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ON WEIGHTED MEANS AND MN-CONVEX FUNCTIONS

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Abstract. In this paper, we give more general definitions of weighted means and MN-convex functions. Using these definitions, we also obtain some generalized results related to the properties of MN-convex functions. The importance of this study is that the results of this paper can be reduced to different convexity classes by considering the special cases of M and N.

1. Introduction

The notions of convexity and concavity of a real-valued function of a real variable are well known [16]. The generalized condition of convexity, i.e. MN-convexity with respect to arbitrary means M and N, was proposed in 1933 by Aumann [2]. Recently many authors have dealt with these generalizations. In particular, Niculescu [15] compared MN-convexity with relative convexity. Andersen et al. [3] examined inequalities implied by MN-convexity. In [3], Anderson et al. studied certain generalizations of these notions for a positive-valued function of a positive variable as follows:

Definition 1.1. A function $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ is called a Mean function if

(M1) $M(u,v) = M(v,u)$,
(M2) $M(u,u) = u$,
(M3) $u < M(u,v) < v$ whenever $u < v$,
(M4) $M(\lambda u, \lambda v) = \lambda M(u,v)$ for all $\lambda > 0$.

Example 1.1. For $u, v \in (0,\infty)$

$$M(u,v) = A(u,v) = A = \frac{u+v}{2}$$

is the Arithmetic Mean,

$$M(u,v) = G(u,v) = G = \sqrt{uv}$$

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is the Geometic Mean,

$$M(u,v) = H(u,v) = H = A^{-1}(u^{-1}, v^{-1}) = \frac{2uv}{u+v}$$

is the Harmonic Mean,

$$M(u,v) = L(u,v) = L = \begin{cases} \frac{u-v}{\ln u - \ln v} & u \neq v \\ u & u = v \end{cases}$$

is the Logarithmic Mean,

$$M(u,v) = I(u,v) = I = \begin{cases} \frac{1}{e}\left(\frac{u^u}{v^v}\right)^{\frac{1}{u-v}} & u \neq v \\ u & u = v \end{cases}$$

is the Identric Mean,

$$M(u,v) = M_p(u,v) = M_p = \begin{cases} A^{1/p}(u^p, v^p) = \left(\frac{u^p+v^p}{2}\right)^{1/p} & p \in \mathbb{R}\backslash\{0\} \\ G(u,v) = \sqrt{uv} & p = 0 \end{cases}$$

is the p-Power Mean, In particular, we have the following inequality

$$M_{-1} = H \leq M_0 = G \leq L \leq I \leq A = M_1.$$

Anderson et al. in [3] developed a systematic study to the classical theory of continuous and midconvex functions, by replacing a given mean instead of the arithmetic mean.

Definition 1.2. Let M and N be two means defined on the intervals $I \subset (0,\infty)$ and $J \subset (0,\infty)$ respectively, a function $f : I \to J$ is called MN-midpoint convex if it satisfies

$$f\left(M(u,v)\right) \leq N\left(f(u), f(v)\right)$$

for all $u, v \in I$.

The concept of MN-convexity has been studied extensively in the literature from various points of view (see e.g. [1, 2, 12, 15]),

Let $A\left(u,v,\lambda\right) = \lambda u + (1-\lambda)v$, $G\left(u,v,\lambda\right) = u^{\lambda}v^{1-\lambda}$, $H\left(u,v,\lambda\right) = uv/(\lambda u + (1-\lambda)v)$ and $M_p\left(u,v,\lambda\right) = (\lambda u^p + (1-\lambda)v^p)^{1/p}$ be the weighted arithmetic, geometric, harmonic , power of order p means of two positive real numbers u and v with $u \neq v$ for $\lambda \in [0,1]$, respectively. $M_p\left(u,v,\lambda\right)$ is continuous and strictly increasing with respect to $\lambda \in \mathbb{R}$ for fixed $p \in \mathbb{R}\backslash\{0\}$ and $u, v > 0$ with $u > v$. See [6, 9, 12–15] for some kinds of convexity obtained by using weighted means.

The aims of this paper, a general definition of weighted means and a general definition of MN-convex functions via the weighted means is to give. In recent years, many studies have been done by considering the special cases of M and N. The importance of this study is that some properties of MN-convex functions and some related inequalities have been proven in general terms.

2. Main Results

Definition 2.1. A function $M:(0,\infty)\times(0,\infty)\times[0,1]\to(0,\infty)$ is called a weighted mean function if

(WM1) $M(u,v,\lambda)=M(v,u,1-\lambda)$,
(WM2) $M(u,u,\lambda)=u$,
(WM3) $u<M(u,v,\lambda)<v$ whenever $u<v$ and $\lambda\in(0,1)$. Also $\{M(u,v,0),M(u,v,1)\}=\{u,v\}$.
(WM4) $M(\alpha u,\alpha v,\lambda)=\alpha M(u,v,\lambda)$ for all $\alpha>0$,
(WM5) let $\lambda\in[0,1]$ be fixed. Then $M(u,v,\lambda)\le M(w,v,\lambda)$ whenever $u\le w$ and $M(u,v,\lambda)\le M(u,\omega,\lambda)$ whenever $v\le\omega$.
(WM6) let $u,v\in(0,\infty)$ be fixed and $u\neq v$. Then $M(u,v,.)$ is a strictly monotone and continuous function on $[0,1]$.
(WM7) $M\left(M(u,v,\lambda),M(z,w,\lambda),s\right)=M\left(M(u,z,s),M(v,w,s),\lambda\right)$ for all $u,v,z,w\in(0,\infty)$ and $s,\lambda\in[0,1]$.
(WM8) $M(u,v,s\lambda_1+(1-s)\lambda_2)=M\left(M(u,v,\lambda_1),M(u,v,\lambda_2),s\right)$ for all $u,v\in(0,\infty)$ and $s,\lambda_1,\lambda_2\in[0,1]$.

Remark 2.1. According to the above definition every weighted mean function is a mean function with $\lambda=1/2$. Also, By (WM6) we can say that for each $x\in[u,v]\subseteq(0,\infty)$ there exists a $\lambda\in[0,1]$ such that $x=M(u,v,\lambda)$. Morever;

i.) If $M(u,v,.)$ is a strictly increasing, then $M(u,v,0)=u$ and $M(u,v,1)=v$ whenever $u<v$ (i.e. $M(u,v,\lambda)$ is in the positive direction)

ii.) If $M(u,v,.)$ is a strictly deccreasing, then $M(u,v,0)=v$ and $M(u,v,1)=u$ whenever $u<v$ (i.e. $M(u,v,\lambda)$ is in the negative direction) and $M(u,v,.)([0,1])=[\min\{u,v\},\max\{u,v\}]$.

Remark 2.2. Throughout this paper, we will assume that different weighted means have the same direction unless otherwise stated.

Example 2.1.

$$M(u,v,\lambda)=A(u,v,\lambda)=A_\lambda=(1-\lambda)u+\lambda v$$

is the Weighted Arithmetic Mean,

$$M(u,v,\lambda)=G(u,v,\lambda)=G_\lambda=u^{1-\lambda}v^\lambda$$

is the Weighted Geometic Mean,

$$M(u,v,\lambda)=H(u,v,\lambda)=H_\lambda=A^{-1}(u^{-1},v^{-1},\lambda)=\frac{uv}{\lambda u+(1-\lambda)v}$$

is the Weighted Harmonic Mean,

$$M(u,v,\lambda)=M_p(u,v,\lambda)=M_{p,\lambda}=\begin{cases} A^{1/p}(u^p,v^p,\lambda)=\left((1-\lambda)x^p+\lambda y^p\right)^{1/p} & p\in\mathbb{R}\backslash\{0\} \\ G(u,v,\lambda)=u^{1-\lambda}v^\lambda & p=0 \end{cases}$$

is the p-Power Mean. In particular, we have the following inequality

$$M_{-1,\lambda} = H_\lambda \leq M_{0,\lambda} = G_\lambda \leq M_{1,\lambda} = A_\lambda \leq M_{p,\lambda}$$

for all $x, y \in (0,\infty), t \in [0,1]$ and $p \geq 1$.

Proposition 2.1. *If $M : (0,\infty) \times (0,\infty) \times [0,1] \to (0,\infty)$ is a weighted mean function, then the following identities hold:*

$$M\left(M\left(a, M(a,b,s), \lambda\right), M\left(b, M(a,b,s), \lambda\right), s\right) = M(a,b,s), \tag{2.1}$$

$$M\left(M(a,b,\lambda), M(b,a,\lambda), 1/2\right) = M(a,b,1/2). \tag{2.2}$$

Proof. If we take $v = w = M(a,b,s),\ u = a$ and $z = b$ in (WM7) and we use the property (WM2), then we obtained the identity (2.1). By using similar method, if we take $u = w = a$, $v = z = b$ and $s = 1/2$ in (WM7) and we use the properties (WM1) and (WM2), then we obtained the identity (2.2). □

Definition 2.2. Let M and N be two weighted means defined on the intervals $I \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively, a function $f : I \to J$ is called MN-convex (concave) if it satisfies

$$f\left(M(u,v,\lambda)\right) \leq (\geq)\, N\left(f(u), f(v), \lambda\right)$$

for all $u, v \in I$ and $\lambda \in [0,1]$.

The condition (WM8) in Definition 2.1 shows us that the function $M(u,v,.)$ is both MM-convex and MM-concave on $[0,1]$ for fixed $u, v \in (0,\infty)$. It is easily seen that weighted means mentioned in the Example 2.1 hold the condition (WM8).

We note that by considering the special cases of M and N, we obtain several different convexity classes as AA-convexity (classical convexity), AG-convexity (log-convexity), GA-convexity, GG-convexity (geometrically convexity), HA-convexity (harmonically convexity), M_pA-convexity (p-convexity),...,etc. For some convexity types, see ([6, 9, 14, 15]).

Definition 2.3. Let M and N be two weighted means defined on the intervals $[u,v] \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively and $f : [u,v] \to J$ be a function. We say that f is symmetric with respect to $M(u,v,1/2)$, if it satisfies

$$f\left(M(u,v,\lambda)\right) = f\left(M(u,v,1-\lambda)\right)$$

for all $\lambda \in [0,1]$.

Theorem 2.1. *Let M and N be two weighted means defined on the intervals $[u,v] \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively. If function $f : [u,v] \to J$ is MN-convex, then the function f is bounded.*

Proof. Let $K = \max\left\{f(u), f(v)\right\}$. For any $z = M(u,v,\lambda)$ in the interval $[u,v]$, By using MN-convexity of f and (WM3) we have

$$f(z) \leq N\left(f(u), f(v), \lambda\right) \leq K.$$

The function f is also bounded from below. For any $z \in (u, v]$, there exists a $\lambda_0 \in (0,1]$ such that $z = M(u, v, \lambda_0)$, then by using MN-convexity of f and (2.2) we have

$$f(M(u,v,1/2)) = f(M(z, M(v,u,\lambda_0), 1/2)) \leq N(f(z), f(M(v,u,\lambda_0)), 1/2). \quad (2.3)$$

On the other hand, if $f(z) = f(M(v,u,\lambda_0))$, then $N(f(z), f(M(v,u,\lambda_0)), 1/2) = f(z)$ and thus the function f is also bounded from below.

If $f(z) \neq f(M(v,u,\lambda_0))$, then there exists $\mu_0 \in (0,1)$ such that

$$N(f(z), f(M(v,u,\lambda_0)), 1/2) = \mu_0 f(z) + (1-\mu_0) f(M(v,u,\lambda_0)).$$

By the inequality (2.3) and using K as the upper bound, we have

$$\begin{aligned} f(z) &\geq \frac{1}{\lambda_0}\left[f(M(u,v,1/2)) - (1-\lambda_0) f(M(v,u,\lambda_0))\right] \\ &\geq \frac{1}{\lambda_0}\left[f(M(u,v,1/2)) - (1-\lambda_0)K\right] = k. \end{aligned}$$

Thus, we obtain $f(z) \geq \max\{k, f(u)\}$ for any $z \in [u, v]$. This completes the proof. □

Theorem 2.2. *Let M and N be two weighted means defined on the intervals $I \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively. If the functions $f, g : I \to J$ are MN-convex, then $N(f(.), g(.), 1/2)$ is a MN -convex function.*

Proof. Since f and g are MN-convex functions, we have

$$f(M(u,v,\lambda)) \leq N(f(u), f(v), \lambda)$$

and

$$g(M(u,v,\lambda)) \leq N(g(u), g(v), \lambda)$$

for all $u, v \in I$ and $\lambda \in [0,1]$. Then by (WM5) and (WM7) we have

$$\begin{aligned} & N(f(.), g(.), 1/2)(M(u,v,\lambda)) \\ = \; & N(f(M(u,v,\lambda)), g(M(u,v,\lambda)), 1/2) \\ \leq \; & N(N(f(u), f(v), \lambda), N(g(u), g(v), \lambda), 1/2) \\ = \; & N(N(f(.), g(.), 1/2)(u), N(f(.), g(.), 1/2)(v), \lambda). \end{aligned}$$

This completes the proof. □

We can give the following results for different convexity classes by considering the special cases of M and N.

Corollary 2.1. *Let $I, J \subseteq (0,\infty)$ and $f, g : I \to J$.*

i.) If f and g are convex functions, then $A(f(.), g(.), 1/2) = (f+g)/2$ is also convex function.

ii.) If f and g are GA-convex functions, then $A(f(.), g(.), 1/2) = (f+g)/2$ is also GA-convex function.

iii.) If f and g are harmonically convex functions, then $A(f(.), g(.), 1/2) = (f+g)/2$ is also harmonically convex function.

iv.) If f and g are p-convex functions, then $A(f(.), g(.), 1/2) = (f+g)/2$ is also p-convex function.

v.) If f and g are log-convex functions, then $G(f(.), g(.), 1/2) = \sqrt{fg}$ is also log-convex function.

vi.) If f and g are GG-convex functions, then $G(f(.), g(.), 1/2) = \sqrt{fg}$ is also GG-convex function.

vii.) If f and g are HG-convex functions, then $G(f(.), g(.), 1/2) = \sqrt{fg}$ is also HG-convex function.

viii.) If f and g are AH-convex functions, then $H(f(.), g(.), 1/2) = 2fg/(f+g)$ is also AH-convex function.

Remark 2.3. In Corollary 2.1, we gave results only for some convexity types. It is possible to increase the results by considering another special cases of M and N.

Theorem 2.3. *Let M and N be two weighted means defined on the intervals $I \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively. If $f : I \to J$ is a MN-convex function and $\alpha > 0$, then αf is a MN-convex function.*

Proof. By using MN-convexity of f and (WM4), we have

$$\alpha f\left(M(u,v,\lambda)\right) \leq \alpha N\left(f(u), f(v), \lambda\right) \leq N\left(\alpha f(u), \alpha f(v), \lambda\right).$$

This completes the proof. □

Theorem 2.4. *Let M, N and K be three weighted means defined on the intervals $I \subseteq (0,\infty)$, $J \subseteq (0,\infty)$ and $L \subseteq (0,\infty)$ respectively. If $f : I \to J$ is a MN-convex function and $g : J \subseteq (0,\infty) \to L$ is nondecreasing and NK-convex function, then $g \circ f$ is a MK-convex function.*

Proof. By using MN-convexity of f , we have

$$f\left(M(u,v,\lambda)\right) \leq N\left(f(u), f(v), \lambda\right).$$

Since g is NK-convex and nondecreasing function

$$g\left(f\left(M(u,v,\lambda)\right)\right) \leq g\left(N\left(f(u), f(v), \lambda\right)\right) \leq K\left(g(f(u)), g(f(v)), \lambda\right).$$

This completes the proof. □

Theorem 2.5. *Let M and N be two weighted means defined on the intervals $I \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively. If the function $f : I \to J$ is MN-convex and $N \leq A$ (A is the weighted arithmetic mean), then f satisfies Lipschitz condition on any closed interval $[a,b]$ contained in the interior I° of I. Consequently, f is absolutely continuous on $[a,b]$ and continuous on I°.*

Proof. Choose $\varepsilon > 0$ so that $a-\varepsilon$ and $b+\varepsilon$ belong to I, and let m_1 and m_2 be the lower and upper bounds for f on $[a-\varepsilon, b+\varepsilon]$. If u and v are distinct points of $[a,b]$ and we choose a point z such that

$$v = M(u,z,\lambda),\ \ \lambda = \frac{|v-u|}{\varepsilon + |v-u|},$$

then

$$f(v) \leq N\left(f(u), f(z), \lambda\right) \leq A\left(f(u), f(z), \lambda\right) = f(u) + \lambda\left[f(z) - f(u)\right]$$

$$f(v)-f(u)\leq\lambda\left[f(z)-f(u)\right]\leq\lambda(m_2-m_1)<\frac{|v-u|}{\varepsilon}(m_2-m_1)=K\left|v-u\right|$$

where $K=(m_2-m_1)/\varepsilon$. Since this is true for any $u,v\in[a,b]$, we conclude that $|f(v)-f(u)|\leq K\left|v-u\right|$ as desired.

Next we recall that f is absolutely continuous on $[a,b]$ if corresponding to any $\varepsilon>0$, we can produce a $\delta>0$ such that for any collection $\{(a_i,b_i)\}_1^n$ of disjoint open subintervals of $[a,b]$ with $\sum_{i=1}^n(b_i-a_i)<\delta$, $\sum_{i=1}^n|f(b_i)-f(a_i)|<\varepsilon$. Clearly the choice $\delta=\varepsilon/K$ meets this requirement.

Finally the continuity of f on I° is a consequence of the arbitrariness. □

Theorem 2.6. *Let M and N be two weighted means defined on the intervals $I\subseteq(0,\infty)$ and $J\subseteq(0,\infty)$ respectively. If function $f_\alpha:I\to J$ be an arbitrary family of MN-convex functions and let $f(u)=\sup_\alpha f_\alpha(u)$. If $K=\{x\in I:f(x)<\infty\}$ is nonempty, then K is an interval and f is MN-convex function on K.*

Proof. Let $\lambda\in[0,1]$ and $u,v\in K$ be arbitrary. Then

$$\begin{aligned}
& f\left(M\left(u,v,\lambda\right)\right)\\
=\ & \sup_\alpha f_\alpha\left(M\left(u,v,\lambda\right)\right)\\
\leq\ & \sup_\alpha\left(N\left(f_\alpha(u),f_\alpha(v),\lambda\right)\right)\\
\leq\ & N\left(\sup_\alpha f_\alpha(u),\sup_\alpha f_\alpha(v),\lambda\right)\\
=\ & N\left(f(u),f(v),\lambda\right)<\infty.
\end{aligned}$$

This shows simultaneously that K is an interval, since it contains every point between any two of its points, and that f is MN-convex function on K. This completes the proof of theorem. □

Theorem 2.7 (Hermite-Hadamard's inequalities for MN-convex functions)**.** *Let M and N be two weighted means defined on the intervals $I\subseteq(0,\infty)$ and $J\subseteq(0,\infty)$ respectively. If function $f:I\to J$ is MN-convex and the following integral exists, then we have*

$$f\left(M(u,v,1/2)\right)\leq\int_0^1N\left(f\left(M(u,v,\lambda)\right),f\left(M(u,v,1-\lambda)\right),1/2\right)d\lambda\leq N\left(f(u),f(v),1/2\right) \tag{2.4}$$

for all $u,v\in I$ with $u<v$.

Proof. Since $f:I\to\mathbb{R}$ is a MN-convex function, by using (2.2) we have

$$\begin{aligned}
f\left(M(u,v,1/2)\right) &= f\left(M\left(M(u,v,\lambda),M(u,v,1-\lambda),1/2\right)\right)\\
&\leq N\left(f\left(M(u,v,\lambda)\right),f\left(M(u,v,1-\lambda)\right),1/2\right)
\end{aligned}$$

for all $u,v\in I$ and $\lambda\in[0,1]$. Further, integrating for $\lambda\in[0,1]$, we have

$$f\left(M(u,v,1/2)\right)\leq\int_0^1N\left(f\left(M(u,v,\lambda)\right),f\left(M(u,v,1-\lambda)\right),1/2\right)d\lambda. \tag{2.5}$$

Thus, we obtain the left-hand side of the inequality (2.4) from (2.5).

Secondly, By using MN-convexity of f and (WM5) with (2.2), we get

$$\begin{aligned} & N\left(f\left(M(u,v,\lambda)\right), f\left(M(u,v,1-\lambda)\right), 1/2\right) \\ \leq\ & N\left(N(f(u),f(v),\lambda), N(f(u),f(v),1-\lambda), 1/2\right) \\ =\ & N\left(f(u),f(v),1/2\right). \end{aligned}$$

Integrating this inequality with respect to λ over $[0,1]$, we obtain the right-hand side of the inequality (2.4). This completes the proof. □

We can give the following some results for different convexity classes by considering the special cases of M and N. It is possible to increase the results by considering another special cases of M and N.

Corollary 2.2. *Let $I, J \subseteq (0,\infty)$ and $f : I \to J$.*

i.) If f is convex function (i.e. if $M = N = A$ (A is the weighted arithmetic mean)), then we have the following well-known celebrated Hermite-Hadamard's inequalities for convex functions

$$\begin{aligned} f\left(A(u,v,1/2)\right) &= f\left(\frac{u+v}{2}\right) \\ &\leq \int_0^1 A\left(f\left(A(u,v,\lambda)\right), f\left(A(u,v,1-\lambda)\right), 1/2\right) d\lambda \\ &= \frac{1}{2(v-u)} \int_u^v f(x) + f(u+v-x) dx \\ &= \frac{1}{v-u} \int_u^v f(x) dx \\ &\leq A\left(f(u), f(v), 1/2\right) = \frac{f(u)+f(v)}{2}. \end{aligned}$$

ii.) If f is GA-convex function, then we have the following Hermite-Hadamard's inequalities for GA-convex functions (see [7, Theorem 3.1. with $s = 1$]*)*

$$\begin{aligned} f\left(G(u,v,1/2)\right) &= f\left(\sqrt{uv}\right) \\ &\leq \int_0^1 A\left(f\left(G(u,v,\lambda)\right), f\left(G(u,v,1-\lambda)\right), 1/2\right) d\lambda \\ &= \frac{1}{2(\ln v - \ln u)} \int_u^v f(x) + f(\frac{uv}{x}) \frac{dx}{x} \\ &= \frac{1}{\ln v - \ln u} \int_u^v \frac{f(x)}{x} dx \\ &\leq A\left(f(u), f(v), 1/2\right) = \frac{f(u)+f(v)}{2}. \end{aligned}$$

iii.) If f is harmonically convex function, then we have the following Hermite-Hadamard's inequalities for harmonically-convex functions (see [6, 2.4. Theorem]*)*

$$\begin{aligned} f\left(H(u,v,1/2)\right) &= f\left(\frac{2uv}{u+v}\right) \\ &\leq \int_0^1 A\left(f\left(H(u,v,\lambda)\right), f\left(H(u,v,1-\lambda)\right), 1/2\right) d\lambda \\ &= \frac{uv}{2(v-u)}\int_u^v f(x)+f\left(\left[u^{-1}+v^{-1}-x^{-1}\right]^{-1}\right)\frac{dx}{x^2} \\ &= \frac{uv}{v-u}\int_u^v \frac{f(x)}{x^2}dx \\ &\leq A\left(f(u), f(v), 1/2\right) = \frac{f(u)+f(v)}{2}. \end{aligned}$$

iv.) If f is p-convex function ($p \neq 0$), then we have the following Hermite-Hadamard's inequalities for p-convex functions (see [9, Theorem 2]*)*

$$\begin{aligned} f\left(M_p(u,v,1/2)\right) &= f\left(\left[\frac{u^p+v^p}{2}\right]^{1/p}\right) \\ &\leq \int_0^1 A\left(f\left(M_p(u,v,\lambda)\right), f\left(M_p(u,v,1-\lambda)\right), 1/2\right) d\lambda \\ &= \frac{p}{2(v^p-u^p)}\int_u^v f(x)+f\left(\left[u^p+v^p-x^p\right]^{1/p}\right)\frac{dx}{x^{1-p}} \\ &= \frac{p}{v^p-u^p}\int_u^v \frac{f(x)}{x^{1-p}}dx \\ &\leq A\left(f(u), f(v), 1/2\right) = \frac{f(u)+f(v)}{2}. \end{aligned}$$

v.) If f is log-convex function, then we have the following Hermite-Hadamard's inequalities for log-convex functions (see [5, Theorem 2.1]*)*

$$\begin{aligned} f\left(A(u,v,1/2)\right) &= f\left(\frac{u+v}{2}\right) \\ &\leq \int_0^1 G\left(f\left(A(u,v,\lambda)\right), f\left(A(u,v,1-\lambda)\right), 1/2\right) d\lambda \\ &= \frac{1}{v-u}\int_u^v \sqrt{f(x)f(u+v-x)}dx \\ &\leq G\left(f(u), f(v), 1/2\right) = \sqrt{f(u)f(v)}. \end{aligned}$$

vi.) If f is GG-convex function, then we have the following Hermite-Hadamard's inequalities for GG-convex functions (see [8, the inequality (7)]*)*

$$\begin{aligned} f\left(G(u,v,1/2)\right) &= f\left(\sqrt{uv}\right) \\ &\leq \int_0^1 G\left(f\left(G(u,v,\lambda)\right), f\left(G(u,v,1-\lambda)\right), 1/2\right) d\lambda \\ &= \frac{1}{\ln v - \ln u}\int_u^v \sqrt{f(x) f\left(\frac{uv}{x}\right)}\frac{dx}{x} \\ &\leq G\left(f(u), f(v), 1/2\right) = \sqrt{f(u) f(v)}. \end{aligned}$$

vii.) If f is HG-convex function, then we have

$$\begin{aligned} f\left(H(u,v,1/2)\right) &= f\left(\frac{2uv}{u+v}\right) \\ &\leq \int_0^1 G\left(f\left(H(u,v,\lambda)\right), f\left(H(u,v,1-\lambda)\right), 1/2\right) d\lambda \\ &= \frac{uv}{v-u}\int_u^v \sqrt{f(x) f\left(\left[u^{-1}+v^{-1}-x^{-1}\right]^{-1}\right)}\frac{dx}{x^2} \\ &\leq G\left(f(u), f(v), 1/2\right) = \sqrt{f(u) f(v)}. \end{aligned}$$

viii.) If f is AH-convex function, then we have

$$\begin{aligned} f\left(A(u,v,1/2)\right) &= f\left(\frac{u+v}{2}\right) \\ &\leq \int_0^1 H\left(f\left(A(u,v,\lambda)\right), f\left(A(u,v,1-\lambda)\right), 1/2\right) d\lambda \\ &= \frac{2}{v-u}\int_u^v \frac{f(x) f(u+v-x)}{f(x)+f(u+v-x)} dx \\ &\leq A\left(f(u), f(v), 1/2\right) = \frac{f(u)+f(v)}{2}. \end{aligned}$$

Theorem 2.8. *Let M and N be two weighted means defined on the intervals $[u,v] \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively. If function $f : [u,v] \to J$ is MN-convex and symmetric with respect to $M(u,v,1/2)$, then we have*

$$f\left(M(u,v,1/2)\right) \leq f(x) \leq N\left(f(u), f(v), 1/2\right) \tag{2.6}$$

for all $x \in I$.

Proof. Let $x \in [u,v]$ be arbitrary point. Then there exists a $\lambda \in [0,1]$ such that $x = M(u,v,\lambda)$. Since $f : [u,v] \to J$ is a MN-convex function and symmetric with respect to $M(u,v,1/2)$, by using (2.2) we have

$$\begin{aligned} f\left(M(u,v,1/2)\right) &= f\left(M\left(M(u,v,\lambda), M(u,v,1-\lambda), 1/2\right)\right) \\ &\leq N\left(f\left(M(u,v,\lambda)\right), f\left(M(u,v,1-\lambda)\right), 1/2\right) \\ &= f(x). \end{aligned}$$

Thus, we obtain the left-hand side of the inequality (2.6). Secondly, By using MN-convexity of f and (WM5) with (2.2), we get

$$\begin{aligned} & f(x)=N\left(f\left(M(u,v,\lambda)\right),f\left(M(u,v,1-\lambda\right),1/2\right) \\ \leq \; & N\left(N(f(u),f(v),\lambda),N(f(u),f(v),1-\lambda),1/2\right) \\ = \; & N\left(f(u),f(v),1/2\right). \end{aligned}$$

This completes the proof. □

We can give the following some results for different convexity classes by considering the special cases of M and N. It is possible to increase the results by considering another special cases of M and N.

Corollary 2.3. *Let $I,J\subseteq(0,\infty)$ and $f:I\to J$.*

i.) If f is a convex function and symmetric with respect to $(u+v)/2$, then we have the following inequalities for convex functions (see [4, Theorem 2]*)*

$$f\left(\frac{u+v}{2}\right)\leq f(x)\leq\frac{f(u)+f(v)}{2}.$$

ii.) If f is a GA-convex function and symmetric with respect to $\sqrt{uv}$, then we have the following inequalities for convex functions (see [10, Theorem 2.9]*)*

$$f\left(\sqrt{uv}\right)\leq f(x)\leq\frac{f(u)+f(v)}{2}.$$

iii.) If f is a p-convex function and symmetric with respect to $\left(\frac{u^p+v^p}{2}\right)^{1/p}$, then we have the following inequalities for convex functions (see [11, Theorem 2.2]*)*

$$f\left(\left[\frac{u^p+v^p}{2}\right]^{1/p}\right)\leq f(x)\leq\frac{f(u)+f(v)}{2}.$$

3. Conclusion

The aim of this article is to determine that a mean is called the weighted mean when it meets what conditions, and also is to give a general definition of MN-convex functions. The importance of this study is that some properties of MN-convex functions and some related inequalities have been proven in general terms via this general definition of MN-convex functions.

References

[1] J. Aczél, *A generalization of the notion of convex functions*, Norske Vid. Selsk. Forhd., Trondhjem, **19**(24) (1947), 87–90.

[2] G. Aumann, *Konvexe Funktionen und Induktion bei Ungleichungen zwischen Mittelverten*, Bayer. Akad. Wiss.Math.-Natur. Kl. Abh., Math. Ann., **109** (1933), 405–413.

[3] G.D. Anderson, M. K. Vamanamurthy, M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl., **335** (2007), 1294–1308.

[4] S.S. Dragomir, *Symmetrized convexity and Hermite-Hadamard type inequalities*, Journal of Mathematical Inequalities, **10**(4) (2016), 901–918.

[5] S.S. Dragomir, B. Mond, *Integral inequalities of Hadamard type for log-convex functions*, Demonstratio Mathematica, **31**(2) (1998), 354–364.

[6] İ. İşcan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacet. J. Math. Stat., **43**(6) (2014), 935–942.

[7] İ. İşcan, *Hermite-Hadamard type inequalities for GA-s-convex functions*, Le Matematiche, **LXIX** (2014), Fasc. II, 129–146.

[8] İ. İşcan, *On Some New Hermite-Hadamard type inequalities for s-geometrically convex functions*, International Journal of Mathematics and Mathematical Sciences, **2014** (2014), Article ID 163901, 8 pages.

[9] İ. İşcan, *Ostrowski type inequalities for p-convex functions, New Trends in Mathematical Sciences*, **4**(3) (2016), 140-150.

[10] İ. İşcan, *Symmetrized GA-convexity and Related Some Integral Inequalities*, Filomat, **33**(13) (2019), 4121-4136.

[11] İ. İşcan, *Symmetrized p-convexity and related some integral inequalities*, TWMS J. App. and Eng. Math., **10**(4) (2020), 1036–1048.

[12] J. Matkowski, *Convex functions with respect to a mean and a characterization of quasi-arithmetic means*, Real Anal. Exchange **29** (2003/2004), 229–246.

[13] T.Z. Mirković, *New inequalities of Wirtinger type for convex and MN-convex functions*, Facta Universitatis Ser. Math. Inform. **34**(2) (2019), 165–173.

[14] C.P. Niculescu, *Convexity according to the geometric mean*, Math. Inequal. Appl.,**3**(2) (2000), 155–167.

[15] C.P. Niculescu, *Convexity according to means*, Math. Inequal. Appl. **6** (2003), 571–579.

[16] A.W. Roberts, D.E. Varberg, *Convex Functions*, Academic Press, New York, 1973.

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