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# SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES VIA NON-CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In this paper, some new inequalities for product of two convex functions have been proved via non-conformable fractional integrals. We also establish several new integral inequalities including non-conformable fractional integrals for quasi-convex functions and $s-$Godunova-Levin functions by using two important integral identities. In order to obtain our results, we have used fairly elementary methodology by using the classical inequalities like power mean inequality and properties of modulus.

## 1. Introduction

In recent years, fractional calculus has become the center of attention of many researchers, both with its theory and its applications in engineering, mathematical biology and modelling. The new fractional derivative and integral operators have brought a new dimension to mathematical analysis and applied mathematics with different features (See [4, 6, 7, 12, 18]). The new operators have been used to generalize some known inequalities, besides they bring out with new trends and calculations in inequality theory (See [1–3, 5, 8, 13–17, 19–21]). Let's start with the definition of the non-conformable integral operator, which has an important place among the new operators.

**Definition 1.1.** [9] Let $\alpha \in \mathbb{R}$ and $0 < a < b$. For each function $f \in L^1[a, b]$, we define

$$_{N_3}J_u^{\alpha} f(x) = \int_u^x t^{-\alpha} f(t)dt$$

for every $x, u \in [a, b]$.

*Key words and phrases.* Fractional integrals, convex functions, Godunova-Levin function, quasi-convex functions.

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**Definition 1.2.** [9] Let $\alpha \in \mathbb{R}$ and $a < b$. For each function $f \in L_{\alpha,0}[a,b]$ let us define the fractional integrals

$$_{N_3}J^{\alpha}_{a+}f(x) = \int_a^x (x-t)^{-\alpha} f(t)dt,$$

$$_{N_3}J^{\alpha}_{b-}f(x) = \int_x^b (t-x)^{-\alpha} f(t)dt$$

for every $x \in [a,b]$.

The following remarkable different types of convex functions are used quite often in inequality theory.

**Definition 1.3.** [10] Let real function $f$ be defined on some nonempty interval $I$ of real numbers line $R$. The function $f$ is said to be quasi-convex on $I$ if inequality

$$f(tx+(1-t)y) \leq \max\{f(x),\ f(y)\}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

**Definition 1.4.** [11] We say that the function $f : C \subset X \to [0,\infty)$ is $s-$Godunova$-$Levin type, with $s \in [0,1]$, if the following inequality holds:

$$f(tx+(1-t)y) \leq t^{-s}f(x) + (1-t)^{-s}f(y)$$

for all $t \in (0,1)$ and $x, y \in C$.

The paper is organized as follows: Firstly, we prove some new inequalities for product of two convex functions via non-conformable fractional integral operator. Then, we establish two new integral identities and prove several new inequalities by using these new identities.

## 2. Main Result

We will start with the following inequality that involves the product of two convex functions.

**Theorem 2.1.** *Let $f, g : [a,b] \to R$ be differentiable functions on $(a,b)$. If $f \in L_{\alpha-1}[a,b]$ with $\alpha \in R$ and $f, g$ are convex functions on $[a,b]$, then we have:*

$$\begin{aligned} &\frac{1}{(b-a)^{-\alpha+1}}\left[_{N_3}J^{\alpha}_{b-}fg(a) +_{N_3} J^{\alpha}_{a+}fg(b)\right] \\ \leq\ &\left[\frac{1}{3-\alpha} + \frac{-2}{\alpha^3-6\alpha^2+11\alpha-6}\right][fg(a)+fg(b)] \\ &+\frac{2}{\alpha^2-5\alpha+6}[f(a)g(b)+f(b)g(a)]. \end{aligned}$$

*Proof.* Since $f, g$ are convex on $[a,b]$, we have

$$f(tb+(1-t)a) \leq tf(b) + (1-t)f(a)$$

and

$$g(tb+(1-t)a) \leq tg(b) + (1-t)g(a).$$

If we multiply these inequalities, we get

$$\begin{aligned} f\left(tb+(1-t)a\right)g\left(tb+(1-t)a\right) &\leq t^{2}fg\left(b\right)+\left(1-t\right)^{2}fg\left(a\right) \\ &\quad +t\left(1-t\right)\left[f\left(a\right)g\left(b\right)+f\left(b\right)g\left(a\right)\right]. \end{aligned}$$

By multiplying the above inequality by $t^{-\alpha}$, we can write the following inequality

$$\begin{aligned} t^{-\alpha}fg\left(tb+(1-t)a\right) &\leq t^{2-\alpha}fg\left(b\right)+t^{-\alpha}\left(1-t\right)^{2}fg\left(a\right) \\ &\quad +t^{1-\alpha}\left(1-t\right)\left[f\left(a\right)g\left(b\right)+f\left(b\right)g\left(a\right)\right]. \end{aligned}$$

Now, by integrating the resulting inequality with respect to $t$ over $[0,1]$, we have

$$\begin{aligned} \int_{0}^{1}t^{-\alpha}fg\left(tb+(1-t)a\right)dt &\leq \int_{0}^{1}t^{2-\alpha}fg\left(b\right)dt+\int_{0}^{1}t^{-\alpha}\left(1-t\right)^{2}fg\left(a\right)dt \\ &\quad +\int_{0}^{1}t^{1-\alpha}\left(1-t\right)\left[f\left(a\right)g\left(b\right)+f\left(b\right)g\left(a\right)\right]dt. \end{aligned}$$

Namely,

$$\begin{aligned} \int_{0}^{1}t^{-\alpha}fg\left(tb+(1-t)a\right)dt &\leq fg\left(b\right)\int_{0}^{1}t^{2-\alpha}dt+fg\left(a\right)\int_{0}^{1}t^{-\alpha}\left(1-t\right)^{2}dt \\ &\quad +\left[f\left(a\right)g\left(b\right)+f\left(b\right)g\left(a\right)\right]\int_{0}^{1}t^{1-\alpha}\left(1-t\right)dt. \end{aligned}$$

By computing the above integrals and changing the variables, we deduce

$$\begin{aligned} \frac{1}{(b-a)^{-\alpha+1}}\,{}_{N_3}J_{b^-}^{\alpha}fg(a) &\leq \frac{1}{3-\alpha}fg\left(b\right)+\frac{-2}{\alpha^{3}-6\alpha^{2}+11\alpha-6}fg\left(a\right) \\ &\quad +\frac{1}{\alpha^{2}-5\alpha+6}\left[f\left(a\right)g\left(b\right)+f\left(b\right)g\left(a\right)\right]. \end{aligned} \tag{2.1}$$

Similarly, we obtain

$$\begin{aligned} \int_{0}^{1}t^{-\alpha}fg\left(ta+(1-t)b\right)dt &\leq fg\left(a\right)\int_{0}^{1}t^{2-\alpha}dt+fg\left(b\right)\int_{0}^{1}t^{-\alpha}\left(1-t\right)^{2}dt \\ &\quad +\left[f\left(a\right)g\left(b\right)+f\left(b\right)g\left(a\right)\right]\int_{0}^{1}t^{1-\alpha}\left(1-t\right)dt. \end{aligned}$$

That is

$$\begin{aligned} \frac{1}{(b-a)^{-\alpha+1}}\,{}_{N_3}J_{a^+}^{\alpha}fg(b) &\leq \frac{1}{3-\alpha}fg\left(a\right)+\frac{-2}{\alpha^{3}-6\alpha^{2}+11\alpha-6}fg\left(b\right) \\ &\quad +\frac{1}{\alpha^{2}-5\alpha+6}\left[f\left(a\right)g\left(b\right)+f\left(b\right)g\left(a\right)\right]. \end{aligned} \tag{2.2}$$

By adding the inequalities (2.1) and (2.2), we get the proof. □

We proceed by giving an important identity involving non-conformable fractional Integrals:

**Lemma 2.1.** *Let $\alpha < 1, a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on $(a,b)$. If $f'' \in L_{\alpha-1}[a,b]$, then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{1-\alpha}{2\,(b-a)^{-\alpha}} \left\{ \begin{array}{c} \alpha \left[ {}_{N_3}J_{b^-}^{\alpha+1} f(a) \ + \ {}_{N_3}J_{a^+}^{\alpha+1} f(b) \right] \\ + \frac{(2-\alpha)}{b-a} \left[ {}_{N_3}J_{b^-}^{\alpha} f(a) \ + \ {}_{N_3}J_{a^+}^{\alpha} f(b) \right] \end{array} \right\} \\ = \ & \frac{(b-a)^2}{2} \left( I_1 + I_2 \right), \end{aligned} \tag{2.3}$$

*where*

$$I_1 = \int_0^1 t(1-t)^{1-\alpha} f''(ta+(1-t)b)dt, \quad I_2 = \int_0^1 t(1-t)^{1-\alpha} f''((1-t)\,a+tb)dt.$$

*Proof.* It's obvious that:

$$\begin{aligned} I_1 &= \int_0^1 t(1-t)^{1-\alpha} f''(ta+(1-t)b)dt \\ &= \int_0^1 t^{1-\alpha}(1-t) f''(tb+(1-t)a)dt \\ &= \int_0^1 (t^{1-\alpha} - t^{2-\alpha}) f''(tb+(1-t)a)dt. \end{aligned}$$

Integration by parts twice gives that the first integral is equal to:

$$\begin{aligned} I_1 &= \frac{t^{1-\alpha}-t^{2-\alpha}}{b-a} f'(tb+(1-t)a)|_0^1 \\ &\quad - \frac{1}{b-a} \int_0^1 ((1-\alpha)t^{-\alpha} - (2-\alpha)\,t^{1-\alpha}) f'(tb+(1-t)a)dt \\ &= -\frac{1}{b-a} \left[ \begin{array}{c} \frac{(1-\alpha)t^{-\alpha}-(2-\alpha)t^{1-\alpha}}{b-a} f(tb+(1-t)a)|_0^1 \\ -\frac{1}{b-a} \int_0^1 (\alpha(1-\alpha)t^{-\alpha-1} - (1-\alpha)\,(2-\alpha)\,t^{-\alpha}) f(tb+(1-t)a)dt \end{array} \right] \\ &= \frac{1}{(b-a)^2} f(b) + \frac{(1-\alpha)}{(b-a)^2} \left[ \begin{array}{c} -\alpha \int_0^1 t^{-\alpha-1} f(tb+(1-t)a)dt \\ -\,(2-\alpha) \int_0^1 t^{-\alpha} f(tb+(1-t)a)dt \end{array} \right] \\ &= \frac{1}{(b-a)^2} f(b) + \frac{(1-\alpha)}{(b-a)^2} \left[ \begin{array}{c} -\alpha \int_a^b \left(\frac{x-a}{b-a}\right)^{-\alpha-1} f(x) \frac{dx}{b-a} \\ -\,(2-\alpha) \int_a^b \left(\frac{x-a}{b-a}\right)^{-\alpha} f(x) \frac{dx}{b-a} \end{array} \right] \\ &= \frac{1}{(b-a)^2} f(b) + \frac{(1-\alpha)}{(b-a)^2} \left[ \begin{array}{c} \frac{-\alpha}{(b-a)^{-\alpha}} \int_a^b (x-a)^{-\alpha-1} f(x)dx \\ -\frac{2-\alpha}{(b-a)^{-\alpha+1}} \int_a^b (x-a)^{-\alpha} f(x)dx \end{array} \right] \\ &= \frac{1}{(b-a)^2} f(b) - \frac{(1-\alpha)}{(b-a)^{2-\alpha}} \left[ \alpha\, {}_{N_3}J_{b^-}^{\alpha+1} f(a) + \frac{2-\alpha}{b-a}\ {}_{N_3}J_{b^-}^{\alpha} f(a) \right]. \end{aligned}$$

Similarly, for the second integral, we can write:

$$\begin{aligned} I_2 &= \int_0^1 t(1-t)^{1-\alpha} f''((1-t)\,a+tb)dt = \int_0^1 t^{1-\alpha}(1-t)f''(ta+(1-t)\,b)dt \\ &= \int_0^1 (t^{1-\alpha}-t^{2-\alpha})f''(ta+(1-t)b)dt \end{aligned}$$

and integration by parts, we get:

$$\begin{aligned} I_2 &= \frac{1}{(b-a)^2}f(a)+\frac{1-\alpha}{(b-a)^2}\left[\begin{array}{c} -\alpha\int_0^1 t^{-\alpha-1}f(ta+(1-t)b)dt \\ -\frac{(2-\alpha)}{b-a}\int_0^1 t^{-\alpha}f(ta+(1-t)b)dt \end{array}\right] \\ &= \frac{1}{(b-a)^2}f(a)+\frac{1-\alpha}{(b-a)^2}\left[\begin{array}{c} -\alpha\int_a^b \left(\frac{b-x}{b-a}\right)^{-\alpha-1} f(x)\frac{dx}{b-a} \\ -\frac{(2-\alpha)}{b-a}\int_a^b \left(\frac{b-x}{b-a}\right)^{-\alpha} f(x)\frac{dx}{b-a} \end{array}\right] \\ &= \frac{1}{(b-a)^2}f(a)-\frac{1-\alpha}{(b-a)^{2-\alpha}}\left[\alpha_{N_3}J_{a^+}^{\alpha+1}f(b)+\frac{2-\alpha}{b-a}\ {}_{N_3}J_{a^+}^{\alpha}f(b)\right]. \end{aligned}$$

Summing these integrals, we get:

$$I_1+I_2=\frac{f(a)+f(b)}{(b-a)^2}-\frac{1-\alpha}{(b-a)^{2-\alpha}}\left\{\begin{array}{c} \alpha\left[{}_{N_3}J_{b^-}^{\alpha+1}f(a)\ +\ {}_{N_3}J_{a^+}^{\alpha+1}f(b)\right] \\ \frac{(2-\alpha)}{b-a}\left[{}_{N_3}J_{b^-}^{\alpha}f(a)\ +\ {}_{N_3}J_{a^+}^{\alpha}f(b)\right] \end{array}\right\}.$$

Multiplying both sides of the last equality by the expression $\frac{(b-a)^2}{2}$, we obtain (2.3). The proof is completed. □

**Theorem 2.2.** *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $I^{\circ}$ where $a,b\in I$. If $f''\in L[a,b]$ and $\left|f''\right|$ is $s-$Godunova-Levin type function, then for all $a\le x<y\le b$ and $\alpha<1, s\in[0,1)$, we have:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1-\alpha}{2\,(b-a)^{-\alpha}}\left\{\begin{array}{c} \alpha\left[{}_{N_3}J_{b^-}^{\alpha+1}f(a)\ +\ {}_{N_3}J_{a^+}^{\alpha+1}f(b)\right] \\ +\frac{(2-\alpha)}{b-a}\left[{}_{N_3}J_{b^-}^{\alpha}f(a)\ +\ {}_{N_3}J_{a^+}^{\alpha}f(b)\right] \end{array}\right\}\right| \\ &\qquad\le \frac{(b-a)^2}{2}\left[\beta\,(2-s,2-\alpha)+\beta\,(2,2-\alpha-s)\right]\left|f''\,(a)\right|+\left|f''\,(b)\right| \end{aligned}$$

*where $\beta\,(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x>1,\ y>0$ is modify of $\beta$.*

*Proof.* From Lemma 2.1, we have

$$\begin{aligned} &\left|\int_0^1 t(1-t)^{1-\alpha}f''\,(ta+(1-t)b)\,dt+\int_0^1 t(1-t)^{1-\alpha}f''\,(tb+(1-t)a)\,dt\right| \\ &\le \int_0^1 t(1-t)^{1-\alpha}\left|f''\,(ta+(1-t)\,b)\right|dt+\int_0^1 t(1-t)^{1-\alpha}\left|f''\,(tb+(1-t)a)\right|dt. \end{aligned}$$

Since $\left|f''\right|$ is $s-$Godunova-Levin type function, by computing the integrals, respectively, we get

$$\begin{aligned}&\int_0^1 t(1-t)^{1-\alpha}\left|f''\left(ta+(1-t)b\right)\right|dt\\&\leq\left|f''(a)\right|\int_0^1 t(1-t)^{1-\alpha}t^{-s}dt+\left|f''\left(b\right)\right|\int_0^1 t(1-t)^{1-\alpha-s}dt\\&=\left[\left|f''(a)\right|\beta\left(2-s,2-\alpha\right)+\left|f''(b)\right|\beta\left(2,2-\alpha-s\right)\right]\end{aligned}$$

and

$$\int_0^1 t(1-t)^{1-\alpha}\left|f''\left(tb+(1-t)a\right)\right|dt\leq\left[\left|f''(b)\right|\beta\left(2-s,2-\alpha\right)+\left|f''(a)\right|\beta\left(2,2-\alpha-s\right)\right].$$

Finally, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1-\alpha}{2\left(b-a\right)^{-\alpha}}\left\{\begin{array}{c}\alpha\left[{}_{N_3}J_{b-}^{\alpha+1}f(a)\ +\ {}_{N_3}J_{a+}^{\alpha+1}f(b)\right]\\+\frac{(2-\alpha)}{b-a}\left[{}_{N_3}J_{b-}^{\alpha}f(a)\ +\ {}_{N_3}J_{a+}^{\alpha}f(b)\right]\end{array}\right\}\right|\\&\leq\frac{\left(b-a\right)^2}{2}\left[\beta\left(2-s,2-\alpha\right)+\beta\left(2,2-\alpha-s\right)\right]\left|f''\left(a\right)\right|+\left|f''\left(b\right)\right|.\end{aligned}$$

□

**Theorem 2.3.** *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $I^\circ$ such that $f''\in L[a,b]$, where $a,b\in I$, $a<b$. If $\left|f''\right|^q$ is quasi-convex on $[a,b]\subset I$ and $q\geq1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1-\alpha}{2\left(b-a\right)^{-\alpha}}\left\{\begin{array}{c}\alpha\left[{}_{N_3}J_{b-}^{\alpha+1}f(a)\ +\ {}_{N_3}J_{a+}^{\alpha+1}f(b)\right]\\+\frac{(2-\alpha)}{b-a}\left[{}_{N_3}J_{b-}^{\alpha}f(a)\ +\ {}_{N_3}J_{a+}^{\alpha}f(b)\right]\end{array}\right\}\right|\\&\leq\frac{\left(b-a\right)^2}{\left(2-\alpha\right)\left(3-\alpha\right)}\left[\max\left(\left|f''\left(a\right)\right|^q,\left|f''\left(b\right)\right|^q\right)\right]^{\frac{1}{q}}\end{aligned}$$

*where $\alpha<1$.*

*Proof.* By using Lemma 2.1 and power-mean inequality with properties of modulus, we can write:

$$U=\frac{f(a)+f(b)}{2}-\frac{1-\alpha}{2\left(b-a\right)^{-\alpha}}\left\{\begin{array}{c}\alpha\left[{}_{N_3}J_{b-}^{\alpha+1}f(a)\ +\ {}_{N_3}J_{a+}^{\alpha+1}f(b)\right]\\ \\+\frac{(2-\alpha)}{b-a}\left[{}_{N_3}J_{b-}^{\alpha}f(a)\ +\ {}_{N_3}J_{a+}^{\alpha}f(b)\right]\end{array}\right\}$$

$$|U|\leq\frac{(b-a)^2}{2}\left[\begin{array}{c}\int_0^1 t(1-t)^{1-\alpha}\left|f''(ta+(1-t)b)\right|dt\\+\int_0^1 t(1-t)^{1-\alpha}\left|f''\left(tb+(1-t)a\right)\right|dt\end{array}\right]$$

From Lemma 1, it's obvious that:

$$|U| \leq \frac{(b-a)^2}{2}\left[\begin{array}{c} \left(\int_0^1 t(1-t)^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^{1-\alpha}\left|f''\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 t(1-t)^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^{1-\alpha}\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right]$$
$$= \frac{(b-a)^2}{(2-\alpha)(3-\alpha)}\left[\max\left(\left|f''(a)\right|^q, \ |f''(b)|^q\right)\right]^{\frac{1}{q}}$$

which completes the proof. Here, we used the quasi-convexity of $|f''|^q$ on $[a,b]$ and it can be easily checked that

$$\int_0^1 (t^{1-\alpha}-t^{2-\alpha})dt = \frac{1}{(2-\alpha)(3-\alpha)}.$$

□

Similarly, we can prove the following lemma that is generalization of Lemma 2.1.

Let $n \in \mathbb{N}$ and $a,b \in \mathbb{R}$ and let $a < b$, $n \geq 1$. The interval $[a,b]$ with a uniform step $h = \frac{b-a}{n}$ is divided into $n$ subintervals: $[a,b] = \bigcup\limits_{k=1}^{n}[\xi_{k-1},\xi_k]$, where $\xi_i = a+ih$, $i = 0,1,2,...,n$.

**Lemma 2.2.** *Let $\alpha < 1, a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on $(a,b)$. If $f'' \in L_{\alpha-1}[a,b]$, then*

$$\sum_{k=1}^{n}\left[\frac{f(\xi_{k-1})+f(\xi_k)}{2}\right] - \frac{1-\alpha}{2h^{-\alpha}} \tag{2.4}$$
$$\times\sum_{k=1}^{n}\left\{\begin{array}{c} \alpha\left[{}_{N_3}J^{\alpha+1}_{\xi_k^-}f(\xi_{k-1}) \ + \ {}_{N_3}J^{\alpha+1}_{\xi_{k-1}^+}f(\xi_k)\right] \\ +\frac{(2-\alpha)}{h}\left[{}_{N_3}J^{\alpha}_{\xi_k^-}f(\xi_{k-1}) \ + \ {}_{N_3}J^{\alpha}_{\xi_{k-1}^+}f(\xi_k)\right] \end{array}\right\}$$
$$= \frac{h^2}{2}\sum_{k=1}^{n}(I_{1k}+I_{2k}),$$

*where*

$$h = \frac{b-a}{n}, \ \xi_i = a+ih, i = 0,1,2,\dots n,$$
$$I_{1k} = \int_0^1 t(1-t)^{1-\alpha}f''(t\xi_{k-1}+(1-t)\xi_k)dt,$$
$$I_{2k} = \int_0^1 t(1-t)^{1-\alpha}f''((1-t)\,\xi_{k-1}+t\xi_k)dt.$$

*Proof.* It's obvious that

$$I_{1k} = \int_0^1 t(1-t)^{1-\alpha}f''(t\xi_{k-1}+(1-t)\xi_k)dt = \int_0^1 t^{1-\alpha}(1-t)f''((1-t)\xi_{k-1}+t\xi_k)dt$$
$$= \int_0^1 (t^{1-\alpha}-t^{2-\alpha})f''((1-t)\xi_{k-1}+t\xi_k)dt.$$

Integration by parts twice gives that the first integral can be calculated as:

$$\begin{aligned}
I_{1k} &= \frac{t^{\alpha-1}-t^{2-\alpha}}{\frac{b-a}{n}}f'((1-t)\xi_{k-1}+t\xi_k)|_0^1 \\
&\quad -\frac{1}{\frac{b-a}{n}}\int_0^1((1-\alpha)t^{-\alpha}-(2-\alpha)\,t^{1-\alpha})f'((1-t)\xi_{k-1}+t\xi_k)dt \\
&= -\frac{1}{\frac{b-a}{n}}\left[\begin{array}{c} \frac{(1-\alpha)t^{-\alpha}-(2-\alpha)t^{1-\alpha}}{\frac{b-a}{n}}f((1-t)\xi_{k-1}+t\xi_k)|_0^1 \\ \\ -\frac{1}{\frac{b-a}{n}}\int_0^1(\alpha(1-\alpha)t^{-\alpha-1}-(1-\alpha)\,(2-\alpha)\,t^{-\alpha})f((1-t)\xi_{k-1}+t\xi_k)dt \end{array}\right] \\
&= \frac{1}{\frac{(b-a)^2}{n^2}}f(\xi_k)+\frac{(1-\alpha)}{\frac{(b-a)^2}{n^2}}\left[\begin{array}{c} -\alpha\int_0^1 t^{-\alpha-1}f((1-t)\xi_{k-1}+t\xi_k)dt \\ \\ -(2-\alpha)\int_0^1 t^{-\alpha}f((1-t)\xi_{k-1}+t\xi_k)dt \end{array}\right] \\
&= \frac{1}{\frac{(b-a)^2}{n^2}}f(\xi_k)+\frac{(1-\alpha)}{\frac{(b-a)^2}{n^2}}\left[\begin{array}{c} -\alpha\int_{\xi_{k-1}}^{\xi_k}\left(\frac{x-\xi_{k-1}}{\frac{b-a}{n}}\right)^{-\alpha-1}f(x)\frac{dx}{\frac{b-a}{n}} \\ \\ -(2-\alpha)\int_{\xi_{k-1}}^{\xi_k}\left(\frac{x-\xi_{k-1}}{\frac{b-a}{n}}\right)^{-\alpha}f(x)\frac{dx}{\frac{b-a}{n}} \end{array}\right] \\
&= \frac{1}{\frac{(b-a)^2}{n^2}}f(\xi_k)+\frac{(1-\alpha)}{\frac{(b-a)^2}{n^2}}\left[\begin{array}{c} \frac{-\alpha}{\left(\frac{b-a}{n}\right)^{-\alpha}}\int_{\xi_{k-1}}^{\xi_k}(x-\xi_{k-1})^{-\alpha-1}f(x)dx \\ \\ -\frac{2-\alpha}{\left(\frac{b-a}{n}\right)^{-\alpha+1}}\int_{\xi_{k-1}}^{\xi_k}(x-\xi_{k-1})^{-\alpha}f(x)dx \end{array}\right] \\
&= \frac{1}{\left(\frac{b-a}{n}\right)^2}f(\xi_k)-\frac{(1-\alpha)}{\left(\frac{b-a}{n}\right)^{2-\alpha}}\left[\alpha_{N_3}J_{\xi_k^-}^{\alpha+1}f(\xi_{k-1})+\frac{2-\alpha}{\frac{b-a}{h}}\;{}_{N_3}J_{\xi_k^-}^{\alpha}f(\xi_{k-1})\right].
\end{aligned}$$

Similarly, for the second integral, we can easily have:

$$\begin{aligned}
I_{2k} &= \int_0^1 t(1-t)^{1-\alpha}f''((1-t)\,\xi_{k-1}+t\xi_k)dt=\int_0^1 t^{1-\alpha}(1-t)f''(t\xi_{k-1}+(1-t)\,\xi_k)dt \\
&= \int_0^1(t^{1-\alpha}-t^{2-\alpha})f''(t\xi_{k-1}+(1-t)\xi_k)dt
\end{aligned}$$

and integration by parts, we get:

$$\begin{aligned}
I_{2k} &= \frac{1}{\left(\frac{b-a}{n}\right)^2}f(\xi_{k-1})+\frac{1-\alpha}{\left(\frac{b-a}{n}\right)^2}\left[\begin{array}{c} -\alpha\int_0^1 t^{-\alpha-1}f(t\xi_{k-1}+(1-t)\xi_k)dt \\ \\ -(2-\alpha)\int_0^1 t^{-\alpha}f(t\xi_{k-1}+(1-t)\xi_k)dt \end{array}\right] \\
&= \frac{1}{\left(\frac{b-a}{n}\right)^2}f(\xi_{k-1})+\frac{1-\alpha}{\left(\frac{b-a}{n}\right)^2}\left[\begin{array}{c} -\alpha\int_{\xi_{k-1}}^{\xi_k}\left(\frac{\xi_k-x}{\frac{b-a}{n}}\right)^{-\alpha-1}f(x)\frac{dx}{b-a} \\ \\ -(2-\alpha)\int_{\xi_{k-1}}^{\xi_k}\left(\frac{\xi_k-x}{\frac{b-a}{n}}\right)^{-\alpha}f(x)\frac{dx}{b-a} \end{array}\right] \\
&= \frac{1}{\left(\frac{b-a}{n}\right)^2}f(\xi_{k-1})-\frac{1-\alpha}{\left(\frac{b-a}{n}\right)^{2-\alpha}}\left[\alpha_{N_3}J_{a^+}^{\alpha+1}f(b)+\frac{2-\alpha}{\frac{b-a}{n}}\;{}_{N_3}J_{a^+}^{\alpha}f(b)\right]
\end{aligned}$$

Necessary actions should be taken by considering the sum from 1 to n, we obtain:

$$I_{1k}+I_{2k}=\frac{f(\xi_{k-1})+f(\xi_k)}{\left(\frac{b-a}{n}\right)^2}-\frac{1-\alpha}{\left(\frac{b-a}{n}\right)^{2-\alpha}}\left\{\begin{array}{c}\alpha\left[{}_{N_3}J^{\alpha+1}_{\xi_k^-}f(\xi_{k-1})\;+\;{}_{N_3}J^{\alpha+1}_{\xi_{k-1}^+}f(\xi_k)\right]\\ \frac{(2-\alpha)}{h}\left[{}_{N_3}J^{\alpha}_{\xi_k^-}f(\xi_{k-1})\;+\;{}_{N_3}J^{\alpha}_{\xi_{k-1}^+}f(\xi_k)\right]\end{array}\right\}.$$

Multiplying both sides last equality by the expression $\frac{(h)^2}{2}$, we obtain (2.4). The proof is completed. □

**Theorem 2.4.** *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $I^\circ$ where $a,b\in I$ with $t\in[0,1]$. If $f''\in L_{\alpha-1}[a,b]$ and $\left|f''\right|$ is $s-$Godunova-Levin type function. then for all $a\le x<y\le b$ and $\alpha<1,\ s\in[0,1)$ we have*

$$\left|\sum_{k=1}^{n}\frac{f(\xi_{k-1})+f(\xi_k)}{2}-\frac{1-\alpha}{2\,(h)^{-\alpha}}\sum_{k=1}^{n}\left\{\begin{array}{c}\alpha\left[{}_{N_3}J^{\alpha+1}_{\xi_k^-}f(\xi_{k-1})\;+\;{}_{N_3}J^{\alpha+1}_{\xi_{k-1}^+}f(\xi_k)\right]\\ +\frac{(2-\alpha)}{h}\left[{}_{N_3}J^{\alpha}_{\xi_k^-}f(\xi_{k-1})\;+\;{}_{N_3}J^{\alpha}_{\xi_{k-1}^+}f(\xi_k)\right]\end{array}\right\}\right|$$
$$\le\frac{(h)^2}{2}\sum_{k=1}^{n}\left[\beta\left(2-s,2-\alpha\right)+\beta\left(2,2-\alpha-s\right)\right]\left|f''\left(\xi_{k-1}\right)\right|+\left|f''\left(\xi_k\right)\right|$$

*where $\beta\left(x,y\right)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x>1,\ y>0$ is modify of $\beta$.*

*Proof.* From Lemma 2.2, we have

$$\left|\int_0^1 t(1-t)^{1-\alpha}f''\left(t\xi_{k-1}+(1-t)\xi_k\right)dt+\int_0^1 t(1-t)^{1-\alpha}f''\left(t\xi_k+(1-t)\xi_{k-1}\right)dt\right|$$
$$\le\int_0^1 t(1-t)^{1-\alpha}\left|f''\left(t\xi_{k-1}+(1-t)\,\xi_k\right)\right|dt+\int_0^1 t(1-t)^{1-\alpha}\left|f''\left(t\xi_k+(1-t)\xi_{k-1}\right)\right|dt.$$

Since $\left|f''\right|$ is $s-$Godunova-Levin type function and by computing the above integrals, we get:

$$\int_0^1 t(1-t)^{1-\alpha}\left|f''\left(t\xi_{k-1}+(1-t)\xi_k\right)\right|dt$$
$$\le\left|f''(\xi_{k-1})\right|\int_0^1 t(1-t)^{1-\alpha}t^{-s}dt+\left|f''\left(\xi_k\right)\right|\int_0^1 t(1-t)^{1-\alpha-s}dt$$
$$=\left[\left|f''(\xi_{k-1})\right|\beta\left(2-s,2-\alpha\right)+\left|f''(\xi_k)\right|\beta\left(2,2-\alpha-s\right)\right]$$

and

$$\int_0^1 t(1-t)^{1-\alpha}\left|f''\left(t\xi_k+(1-t)\xi_{k-1}\right)\right|dt$$
$$=\left[\left|f''(\xi_k)\right|\beta\left(2-s,2-\alpha\right)+\left|f''(\xi_{k-1})\right|\beta\left(2,2-\alpha-s\right)\right].$$

Finally, we have

$$\left|\sum_{k=1}^{n}\frac{f(\xi_{k-1})+f(\xi_k)}{2}-\frac{1-\alpha}{2(h)^{-\alpha}}\sum_{k=1}^{n}\left\{\begin{array}{c}\alpha\left[{}_{N_3}J^{\alpha+1}_{\xi_k^-}f(\xi_{k-1})\ +\ {}_{N_3}J^{\alpha+1}_{\xi_{k-1}^+}f(\xi_k)\right]\\+\frac{(2-\alpha)}{h}\left[{}_{N_3}J^{\alpha}_{\xi_k^-}f(\xi_{k-1})\ +\ {}_{N_3}J^{\alpha}_{\xi_{k-1}^+}f(\xi_k)\right]\end{array}\right\}\right|$$
$$\leq\frac{(h)^2}{2}\sum_{k=1}^{n}\left[\beta(2-s,2-\alpha)+\beta(2,2-\alpha-s)\right]\left|f''(\xi_{k-1})\right|+\left|f''(\xi_k)\right|.$$

□

**Theorem 2.5.** *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $I^\circ$ such that $f''\in L_{\alpha-1}[a,b]$ ,where $a,b\in I$ , $a<b$ with $t\in[0,1]$. If $\left|f''\right|^q$ is quasi-convex on $[a,b]\subset I$ and $q\geq 1$, then the following inequality for fractional integrals holds:*

$$\left|\sum_{k=1}^{n}\frac{f(\xi_{k-1})+f(\xi_k)}{2}-\frac{1-\alpha}{2(h)^{-\alpha}}\sum_{k=1}^{n}\left\{\begin{array}{c}\alpha\left[{}_{N_3}J^{\alpha+1}_{\xi_k^-}f(\xi_{k-1})\ +\ {}_{N_3}J^{\alpha+1}_{\xi_{k-1}^+}f(\xi_k)\right]\\+\frac{(2-\alpha)}{h}\left[{}_{N_3}J^{\alpha}_{\xi_k^-}f(\xi_{k-1})\ +\ {}_{N_3}J^{\alpha}_{\xi_{k-1}^+}f(\xi_k)\right]\end{array}\right\}\right|$$
$$\leq\frac{(h)^2}{(2-\alpha)(3-\alpha)}\sum_{k=1}^{n}\left[\max\left(\left|f''(a)\right|^q,\left|f''(b)\right|^q\right)\right]^{\frac{1}{q}}$$

*for $\alpha<1$.*

*Proof.* By using Lemma 2.2 and power-mean inequality with properties of modulus, we can write,

$$V=\sum_{k=1}^{n}\frac{f(\xi_{k-1})+f(\xi_k)}{2}-\frac{1-\alpha}{2(h)^{-\alpha}}\sum_{k=1}^{n}\left\{\begin{array}{c}\alpha\left[{}_{N_3}J^{\alpha+1}_{\xi_k^-}f(\xi_{k-1})\ +\ {}_{N_3}J^{\alpha+1}_{\xi_{k-1}^+}f(\xi_k)\right]\\+\frac{(2-\alpha)}{h}\sum_{k=1}^{n}\left[{}_{N_3}J^{\alpha}_{\xi_k^-}f(\xi_{k-1})\ +\ {}_{N_3}J^{\alpha}_{\xi_{k-1}^+}f(\xi_k)\right]\end{array}\right\}$$

$$|V|\leq\frac{(h)^2}{2}\sum_{k=1}^{n}\left[\begin{array}{c}\int_0^1 t(1-t)^{1-\alpha}\left|f''(t\xi_{k-1}+(1-t)\xi_k)\right|dt\\+\int_0^1 t(1-t)^{1-\alpha}\left|f''(t\xi_k+(1-t)\xi_{k-1})\right|dt\end{array}\right]$$

It's obvious that

$$\begin{aligned}I_1&=\int_0^1 t(1-t)^{1-\alpha}f''(t\xi_{k-1}+(1-t)\xi_k)dt=\int_0^1 t^{1-\alpha}(1-t)f''(t\xi_k+(1-t)\xi_{k-1})dt\\&=\int_0^1(t^{1-\alpha}-t^{2-\alpha})f''(t\xi_k+(1-t)\xi_{k-1})dt\end{aligned}$$

and

$$\begin{aligned}I_2&=\int_0^1 t(1-t)^{1-\alpha}f''((1-t)\xi_{k-1}+t\xi_k)dt=\int_0^1 t^{1-\alpha}(1-t)f''(t\xi_{k-1}+(1-t)\xi_k)dt\\&=\int_0^1(t^{1-\alpha}-t^{2-\alpha})f''(t\xi_{k-1}+(1-t)\xi_k)dt\end{aligned}$$

$$
\begin{aligned}
|V| &\leq \frac{(h)^2}{2}\sum_{k=1}^{n}\left[\begin{array}{c}\left(\int_0^1 t(1-t)^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^{1-\alpha}\left|f''\left(t\xi_{k-1}+(1-t)\xi_k\right)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 t(1-t)^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^{1-\alpha}\left|f''(t\xi_k+(1-t)\xi_{k-1})\right|^q dt\right)^{\frac{1}{q}}\end{array}\right] \\
&= \frac{(h)^2}{(2-\alpha)(3-\alpha)}\sum_{k=1}^{n}\left[\max\left(\left|f''\left(\xi_{k-1}\right)\right|^q,\ \left|f''\left(\xi_k\right)\right|^q\right)\right]^{\frac{1}{q}}
\end{aligned}
$$

which completes the proof. □

**Theorem 2.6.** *Let $f:[a,b]\to R$ be a differentiable function on $[a,b]$. If $f\in L_{\alpha-1}[a,b]$ with $\alpha\in R$ and $f$ is convex on $[a,b]$, then we have*

$$
f\left(\frac{a+b}{2}\right)\leq\frac{1-\alpha}{2\left(b-a\right)^{1-\alpha}}\left[N_3J_{b-}^{a}f\left(a\right)+N_3J_{a+}^{b}f\left(b\right)\right]\leq\frac{f\left(a\right)+f\left(b\right)}{2}. \tag{2.5}
$$

*Proof.* Since $f$ is convex on $[a,b]$, we have $f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2}$. If we choose $x=ta+(1-t)\,b$ *and* $y=(1-t)\,a+tb$ and multiplying both sides of the inequality by $t^{-\alpha}$, we can write the following simple inequality

$$
2f\left(\frac{a+b}{2}\right)t^{-\alpha}\leq t^{-\alpha}f\left(ta+(1-t)\,b\right)+t^{-\alpha}f\left((1-t)\,a+tb\right).
$$

Now, by integrating the resulting inequality with respect to $t$ over $[0,1]$, we can obtain

$$
\begin{aligned}
\frac{2}{1-\alpha}f\left(\frac{a+b}{2}\right) &\leq \int_0^1 t^{-\alpha}f\left(ta+(1-t)\,b\right)dt+\int_0^1 t^{-\alpha}f\left((1-t)\,a+tb\right)dt \\
&= \frac{1}{(b-a)^{1-\alpha}}\left[\int_a^b (b-x)^{-\alpha}f\left(x\right)dx+\int_a^b (x-a)^{-\alpha}f\left(x\right)dx\right] \\
&= \frac{1}{(b-a)^{1-\alpha}}\left[N_3J_{b-}^{a}f\left(a\right)+N_3J_{a+}^{b}f\left(b\right)\right].
\end{aligned}
$$

.

For the proof of the second inequality, we can write

$$
f\left(ta+(1-t)\,b\right)\leq tf\left(a\right)+(1-t)\,f\left(b\right)
$$

$$
f\left((1-t)\,a+tb\right)\leq(1-t)\,f\left(a\right)+tf\left(b\right).
$$

By adding these inequalities, we obtain

$$
f\left(ta+(1-t)\,b\right)+f\left((1-t)\,a+tb\right)\leq tf\left(a\right)+(1-t)\,f\left(b\right)+(1-t)\,f\left(a\right)+tf\left(b\right).
$$

Now, if we multiply the resulting inequality by $t^{-\alpha}$and integrating with respect to $t$ over $[0,1]$ we obtain the following required inequality.

$$
\frac{1}{(b-a)^{1-\alpha}}\left[N_3J_{b-}^{a}f\left(a\right)+N_3J_{a+}^{b}f\left(b\right)\right]\leq\frac{1}{1-\alpha}\left[f\left(a\right)+f\left(b\right)\right]
$$

or

$$
\frac{1-\alpha}{2\left(b-a\right)^{1-\alpha}}\left[N_3J_{b-}^{a}f\left(a\right)+N_3J_{a+}^{b}f\left(b\right)\right]\leq\frac{1}{2}\left[f\left(a\right)+f\left(b\right)\right]
$$

which prove the second inequality of (2.5). □

**Corollary 2.1.** *If we choose* $\alpha = 0$ *in* (2.5) *then the inequality* (2.5) *reduces to the following inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\left[N_3 J^a_{b^-} f(a) + N_3 J^b_{a^+} f(b)\right] \leq \frac{f(a)+f(b)}{2}.$$

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