late, Atangana and Baleanu presented another fractional operator involving the special Mittag–Leffler function, which tackles the issue of recovering the original function. It is seen that Mittag–Leffler's function is more reasonable than a power law in demonstrating the physical phenomenon around us. This made the operator more powerful and accommodating. Thus, numerous researchers have shown a keen fascination for using this special operator. Atangana and Baleanu presented the derivative in both the Caputo and the Reimann–Liouville sense:

$B(\alpha)$ is normalization function with $B(0) = B(1) = 1$.

Definition 1.4. [4] Let $\kappa \in H^1(\mu, \nu)$, $\nu > \mu$, $\alpha \in [0, 1]$ then, the definition of the new fractional derivative is given:

$${}^{ABC}_{\ \ \mu}D^{\alpha}_{\rho}\left[\kappa(\rho)\right] = \frac{B(\alpha)}{1-\alpha}\int_{\mu}^{\rho} \kappa'(x)E_{\alpha}\left[-\alpha\frac{(\rho-x)^{\alpha}}{(1-\alpha)}\right]dx. \tag{1.3}$$

Here $H^1(\mu, \nu)$ can be defined as $H^1(\mu, \nu) = \{\kappa : \kappa \in L_1[\mu, \nu] \; and \; \kappa' \in L_1[\mu, \nu]\}$.

Definition 1.5. [4] Let $\kappa \in H^1(\mu,\nu)$, $\nu > \mu$, $\alpha \in [0,1]$ then, the definition of the new fractional derivative is given:

$$ {}^{ABR}_{\ \ \mu}D^{\alpha}_{\rho}\left[\kappa(\rho)\right] = \frac{B(\alpha)}{1-\alpha}\frac{d}{d\rho}\int_{\mu}^{\rho}\kappa(x)E_{\alpha}\left[-\alpha\frac{(\rho-x)^{\alpha}}{(1-\alpha)}\right]dx. \tag{1.4} $$

Equations (1.3) and (1.4) have a non-local kernel. Also in equation (1.4) when the function is constant we get zero.

However, in the same paper they provide the corresponding Atangana–Baleanu AB–fractional integral operator as:

Definition 1.6. [4] The fractional integral associate to the new fractional derivative with non-local kernel of a function $\kappa \in H^1(\mu,\nu)$ as defined:

$$ {}^{AB}_{\ \mu}I^{\alpha}\left\{\kappa(\rho)\right\} = \frac{1-\alpha}{B(\alpha)}\kappa(\rho) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\mu}^{\rho}\kappa(y)(\rho-y)^{\alpha-1}dy $$

where $\nu > \mu, \alpha \in [0,1]$.

In [1], Abdeljawad and Baleanu introduced right hand side of integral operator as following; The right fractional new integral with ML kernel of order $\alpha \in [0,1]$ is defined by

$$ {}^{AB}I^{\alpha}_{\nu}\left\{\kappa(\rho)\right\} = \frac{1-\alpha}{B(\alpha)}\kappa(\rho) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\rho}^{\nu}\kappa(y)(y-\rho)^{\alpha-1}dy. $$

Some recent development in theory of integral inequalities involving AB operators can be seen in [3, 6, 12, 20].

The main purpose of this article is to present some new integral inequalities for quasi-convex and P-function including Atangana-Baleanu integral operator with the help of the identity given earlier by Set et al. in [20].

2. Main Results

Let $\kappa : [\mu,\nu] \to \mathbb{R}$ be differentiable function on (μ,ν) with $\mu < \nu$. Throughout this section we will take

$$ \begin{aligned} & {}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu) \\ = \; & {}^{AB}_{\ \mu}I^{\alpha}\left\{\kappa\left(\frac{\mu+\rho}{2}\right)\right\} + {}^{AB}I^{\alpha}_{\rho}\left\{\kappa\left(\frac{\mu+\rho}{2}\right)\right\} + {}^{AB}_{\ \rho}I^{\alpha}\left\{\kappa\left(\frac{\nu+\rho}{2}\right)\right\} + {}^{AB}I^{\alpha}_{\nu}\left\{\kappa\left(\frac{\nu+\rho}{2}\right)\right\} \\ & -\frac{(\rho-\mu)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\kappa(\rho)+\kappa(\mu)\right] - \frac{(\nu-\rho)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\kappa(\rho)+\kappa(\nu)\right] \\ & -\frac{2(1-\alpha)}{B(\alpha)}\left[\kappa\left(\frac{\mu+\rho}{2}\right)+\kappa\left(\frac{\nu+\rho}{2}\right)\right]. \end{aligned} $$

In [20], Set et al. established a new identity via Atangana-Baleanu fractional fractional operators as follows.

Lemma 2.1. *$\kappa : [\mu, \nu] \to \mathbb{R}$ be differentiable function on (μ, ν) with $\mu < \nu$. Then we have the following identity for Atangana-Baleanu fractional integral operators*

$$\begin{aligned} & {}^{AB}I_\kappa(\rho, \alpha, \mu, \nu) \\ = & \frac{(\rho-\mu)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)} \left[\int_0^1 \frac{\omega^\alpha}{2} \kappa'\left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\mu\right) d\omega - \int_0^1 \frac{\omega^\alpha}{2} \kappa'\left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\mu\right) d\omega \right] \\ & + \frac{(\nu-\rho)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)} \left[\int_0^1 \frac{\omega^\alpha}{2} \kappa'\left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\nu\right) d\omega - \int_0^1 \frac{\omega^\alpha}{2} \kappa'\left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\nu\right) d\omega \right] \end{aligned}$$

where $\alpha \in [0, 1]$, $\rho \in [\mu, \nu]$ and $\Gamma(.)$ is Gamma function.

Theorem 2.1. *$\kappa : [\mu, \nu] \to \mathbb{R}$ be differentiable function on (μ, ν) with $\mu < \nu$ and $\kappa' \in L_1[\mu, \nu]$. If $|\kappa'|$ is a quasi-convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned} & \left| {}^{AB}I_\kappa(\rho, \alpha, \mu, \nu) \right| \\ \leq & \frac{(\rho-\mu)^{\alpha+1} \max\{|\kappa'(\rho)|, |\kappa'(\mu)|\}}{2^\alpha B(\alpha)\Gamma(\alpha+1)} + \frac{(\nu-\rho)^\alpha \max\{|\kappa'(\rho)|, |\kappa'(\nu)|\}}{2^\alpha B(\alpha)\Gamma(\alpha+1)} \end{aligned} \tag{2.1}$$

where $\rho \in [\mu, \nu]$, $\alpha \in [0, 1]$, $B(\alpha)$ is normalization function.

Proof. By using the identity that is given in Lemma 2.1, we can write

$$\begin{aligned} & \left| {}^{AB}I_\kappa(\rho, \alpha, \mu, \nu) \right| \\ = & \left| \frac{(\rho-\mu)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)} \left[\int_0^1 \frac{\omega^\alpha}{2} \kappa'\left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\mu\right) d\omega - \int_0^1 \frac{\omega^\alpha}{2} \kappa'\left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\mu\right) d\omega \right] \right. \\ & \left. + \frac{(\nu-\rho)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)} \left[\int_0^1 \frac{\omega^\alpha}{2} \kappa'\left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\nu\right) d\omega - \int_0^1 \frac{\omega^\alpha}{2} \kappa'\left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\nu\right) d\omega \right] \right| \\ \leq & \frac{(\rho-\mu)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)} \left[\int_0^1 \frac{\omega^\alpha}{2} \left| \kappa'\left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\mu\right) \right| d\omega + \int_0^1 \frac{\omega^\alpha}{2} \left| \kappa'\left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\mu\right) \right| d\omega \right] \\ & + \frac{(\nu-\rho)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)} \left[\int_0^1 \frac{\omega^\alpha}{2} \left| \kappa'\left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\nu\right) \right| d\omega + \int_0^1 \frac{\omega^\alpha}{2} \left| \kappa'\left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\nu\right) \right| d\omega \right]. \end{aligned} \tag{2.2}$$

By using quasi-convexity of $|\kappa'|$, we get

$$\begin{aligned} & \left| {}^{AB}I_\kappa(\rho, \alpha, \mu, \nu) \right| \\ \leq & \frac{(\rho-\mu)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)} \left[\int_0^1 \frac{\omega^\alpha}{2} \max\{|\kappa'(\rho)|, |\kappa'(\mu)|\} d\omega + \int_0^1 \frac{\omega^\alpha}{2} \max\{|\kappa'(\rho)|, |\kappa'(\mu)|\} d\omega \right] \\ & + \frac{(\nu-\rho)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)} \left[\int_0^1 \frac{\omega^\alpha}{2} \max\{|\kappa'(\rho)|, |\kappa'(\nu)|\} d\omega + \int_0^1 \frac{\omega^\alpha}{2} \max\{|\kappa'(\rho)|, |\kappa'(\nu)|\} d\omega \right] \\ = & \frac{(\rho-\mu)^{\alpha+1} \max\{|\kappa'(\rho)|, |\kappa'(\mu)|\}}{2^\alpha B(\alpha)\Gamma(\alpha+1)} + \frac{(\nu-\rho)^{\alpha+1} \max\{|\kappa'(\rho)|, |\kappa'(\nu)|\}}{2^\alpha B(\alpha)\Gamma(\alpha+1)} \end{aligned}$$

and the proof is completed. □

Corollary 2.1. *In Theorem 2.1, if we take $\alpha = 1$, then the inequality (2.1) reduces to the inequality*

$$\begin{aligned} & \left| \int_{\mu}^{\nu} \kappa(x)dx - \frac{(\rho-\mu)}{2}\left[\kappa(\rho)+\kappa(\mu)\right] - \frac{(\nu-\rho)}{2}\left[\kappa(\rho)+\kappa(\nu)\right] \right| \\ \leq & \frac{(\rho-\mu)^2}{2} \max\left\{ \left|\kappa'(\rho)\right|, \left|\kappa'(\mu)\right| \right\} + \frac{(\nu-\rho)^2}{2} \max\left\{ \left|\kappa'(\rho)\right|, \left|\kappa'(\nu)\right| \right\}. \end{aligned}$$

Theorem 2.2. *$\kappa : [\mu, \nu] \to \mathbb{R}$ be differentiable function on (μ, ν) with $\mu < \nu$ and $\kappa' \in L_1[\mu, \nu]$. If $|\kappa'|^q$ is a quasi-convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned} & \left| {}^{AB}I_{\kappa}(\rho, \alpha, \mu, \nu) \right| \\ \leq & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha p+1} \right)^{\frac{1}{p}} \left(\max\left\{ \left|\kappa'(\rho)\right|^q, \left|\kappa'(\mu)\right|^q \right\} \right)^{\frac{1}{q}} \\ & + \frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha p+1} \right)^{\frac{1}{p}} \left(\max\left\{ \left|\kappa'(\rho)\right|^q, \left|\kappa'(\nu)\right|^q \right\} \right)^{\frac{1}{q}} \end{aligned} \tag{2.3}$$

where $p^{-1} + q^{-1} = 1$, $\rho \in [\mu, \nu]$, $\alpha \in [0, 1]$, $q > 1$, $B(\alpha)$ is normalization function.

Proof. By applying Hölder inequality to the inequality (2.2), we have

$$\begin{aligned} & \left| {}^{AB}I_{\kappa}(\rho, \alpha, \mu, \nu) \right| \\ \leq & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 \left(\frac{\omega^{\alpha}}{2} \right)^p d\omega \right)^{\frac{1}{p}} \left(\int_0^1 \left| \kappa'\left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\mu \right) \right|^q d\omega \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 \left(\frac{\omega^{\alpha}}{2} \right)^p d\omega \right)^{\frac{1}{p}} \left(\int_0^1 \left| \kappa'\left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\mu \right) \right|^q d\omega \right)^{\frac{1}{q}} \right] \\ & + \frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 \left(\frac{\omega^{\alpha}}{2} \right)^p d\omega \right)^{\frac{1}{p}} \left(\int_0^1 \left| \kappa'\left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\nu \right) \right|^q d\omega \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 \left(\frac{\omega^{\alpha}}{2} \right)^p d\omega \right)^{\frac{1}{p}} \left(\int_0^1 \left| \kappa'\left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\nu \right) \right|^q d\omega \right)^{\frac{1}{q}} \right]. \end{aligned} \tag{2.4}$$

By using quasi-convexity of $|\kappa'|^q$, we have

$$\begin{aligned}
&\left|{}^{AB}I_\kappa(\rho,\alpha,\mu,\nu)\right| \\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\left(\frac{\omega^\alpha}{2}\right)^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1 \max\left\{|\kappa'(\rho)|^q, |\kappa'(\mu)|^q\right\} d\omega\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left(\frac{\omega^\alpha}{2}\right)^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1 \max\left\{|\kappa'(\rho)|^q, |\kappa'(\mu)|^q\right\} d\omega\right)^{\frac{1}{q}}\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\left(\frac{\omega^\alpha}{2}\right)^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1 \max\left\{|\kappa'(\rho)|^q, |\kappa'(\nu)|^q\right\} d\omega\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left(\frac{\omega^\alpha}{2}\right)^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1 \max\left\{|\kappa'(\rho)|^q, |\kappa'(\nu)|^q\right\} d\omega\right)^{\frac{1}{q}}\right] \\
=\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha-1} B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^p(\alpha p+1)}\right)^{\frac{1}{p}}\left(\max\left\{|\kappa'(\rho)|^q, |\kappa'(\mu)|^q\right\}\right)^{\frac{1}{q}} \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha-1} B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^p(\alpha p+1)}\right)^{\frac{1}{p}}\left(\max\left\{|\kappa'(\rho)|^q, |\kappa'(\nu)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

So, the proof is completed. □

Corollary 2.2. *In Theorem 2.2, if we take $\alpha = 1$, then the inequality (2.3) reduces to the inequality*

$$\begin{aligned}
&\left|\int_\mu^\nu \kappa(x)dx - \frac{(\rho-\mu)}{2}\left[\kappa(\rho)+\kappa(\mu)\right] - \frac{(\nu-\rho)}{2}\left[\kappa(\rho)+\kappa(\nu)\right]\right| \\
\leq\ & \frac{(\rho-\mu)^2}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\max\left\{|\kappa'(\rho)|^q, |\kappa'(\mu)|^q\right\}\right)^{\frac{1}{q}} \\
&+\frac{(\nu-\rho)^2}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\max\left\{|\kappa'(\rho)|^q, |\kappa'(\nu)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.3. *$\kappa : [\mu,\nu] \to \mathbb{R}$ be differentiable function on (μ,ν) with $\mu < \nu$ and $\kappa' \in L_1[\mu,\nu]$. If $|\kappa'|^q$ is a quasi-convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&\left|{}^{AB}I_\kappa(\rho,\alpha,\mu,\nu)\right| \\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)}\left[\frac{1}{2^{p-1}\,p\,(\alpha p+1)} + \frac{\max\left\{|\kappa'(\rho)|^q, |\kappa'(\mu)|^q\right\}}{q}\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)}\left[\frac{1}{2^{p-1}\,p\,(\alpha p+1)} + \frac{\max\left\{|\kappa'(\rho)|^q, |\kappa'(\nu)|^q\right\}}{q}\right]
\end{aligned} \tag{2.5}$$

where $p^{-1}+q^{-1} = 1$, $\rho \in [\mu,\nu]$, $\alpha \in [0,1]$, $q > 1$, $B(\alpha)$ is normalization function.

Proof. By applying Young inequality as $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$ to the inequality (2.2), we have

$$\begin{aligned}
& \left| {}^{AB}I_\kappa(\rho,\alpha,\mu,\nu) \right| \\
\leq \; & \frac{(\rho-\mu)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)} \left[\frac{1}{p}\int_0^1 \left(\frac{\omega^\alpha}{2}\right)^p d\omega + \frac{1}{q}\int_0^1 \left| \kappa'\left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\mu\right)\right|^q d\omega \right. \\
& \left. + \frac{1}{p}\int_0^1 \left(\frac{\omega^\alpha}{2}\right)^p d\omega + \frac{1}{q}\int_0^1 \left| \kappa'\left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\mu\right)\right|^q d\omega \right] \\
& + \frac{(\nu-\rho)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha)} \left[\frac{1}{p}\int_0^1 \left(\frac{\omega^\alpha}{2}\right)^p d\omega + \frac{1}{q}\int_0^1 \left| \kappa'\left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\nu\right)\right|^q d\omega \right. \\
& \left. + \frac{1}{p}\int_0^1 \left(\frac{\omega^\alpha}{2}\right)^p d\omega + \frac{1}{q}\int_0^1 \left| \kappa'\left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\nu\right)\right|^q d\omega \right].
\end{aligned}$$

By using quasi-convexity of $|\kappa'|^q$ and by a simple computation, we have the desired result. □

Corollary 2.3. *In Theorem 2.3, if we take $\alpha = 1$, then the inequality (2.5) reduces to the inequality*

$$\begin{aligned}
& \left| \int_\mu^\nu \kappa(x)dx - \frac{(\rho-\mu)}{2}\left[\kappa(\rho)+\kappa(\mu)\right] - \frac{(\nu-\rho)}{2}\left[\kappa(\rho)+\kappa(\nu)\right] \right| \\
\leq \; & \frac{(\rho-\mu)^2}{2}\left[\frac{1}{2^{p-1}\, p\,(p+1)} + \frac{\max\left\{|\kappa'(\rho)|^q, |\kappa'(\mu)|^q\right\}}{q} \right] \\
& + \frac{(\nu-\rho)^2}{2}\left[\frac{1}{2^{p-1}\, p\,(p+1)} + \frac{\max\left\{|\kappa'(\rho)|^q, |\kappa'(\nu)|^q\right\}}{q} \right].
\end{aligned}$$

Theorem 2.4. *$\kappa : [\mu,\nu] \to \mathbb{R}$ be differentiable function on (μ,ν) with $\mu < \nu$ and $\kappa' \in L_1[\mu,\nu]$. If $|\kappa'|^q$ is a quasi-convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
& \left| {}^{AB}I_\kappa(\rho,\alpha,\mu,\nu) \right| \\
\leq \; & \frac{(\rho-\mu)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha+1)}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\max\left\{|\kappa'(\rho)|^q, |\kappa'(\mu)|^q\right\}\right)^{\frac{1}{q}} \\
& + \frac{(\nu-\rho)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha+1)}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\max\left\{|\kappa'(\rho)|^q, |\kappa'(\nu)|^q\right\}\right)^{\frac{1}{q}}
\end{aligned} \tag{2.6}$$

where $\rho \in [\mu,\nu]$, $\alpha \in [0,1]$, $q \geq 1$, $B(\alpha)$ is normalization function.

Proof. By using power mean inequality in the inequality (2.2), we have

$$\begin{aligned}
&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 \frac{\omega^{\alpha}}{2}d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)\right|^q d\omega\right)^{\frac{1}{q}}\right.\\
&+\left.\left(\int_0^1 \frac{\omega^{\alpha}}{2}d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)\right|^q d\omega\right)^{\frac{1}{q}}\right]\\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 \frac{\omega^{\alpha}}{2}d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)\right|^q d\omega\right)^{\frac{1}{q}}\right.\\
&+\left.\left(\int_0^1 \frac{\omega^{\alpha}}{2}d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)\right|^q d\omega\right)^{\frac{1}{q}}\right]. \qquad (2.7)
\end{aligned}$$

By using quasi-convexity of $|\kappa'|^q$ and by a simple computation, we have the desired result. □

Corollary 2.4. *In Theorem 2.4, if we take $\alpha=1$, then the inequality (2.6) reduces to the inequality*

$$\begin{aligned}
&\left|\int_{\mu}^{\nu}\kappa(x)dx-\frac{(\rho-\mu)}{2}\left[\kappa(\rho)+\kappa(\mu)\right]-\frac{(\nu-\rho)}{2}\left[\kappa(\rho)+\kappa(\nu)\right]\right|\\
\leq\ & \frac{(\rho-\mu)^2}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\max\left\{\left|\kappa'(\rho)\right|^q,\left|\kappa'(\mu)\right|^q\right\}\right)^{\frac{1}{q}}\\
&+\frac{(\nu-\rho)^2}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\max\left\{\left|\kappa'(\rho)\right|^q,\left|\kappa'(\nu)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.5. *$\kappa:[\mu,\nu]\to\mathbb{R}$ be differentiable function on (μ,ν) with $\mu<\nu$ and $\kappa'\in L_1[\mu,\nu]$. If $|\kappa'|$ is a P-function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right|\leq\frac{(\rho-\mu)^{\alpha+1}\left(|\kappa'(\rho)|+|\kappa'(\mu)|\right)}{2^{\alpha}B(\alpha)\Gamma(\alpha+1)}+\frac{(\nu-\rho)^{\alpha+1}\left(|\kappa'(\rho)|+|\kappa'(\nu)|\right)}{2^{\alpha}B(\alpha)\Gamma(\alpha+1)}$$

where $\rho\in[\mu,\nu]$, $\alpha\in[0,1]$, $B(\alpha)$ is normalization function.

Proof. By using the identity that is given in Lemma 2.1 and since $|\kappa'|$ is P-function, we can write

$$\begin{aligned}
&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right|\\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1\frac{\omega^{\alpha}}{2}(|\kappa'(\rho)|+|\kappa'(\mu)|)d\omega+\int_0^1\frac{\omega^{\alpha}}{2}(|\kappa'(\rho)|+|\kappa'(\mu)|)d\omega\right]\\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1\frac{\omega^{\alpha}}{2}(|\kappa'(\rho)|+|\kappa'(\nu)|)d\omega+\int_0^1\frac{\omega^{\alpha}}{2}(|\kappa'(\rho)|+|\kappa'(\nu)|)d\omega\right]\\
=\ & \frac{(\rho-\mu)^{\alpha+1}\left(|\kappa'(\rho)|+|\kappa'(\mu)|\right)}{2^{\alpha}B(\alpha)\Gamma(\alpha+1)}+\frac{(\nu-\rho)^{\alpha+1}\left(|\kappa'(\rho)|+|\kappa'(\nu)|\right)}{2^{\alpha}B(\alpha)\Gamma(\alpha+1)}
\end{aligned}$$

and the proof is completed. □

Theorem 2.6. *$\kappa : [\mu, \nu] \to \mathbb{R}$ be differentiable function on (μ, ν) with $\mu < \nu$ and $\kappa' \in L_1[\mu, \nu]$. If $|\kappa'|^q$ is a P-function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned} &\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\ \leq\ & \frac{(\rho-\mu)^{\alpha+1}\left(|\kappa'(\rho)|^q+|\kappa'(\mu)|^q\right)^{\frac{1}{q}}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \\ &+\frac{(\nu-\rho)^{\alpha+1}\left(|\kappa'(\rho)|^q+|\kappa'(\nu)|^q\right)^{\frac{1}{q}}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \end{aligned}$$

where $p^{-1}+q^{-1}=1$, $\rho \in [\mu,\nu]$, $\alpha \in [0,1]$, $q>1$, $B(\alpha)$ is normalization function.

Proof. By using the definition of $P-$function in the inequality (2.4), we have

$$\begin{aligned} &\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\ \leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1(|\kappa'(\rho)|^q+|\kappa'(\mu)|^q)d\omega\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1(|\kappa'(\rho)|^q+|\kappa'(\mu)|^q)d\omega\right)^{\frac{1}{q}}\right] \\ &+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1(|\kappa'(\rho)|^q+|\kappa'(\nu)|^q)d\omega\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1(|\kappa'(\rho)|^q+|\kappa'(\nu)|^q)d\omega\right)^{\frac{1}{q}}\right] \\ =\ & \frac{(\rho-\mu)^{\alpha+1}\left(|\kappa'(\rho)|^q+|\kappa'(\mu)|^q\right)^{\frac{1}{q}}}{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^p(\alpha p+1)}\right)^{\frac{1}{p}} \\ &+\frac{(\nu-\rho)^{\alpha+1}\left(|\kappa'(\rho)|^q+|\kappa'(\nu)|^q\right)^{\frac{1}{q}}}{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^p(\alpha p+1)}\right)^{\frac{1}{p}}. \end{aligned}$$

So, the proof is completed. □

Theorem 2.7. *$\kappa : [\mu, \nu] \to \mathbb{R}$ be differentiable function on (μ, ν) with $\mu < \nu$ and $\kappa' \in L_1[\mu, \nu]$. If $|\kappa'|^q$ is a P-convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned} &\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\ \leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\frac{1}{2^{p-1}\,p\,(\alpha p+1)}+\frac{|\kappa'(\rho)|^q+|\kappa'(\mu)|^q}{q}\right] \\ &+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\frac{1}{2^{p-1}\,p\,(\alpha p+1)}+\frac{|\kappa'(\rho)|^q+|\kappa'(\nu)|^q}{q}\right] \end{aligned}$$

where $p^{-1}+q^{-1}=1$, $\rho \in [\mu,\nu]$, $\alpha \in [0,1]$, $q>1$, $B(\alpha)$ is normalization function.

Proof. By using Young inequality as $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$ in the inequality (2.2), by the definition of P-function and by a simple computation, we have the desired result. □

Theorem 2.8. *$\kappa : [\mu, \nu] \to \mathbb{R}$ be differentiable function on (μ, ν) with $\mu < \nu$ and $\kappa' \in L_1[\mu, \nu]$. If $|\kappa'|^q$ is a $P-$convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned} & \left| {}^{AB}I_\kappa(\rho, \alpha, \mu, \nu) \right| \\ \leq \; & \frac{(\rho-\mu)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha+1)} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(|\kappa'(\rho)|^q + |\kappa'(\mu)|^q \right)^{\frac{1}{q}} \\ & + \frac{(\nu-\rho)^{\alpha+1}}{2^\alpha B(\alpha)\Gamma(\alpha+1)} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(|\kappa'(\rho)|^q + |\kappa'(\nu)|^q \right)^{\frac{1}{q}} \end{aligned}$$

where $\rho \in [\mu, \nu]$, $\alpha \in [0, 1]$, $q \geq 1$, $B(\alpha)$ is normalization function.

Proof. In the inequality (2.7), by using the definition of P-function and by a simple computation, we have the desired result. □

3. Conclusion

The study dealt with investigating new Hermite-Hadamard type inequalities for AB-fractional integral operators. We extend the study of Hermite-Hadamard type inequalities via AB-fractional integral operators for differentiable mapping whose derivatives in the absolute values are quasi-convex and P-function. All these integral inequalities are open to being investigated for other classes of convexity functions.

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