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A COMPREHENSIVE FAMILY OF BI-UNIVALENT FUNCTIONS LINKED WITH GEGENBAUER POLYNOMIALS

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Abstract. Making use of Gegenbauer polynomials, we initiate and explore a comprehensive family of regular and bi-univalent (or bi-Schlicht) functions in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$. We investigate certain coefficients bounds and the Fekete-Szegö functional for functions in this family. We also present few interesting observations and provide relevant connections of the result investigated.

1. Introduction and Preliminaries

Let the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ be symbolized by $\mathfrak{D}$, where $\mathbb{C}$ is the collection of all complex numbers. Let $\mathbb{N} := \mathbb{N}_0 \backslash \{0\} = \{1, 2, 3, ...\}$ and $\mathbb{R}$ be the set of real numbers. The set of normalized regular functions in $\mathfrak{D}$ that have the power series of the form

$$g(z) = z + d_2 z^2 + d_3 z^3 + ... = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1.1}$$

be indicated by $\mathcal{A}$ and the set of all functions of $\mathcal{A}$ that are univalent (or schlicht) in $\mathfrak{D}$ is symbolized by $\mathcal{S}$. As per the Koebe theorem (see [9]) any function $g \in \mathcal{S}$ has an inverse function given by

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2 d_3 + d_4)\omega^4 + \dots , \tag{1.2}$$

such that $z = g^{-1}(g(z))$, $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$ and $r_0(g) \geq 1/4$, $z, \omega \in \mathfrak{D}$.

A function g of $\mathcal{A}$ is called bi-univalent (or bi-schlicht) in $\mathfrak{D}$ if g and its inverse g^{-1} are both univalent (or schlicht) in $\mathfrak{D}$. Let $\sum$ stands for the set of bi-univalent functions having the form (1.1). Investigations of the family $\sum$ begun few decades ago by Lewin [20] and Brannan and Clunie [7]. Later, Tan [32] found some initial coefficient estimates of

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bi-univalent functions. Moreover, Brannan and Taha [6] examined certain classical subsets of $\sum$ in $\mathfrak{D}$. Some interesting outcomes concerning initial bounds for certain special sets of $\sum$ have been appeared in [1], [2], [8], [14], [15] and [24].

Recently, Kiepiela et al. [19] examined the Gegenbauer polynomials (or ultraspherical polynomials) $C_j^{\alpha}(x)$. They are orthogonal polynomials on [-1,1] that can be defined by the recurrence relation

$$C_j^{\alpha}(x)=\frac{2x(j+\alpha-1)C_{j-1}^{\alpha}(x)-(j+2\alpha-2)C_{j-2}^{\alpha}(x)}{j},\, C_0^{\alpha}(x)=1\,,\, C_1^{\alpha}(x)=2\alpha x, \quad (1.3)$$

where $j\in\mathbb{N}\backslash\{1\}$. It is easy to see from (1.3) that $C_2^{\alpha}(x)=2\alpha(1+\alpha)x^2-\alpha$. For $\alpha\in\mathbb{R}\backslash\{0\}$, a generating function of the sequence $C_j^{\alpha}(x)$, $j\in\mathbb{N}$, is defined by (see [3]):

$$\mathcal{H}_{\alpha}(x,z):=\sum_{j=0}^{\infty}C_j^{\alpha}(x)z^j=\frac{1}{(1-2xz+z^2)^{\alpha}}, \quad (1.4)$$

where $z\in\mathfrak{D}$ and $x\in$ [-1,1].

Two particular cases of $C_j^{\alpha}(x)$ are $i)\, C_j^{1}(x)$ the second kind Chebyshev polynomials and $ii)\, C_j^{\frac{1}{2}}(x)$ the Legendre polynomials (See [4]).

Gegenbauer polynomials, Fibonacci polynomials, Pell-Lucas polynomials, Chebyshev polynomials, Horadam polynomials, Fermat-Lucas polynomials and generalizations of them have potential applications in branches such as architecture, physics, combinatorics, number theory, statistics and engineering. Additional information about these polynomials can be found in [12],[13], [16], [17] and [36]. More details about the famous Fekete-Szegö problem associated with Gegenbauer polynomials are available in the works of [3], [4], [35] and [31].

The recent research trends are the outcomes of the study of function in the class $\sum$ linked with any of the above mentioned polynomials, can be seen in [5], [21], [25], [26], [27], [29], [30], [33] and [34]. Generally interest was shown to estimate the initial Taylor-Maclaurin coefficients and the celebrated inequality of Fekete-Szegö for the special subfamilies of $\sum$. However, there is little work on bi-univalent functions linked with Gegenbauer polynomials. To initiate and explore the study on bi-univalent functions linked with Gegenbauer polynomials, we present a comprehensive family of $\sum$ subordinate to Gegenbauer polynomials $C_j^{\alpha}(x)$ as in (1.3) with the generating function (1.4).

For regular functions g and f in $\mathfrak{D}$, g is said to subordinate to f, if there is a Schwarz function ψ in $\mathfrak{D}$, such that $\psi(0)=0$, $|\psi(z)|<1$ and $g(z)=f(\psi(z))$, $z\in\mathfrak{D}$. This subordination is indicated as $g\prec f$ or $g(z)\prec f(z)$. Specifically, when $f\in\mathcal{S}$ in $\mathfrak{D}$, then $g(z)\prec f(z) \iff g(0)=f(0)$ and $g(\mathfrak{D})\subset f(\mathfrak{D})$.

Throughout this paper, the inverse function $g^{-1}(\omega)=f(\omega)$ is as in (1.2) and $\mathcal{H}_{\alpha}(x,z)$ is as in (1.4).

Definition 1.1. A function g in $\sum$ having the power series (1.1) is said to be in the family $S\mathfrak{G}_{\sum}^{\alpha}(\gamma,\tau,\mu,x)$, $0\le\gamma\le1$, $\tau\ge1$, $\mu\ge0, 1/2<x\le1$ and $\alpha\in\mathbb{R}\backslash\{0\}$, if

$$\frac{z(g'(z))^{\tau}+\mu z^2g''(z)}{\gamma g(z)+(1-\gamma)z}\prec\mathcal{H}_{\alpha}(x,z),\ z\in\mathfrak{D}$$

and

$$\frac{\omega(f'(\omega))^{\tau}+\mu\omega^{2}f''(\omega)}{\gamma f(\omega)+(1-\gamma)\omega}\prec \mathcal{H}_{\alpha}(x,\omega),\ \omega\in\mathfrak{D}.$$

The family $S\mathfrak{S}^{\alpha}_{\sum}(\gamma,\tau,\mu,x)$ is of special interest for it contains many well-known as well as new subfamilies of $\sum$ for particular values of γ,τ and μ, as illustrated below:

1. $SK^{\alpha}_{\sum}(\tau,\mu,x)\equiv S\mathfrak{S}^{\alpha}_{\sum}(0,\tau,\mu,x)$ is the set of functions $g\in\sum$ satisfying

$$(g'(z))^{\tau}+\mu z g''(z)\prec \mathcal{H}_{\alpha}(x,z)\quad\text{and}\quad (f'(\omega))^{\tau}+\mu\omega f''(\omega)\prec \mathcal{H}_{\alpha}(x,\omega),\ z,\,\omega\in\mathfrak{D}.$$

2. $SL^{\alpha}_{\sum}(\tau,\mu,x)\equiv S\mathfrak{S}^{\alpha}_{\sum}(1,\tau,\mu,x)$ is the collection of functions $g\in\sum$ satisfying

$$\frac{z(g'(z))^{\tau}}{g(z)}+\mu\left(\frac{z^{2}g''(z)}{g(z)}\right)\prec \mathcal{H}_{\alpha}(x,z),\ z\in\mathfrak{D}$$

and

$$\frac{\omega(f'(\omega))^{\tau}}{f(\omega)}+\mu\left(\frac{\omega^{2}f''(\omega)}{f(\omega)}\right)\prec \mathcal{H}_{\alpha}(x,\omega),\ \omega\in\mathfrak{D}.$$

3. $SM^{\alpha}_{\sum}(\gamma,\tau,x)\equiv S\mathfrak{S}^{\alpha}_{\sum}(\gamma,\tau,1,x)$ is the family of functions $g\in\sum$ satisfying

$$\frac{z(g'(z))^{\tau}+z^{2}g''(z)}{\gamma g(z)+(1-\gamma)z}\prec \mathcal{H}_{\alpha}(x,z),\ z\in\mathfrak{D}$$

and

$$\frac{\omega(f'(\omega))^{\tau}+\omega^{2}f''(\omega)}{\gamma f(\omega)+(1-\gamma)\omega}\prec \mathcal{H}_{\alpha}(x,\omega),\ \omega\in\mathfrak{D}.$$

4. The function classes $S\mathfrak{S}^{\alpha}_{\sum}(\gamma,1,\mu,x)$ and $S\mathfrak{S}^{\alpha}_{\sum}(\gamma,0,\mu,x)$ were investigated in [31].

Remark 1.1. We note that

i) $SK^{\alpha}_{\sum}(\tau,1,x)\equiv SM^{\alpha}_{\sum}(0,\tau,x)$.

ii) $SL^{\alpha}_{\sum}(\tau,1,x)\equiv SM^{\alpha}_{\sum}(1,\tau,x)$.

Remark 1.2. i) For $\mu=0$ and $\tau=1$, the class $SK^{\alpha}_{\sum}(1,0,x)\equiv\mathcal{H}^{\alpha}_{\sum}(x)$ was studied by Amourah et al. [3].

ii) For $\mu=0$ and $\tau=1$, the family $SL^{\alpha}_{\sum}(1,0,x)\equiv S^{\alpha}_{\sum}(x)$ was introduced by Amourah et al. [4].

In Section 2, we derive the estimates for $|d_2|$, $|d_3|$ and the inequality of Fekete- Szegö [11] for functions of the form (1.1) $\in S\mathfrak{S}^{\alpha}_{\sum}(\gamma,\tau,\mu,x)$. In Section 3, few interesting consequences and relevant connections of the result are mentioned.

2. Coefficient bounds and Fekete-Szegö inequality

We determine the initial coefficients bounds and the inequality of Fekete-Szegö for functions in $S\mathfrak{S}^{\alpha}_{\sum}(\gamma,\tau,\mu,x)$, in the following theorem:

Theorem 2.1. *Let* $0 \leq \gamma \leq 1$, $\tau \geq 1$, $\mu \geq 0, 1/2 < x \leq 1$ *and* $\alpha \in \mathbb{R} \backslash \{0\}$. *If the function* $g \in S\mathfrak{S}^{\alpha}_{\sum}(\gamma, \tau, \mu, x)$, *then*

$$|d_2| \leq \frac{2|\alpha| x \sqrt{2x}}{\sqrt{|(2(\mu+\tau)-\gamma)^2(1-2x^2) + 2(\gamma^2 + 2(\tau-\gamma) - 4\mu(2\tau+\mu-3))\alpha x^2|}}, \tag{2.1}$$

$$|d_3| \leq \frac{4\alpha^2 x^2}{(2(\mu+\tau)-\gamma)^2} + \frac{2|\alpha| x}{(3(2\mu+\tau)-\gamma)} \tag{2.2}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{2|\alpha|x}{(3(2\mu+\tau)-\gamma)} & ; |1-\delta| \leq \mathfrak{J} \\ \frac{8\alpha^2 x^3 \, |1-\delta|}{|(2(\mu+\tau)-\gamma)^2(1-2x^2)+2(\gamma^2+2(\tau-\gamma)-4\mu(2\tau+\mu-3))\alpha x^2|} & ; |1-\delta| \geq \mathfrak{J}, \end{cases} \tag{2.3}$$

where

$$\mathfrak{J} = \left| \frac{(2(\mu+\tau)-\gamma)^2(1-2x^2) + 2(\gamma^2 + 2(\tau-\gamma) - 4\mu(2\tau+\mu-3))\alpha x^2}{4(3(2\mu+1)-\gamma)\alpha x^2} \right|. \tag{2.4}$$

Proof. Let $g \in S\mathfrak{S}^{\alpha}_{\sum}(\gamma, \tau, \mu, x)$. Then, for two regular functions $\mathfrak{M}$, $\mathfrak{N}$ given by

$$\mathfrak{M}(z) = \mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + \ldots \quad z \in \mathfrak{D}$$

and

$$\mathfrak{N}(\omega) = \mathfrak{n}_1 \omega + \mathfrak{n}_2 \omega^2 + \mathfrak{n}_3 \omega^3 + \ldots, \quad \omega \in \mathfrak{D}$$

with $\mathfrak{M}(0) = 0, \mathfrak{N}(0) = 0, |\mathfrak{M}(z)| < 1$ and $|\mathfrak{N}(\omega)| < 1, z, \omega \in \mathfrak{D}$ and on account of Definition 1.1, we can write

$$\frac{z(g'(z))^{\tau} + \mu z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} = \mathcal{H}_{\alpha}(x, \mathfrak{M}(z))$$

and

$$\frac{\omega(f'(\omega))^{\tau} + \mu \omega^2 f''(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} = \mathcal{H}_{\alpha}(x, \mathfrak{N}(\omega)).$$

Or, equivalently

$$\frac{z(g'(z))^{\tau} + \mu z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} = 1 + C_1^{\alpha}(x) + C_2^{\alpha}(x)\mathfrak{m}(z) + C_3^{\alpha}(x)(\mathfrak{m}(z))^2 + \ldots \tag{2.5}$$

and

$$\frac{\omega(f'(\omega))^{\tau} + \mu \omega^2 f''(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} = 1 + C_1^{\alpha}(x) + C_2^{\alpha}(x)\mathfrak{n}(\omega) + C_3^{\alpha}(x)(\mathfrak{n}(\omega))^2 + \ldots . \tag{2.6}$$

From (2.5) and (2.6), in view of (1.3), we find

$$\frac{z(g'(z))^{\tau} + \mu z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} = 1 + C_1^{\alpha}(x)\mathfrak{m}_1 z + [C_1^{\alpha}(x)\mathfrak{m}_2 + C_2^{\alpha}(x)\mathfrak{m}_1^2]z^2 + \ldots \tag{2.7}$$

and

$$\frac{\omega(f'(\omega))^{\tau} + \mu \omega^2 f''(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} = 1 + C_1^{\alpha}(x)\mathfrak{n}_1 \omega + [C_1^{\alpha}(x)\mathfrak{n}_2 + C_1^{\alpha}(x)\mathfrak{n}_1^2]\omega^2 + \ldots . \tag{2.8}$$

Clearly, if $|\mathfrak{M}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + \ldots| < 1, z \in \mathfrak{D}$ and $|\mathfrak{N}(\omega)| = |\mathfrak{n}_1 \omega + \mathfrak{n}_2 \omega^2 + \mathfrak{n}_3 \omega^3 + \ldots| < 1, \omega \in \mathfrak{D}$, then

$$|\mathfrak{m}_i| \leq 1 \, \text{and} \, |\mathfrak{n}_i| \leq 1 \, (i \in \mathbb{N}). \tag{2.9}$$

We get the following by equating the corresponding coefficients in (2.7) and (2.8):

$$(2(\mu+\tau)-\gamma)d_2 = C_1^\alpha(x)\mathfrak{m}_1, \tag{2.10}$$

$$(3(2\mu+\tau)-\gamma)d_3 + \left(\gamma^2 - 2\gamma(\mu+\tau) + 2\tau(\tau-1)\right)d_2^2 = C_1^\alpha(x)\mathfrak{m}_2 + C_2^\alpha(x)\mathfrak{m}_1^2, \tag{2.11}$$

$$-\left(2(\mu+\tau)-\gamma\right)d_2 = C_1^\alpha(x)\mathfrak{n}_1 \tag{2.12}$$

and

$$(3(2\mu+\tau)-\gamma)(2d_2^2-d_3) + (\gamma^2 - 2\gamma(\mu+\tau) + 2\tau(\tau-1))d_2^2 = C_1^\alpha(x)\mathfrak{n}_2 + C_2^\alpha(x)\mathfrak{n}_1^2. \tag{2.13}$$

It follows from (2.10) and (2.12) that

$$\mathfrak{m}_1 = -\mathfrak{n}_1, \tag{2.14}$$

$$2(2(\mu+1)-\gamma)^2 d_2^2 = (\mathfrak{m}_1^2+\mathfrak{n}_1^2)(C_1^\alpha(x))^2. \tag{2.15}$$

If we add (2.11) and (2.13), then we obtain

$$2(\gamma^2 + (\tau-\gamma)(2\tau+1) + 2\mu(3-\gamma))d_2^2 = C_1^\alpha(x)(\mathfrak{m}_2+\mathfrak{n}_2) + C_2^\alpha(x)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \tag{2.16}$$

Substituting the value of $\mathfrak{m}_1^2+\mathfrak{n}_1^2$ from (2.15) in (2.16), we get

$$d_2^2 = \frac{(C_1^\alpha(x))^3(\mathfrak{m}_2+\mathfrak{n}_2)}{2\left[(\gamma^2 + (\tau-\gamma)(2\tau+1) + 2\mu(3-\gamma))(C_1^\alpha(x))^2 - (2(\mu+\tau)-\gamma)^2 C_2^\alpha(x)\right]}, \tag{2.17}$$

which yields (2.1) on using (2.9).

After subtracting (2.13) from (2.11) and then using (2.14), we obtain

$$d_3 = d_2^2 + \frac{C_1^\alpha(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{2(3(2\mu+\tau)-\gamma)}. \tag{2.18}$$

Then in view of (2.15), equation (2.18) becomes

$$d_3 = \frac{(C_1^\alpha(x))^2(\mathfrak{m}_1^2+\mathfrak{n}_1^2)}{2(2(\mu+\tau)-\gamma)^2} + \frac{C_1^\alpha(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{2(3(2\mu+\tau)-\gamma)},$$

which gets (2.2) on applying (2.9).

From (2.17) and (2.18), for $\delta \in \mathbb{R}$, we get

$$|d_3 - \delta d_2^2| = |C_1^\alpha(x)| \left| \left(\mathfrak{T}(\delta,x) + \frac{1}{2(3(2\mu+\tau)-\gamma)}\right)\mathfrak{m}_2 + \left(\mathfrak{T}(\delta,x) - \frac{1}{2(3(2\mu+\tau)-\gamma)}\right)\mathfrak{n}_2 \right|,$$

where

$$\mathfrak{T}(\delta,x) = \frac{(1-\delta)\,(C_1^\alpha(x))^2}{2\left[(\gamma^2 + (\tau-\gamma)(2\tau+1) + 2\mu(3-\gamma))(C_1^\alpha(x))^2 - (2(\mu+1)-\gamma)^2 C_2^\alpha(x)\right]}.$$

In view of (1.3), we conclude that

$$|d_3 - \delta d_2^2| \le \begin{cases} \frac{|C_1^\alpha(x)|}{(3(2\mu+\tau)-\gamma)} & ;\ 0 \le |\mathfrak{T}(\delta,x)| \le \frac{1}{2(3(2\mu+\tau)-\gamma)} \\ 2|C_1^\alpha(x)||\mathfrak{T}(\delta,x)| & ;\ |\mathfrak{T}(\delta,x)| \ge \frac{1}{2(3(2\mu+\tau)-\gamma)}, \end{cases}$$

which enable us to conclude (2.3) with $\mathfrak{J}$ as in (2.4). Thus the proof of Theorem 2.1 is completed. □

Remark 2.1. a) By taking $\tau = 1$ in the above theorem, we obtain a result of the authors [31, Theorem 2.1]. Further, setting i) $\mu = 0$, ii) $\gamma = 0$ and iii) $\gamma = 1$, we obtain Corollaries 2.1, 2.2 and 2.3 of [31], respectively.
b) If we let $\mu = 0$ in the above theorem, we get another result of the authors [31, Theorem 3.1]. Further, letting i) $\gamma = 0$ and ii) $\gamma = 1$, we get [31, Corollary 3.1 and Corollary 3.2].

3. Outcome of the main result

Theorem 2.1 would yield the following outcome,when $\gamma = 0$.

Corollary 3.1. *If the function $g \in SK^{\alpha}_{\sum}(\tau, \mu, x)$, then*

$$|d_2| \leq \frac{|\alpha| x\sqrt{2x}}{\sqrt{|(\mu+\tau)^2(1-2x^2) - (2\mu(\mu+2\tau-3)-\tau)\alpha x^2|}},$$

$$|d_3| \leq \frac{\alpha^2 x^2}{(\mu+\tau)^2} + \frac{2|\alpha| x}{3(2\mu+\tau)}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{2|\alpha|x}{3(2\mu+\tau)} & ; \ |1-\delta| \leq \left|\frac{(\mu+\tau)^2(1-2x^2)-(2\mu(\mu+2\tau-3)-\tau)\alpha x^2}{3(2\mu+\tau)\alpha x^2}\right| \\ \frac{2\alpha^2 x^3\,|1-\delta|}{|(\mu+1)^2(1-2x^2)-(2\mu(\mu+2\tau-3)-\tau)\alpha x^2|} & ; \ |1-\delta| \geq \left|\frac{(\mu+\tau)^2(1-2x^2)-(2\mu(\mu+2\tau-3)-\tau)\alpha x^2}{3(2\mu+\tau)\alpha x^2}\right|. \end{cases}$$

Remark 3.1. Corollary 3.1 reduces to Corollary 9 of Amurah et al. [4], when $\tau = 1$ and $\mu = 0$.

Allowing $\gamma = 1$ in Theorem 2.1, we arrive at the following:

Corollary 3.2. *If the function $g \in SL^{\alpha}_{\sum}(\tau, \mu, x)$, then*

$$|d_2| \leq \frac{2|\alpha| x\sqrt{2x}}{\sqrt{|(2(\mu+\tau)-1)^2(1-2x^2) - 2(4\mu(2\tau+\mu-3)-2\tau+1)\alpha x^2|}},$$

$$|d_3| \leq \frac{4\alpha^2 x^2}{(2(\mu+\tau)-1)^2} + \frac{2|\alpha| x}{3(2\mu+\tau)-1}$$

and for some $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{2|\alpha|x}{3(2\mu+\tau)-1} & ; \ |1-\delta| \leq \mathfrak{J}_1 \\ \frac{8\alpha^2 x^2\,|1-\delta|}{|(2(\mu+\tau)-1)^2(1-2x)^2-2(4\mu(2\tau+\mu-3)-2\tau+1)\alpha x^2|} & ; \ |1-\delta| \geq \mathfrak{J}_1. \end{cases}$$

where $\mathfrak{J}_1 = \left|\frac{(2(\mu+\tau)-1)^2(1-2x^2)-2(4\mu(2\tau+\mu-3)-2\tau+1)\alpha x^2}{4(3-\gamma)\alpha x^2}\right|.$

Remark 3.2. Corollary 3.2 reduces to Corollary 8 of Amurah et al. [4] (also see [3]), when $\tau = 1$ and $\mu = 0$.

Setting $\mu = 1$ in Theorem 2.1, we have

Corollary 3.3. *If the function $g \in SM^{\alpha}_{\sum}(\gamma, \tau, x)$, then*

$$|d_2| \leq \frac{2|\alpha| x\sqrt{2x}}{\sqrt{|2(1+\tau)-\gamma)^2(1-2x^2) + 2(\gamma^2-\gamma-6\tau+8)\alpha x^2|}},$$

$$|d_3| \leq \frac{4\alpha^2 x^2}{(2(1+\tau)-\gamma)^2} + \frac{|\alpha| x}{3(2+\tau)-\gamma}$$

and for $\delta \in \mathbb{R}$*,*

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|\alpha| x}{3(2+\tau)-\gamma} & ; \ |1-\delta| \leq \mathfrak{J}_2 \\ \frac{8\alpha^2 x^3 \, |1-\delta|}{|2(1+\tau)-\gamma)^2(1-2x^2)+2(\gamma^2-2\gamma-6\tau+8)\alpha x^2]|} & ; \ |1-\delta| \geq \mathfrak{J}_2, \end{cases}$$

where $\mathfrak{J}_2 = \left| \frac{2(1+\tau)-\gamma)^2(1-2x^2)+2(\gamma^2-2\gamma-6\tau+8)\alpha x^2}{4(3(2+\tau)-\gamma)\alpha x^2} \right|$.

4. Conclusion

A comprehensive family of regular and bi-univalent (or bi-schlicht) functions linked with Gegenbauer polynomials are initiated and explored. Bounds of the first two coefficients $|d_2|$, $|d_3|$ and the celebrated Fekete- Szegö functional have been fixed for the defined family. Through corollaries of our main results, we have highlighted many interesting new consequences.

The contents of the paper on a comprehensive family could inspire further research related to other trends such as families using q - derivative operator [10], [28], q - integral operator [18], meromorphic bi-univalent function families associated with Al-Oboudi differential operator [23] and families using integro-differential operators [22].

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References

[1] I. Aldawish, T. Al-Hawary, B.A. Frasin, *Subclasses of bi-univalent functions defined by Frasin differential operator,* Mathematics, **8**(5) (2020), 1–11.

[2] Ş. Altınkaya and S. Yalçın, *On the Chebyshev polynomial coefficient problem of some subclasses of bi-univalent functions,* Gulf J. Math., **5**(3) (2017), 34-40.

[3] A. Amourah, A. Alamoush, M. Al-Kaseasbeh, *Gegenbauer polynomials and biunivalent functions,* Palestine J. Math., **10**(2) (2021), 625–632.

[4] A. Amourah, B.A. Frasin, T. Abdeljawad, *Fekete-Szegö inequality for analytic and biunivalent functions subordinate to Gegenbauer polynomials,* J. Funct. Spaces, **2021** (2021), Art. ID 5574673, 1–7. https://doi.org/10.1155/2021/5574673.

[5] I.T. Awolere, A.T. Oladipo, *Coefficients of bi-univalent functions involving pseudo-starlikeness associated with Chebyshev polynomials,* Khayyam J. Math., **5**(1) (2019), 140–149.

[6] D. A. Brannan, T. S. Taha, *On some classes of bi-univalent functions,* In: S. M. Mazhar, A. Hamoui, N.S. Faour (eds) Mathematical analysis and its applications. Kuwait, pp 53-60, KFAS Proceedings Series, Vol. 3 (1985), Pergamon Press (Elsevier Science Limited), Oxford, 1988; see also Studia Univ. Babeş-Bolyai Math., **31**(2) (1986), 70–77.

[7] D.A. Brannan, J.G. Clunie, *Aspects of contemporary complex analysis,* Proceedings of the NATO Advanced study institute held at University of Durhary, Newyork: Academic press, 1979.

[8] S. Bulut, *Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions,* C R Acad. Sci. Paris Sér I, **352** (2014), 479–484.

[9] P. L. Duren, *Univalent Functions,* Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, 1983.

[10] S.M. El-Deeb, T. Bulboacă, B.M. El-Matary, *Maclaurin coefficient estimates of bi-univalent functions connected with the q-derivative,* Mathematics, **8** (2020), 1–14.

[11] M. Fekete, G. Szegö, *Eine Bemerkung Über Ungerade Schlichte Funktionen,* J. Lond. Math. Soc., **89** (1933), 85–89.

[12] P. Filipponi, A.F. Horadam, *Derivative sequences of Fibonacci and Lucas polynomials.* In: G. E. Bergum, A. N. Philippou, A. F. Horadam (eds) Applications of Fibonacci Numbers, vol 4 (1990), pp 99–108, Proceedings of the fourth international conference on Fibonacci numbers and their applications, Wake Forest University, Winston-Salem, North Carolina; Springer (Kluwer Academic Publishers), Dordrecht, Boston and London, **4** (1991), 99–108.

[13] P. Filipponi, A.F. Horadam, *Second derivative sequence of Fibonacci and Lucas polynomials,* Fibonacci Quart., **31** (1993), 194–204.

[14] B.A. Frasin, M.K. Aouf, *New subclasses of bi-univalent functions,* Appl. Math., **24** (2011), 1569–1573.

[15] B.A. Frasin, S.R. Swamy, I. Aldawish, *A comprehensive family of bi-univalent functions defined by k-Fibonacci numbers,* J. Funct. Spaces, **2021** (2020), Art. ID 4249509, 1–7.

[16] A.F. Horadam, J.M. Mahon, *Pell and Pell-Lucas polynomials,*Fibonacci Quart., **23** (1985), 7–20.

[17] T. Hörçum, E. Gökçen Koçer, *On some properties of Horadam polynomials,* Int. Math. Forum., **4** (2009), 1243–1252.

[18] B. Khan, H.M. Srivastava, M. Tahir, M. Darus, Q.Z. Ahmed, N. Khan, *Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions,* AIMS Mathematics, **6**(1) (2020), 1024–1039.

[19] K. Kiepiela, I. Naraniecka, J. Szynal, *The Gegenbauer polynomials and typically real functions,* J. Compu. Appl. Math., **153**(1-2) (2003), 273–282.

[20] M. Lewin, *On a coefficient problem for bi-univalent functions,* Proc. Amer. Math. Soc., **18** (1967), 63-68.

[21] N. Magesh, S. Bulut, *Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functons related to pseudo-starlike functions,* Afr. Mat., **29**(1-2) (2018), 203–209.

[22] A.O. Páll-Szabó, G. I. Oros, *Coefficient related studies for new classes of bi-univalent functions,* Mathematics, **8** (2020), 1–13.

[23] T.G. Shaba, M.G. Khan, B. Ahmed, *Coefficient bounds for certain subclasses of meromorphic bi-univalent functions associated with Al-Oboudi differential operator,* Palestine J. Math., **9**(2) (2020), 1–11.

[24] H.M. Srivastava, A.K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions,* Appl. Math. Lett., **23** (2010), 1188–1192.

[25] H.M. Srivastava, Ş. Altınkaya, S. Yalçın, *Certain Subclasses of bi-univalent functions associated with the Horadam polynomials,* Iran J. Sci. Technol. Trans. A Sci., **43** (2019), 1873–1879.

[26] S.R. Swamy, *Coefficient bounds for Al-Oboudi type bi-univalent functions based on a modified sigmoid activation function and Horadam polynimials,* Earthline J. Math. Sci., **7**(2) (2021), 251–270.

[27] S.R. Swamy, S. Bulut, Y. Sailaja, *Some special families of holomorphic and Sălăgean type bi-univalent functions associated with Horadam polynomials involving modified sigmoid activation function,* Hacet. J. Math. Stat., **50**(3) (2021), 710–720.

[28] S.R. Swamy, P.K. Mamatha, *Certain classes of bi-univalent functions associated with q-analogue of Bessel functions,* South East Asian J. Math. Math. Sci., **16**(3) (2020), 61–82.

[29] S.R. Swamy, A.K. Wanas, Y. Sailaja, *Some special families of holomorphic and Sălăgean type bi-univalent functions associated with (m, n)-Lucas polynomials,* Commun. Math. Appl., **11**(4) (2020), 563–574. doi.org/10.26713/cma. v11i4.1411

[30] S.R. Swamy, Y. Sailaja, *Horadam polynomial coefficient estimates for two families of holomorphic and bi-univalent functions,* Int. J. Math. Trends and Tech., **66**(8) (2020), 131–138.

[31] S.R. Swamy, S. Yalçın, *Coefficient bounds for regular and bi-univalent functions linked with Gegenbauer polynomials,* Problemy Analiza, DOI: 10.15393/j3.art.2022.10351.

[32] D.L. Tan, *Coefficient estimates for bi-univalent functions,* Chin. Ann. Math. Ser. A, **5** (1984), 559–568.

[33] A.K. Wanas, *Applications of (M,N)-Lucas polynomials for holomorphic bi-univalent functions,* Filomat, **34** (2020), 3361–3368.

[34] A.K. Wanas, *Horadam polynomials for a new family of λ-pseudo bi-univalent functions associated with Sakaguchi type functions,* Afr. Mat., **32** (2021), 879–889.

[35] A.K. Wanas, *New families of bi-univalent functions governed by Gegenbauer polynomials,* Earthline J. Math. Sci., **7**(2) (2021), 403–427.

[36] T.-T. Wang, W.-P. Zhang, *Some identities involving Fibonacci, Lucas polynomials and their applications,* Bull. Math. Soc. Sci. Math. Roumanie (New Ser.), **55**(103) (2012), 95–103.

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