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ON HARDY-TYPE INTEGRAL INEQUALITIES WITH NEGATIVE PARAMETER

BOUHARKET BENAISSA[1] AND HUSEYIN BUDAK[2]

Abstract. In 2007, Bicheng Yang [4] introduced a new Hardy-type integral inequality of $p < 0$. In this paper, we give new solid-type integral inequalities with two negative parameters $p, q < 0$.

1. Introduction

In 2007, Bicheng Yang [4] presented a new Hardy-type integral inequality with negative parameter:

Theorem 1.1. *If $p < 0$, $r \neq 0$, $f \geq 0$ and $0 < \int_0^\infty t^r (tf(t))^p dt < \infty$, define the function $F(x)$ as*

$$F(x) = \int_0^x f(t)dt, \ for \ r < 1; \quad F(x) = \int_x^\infty f(t)dt, \ for \ r > 1.$$

Then one has

$$\int_0^\infty x^{-r} F^p(x)dx \leq \left(\frac{-p}{|r-1|}\right)^p \int_0^\infty t^{-r}(tf(t))^p dt, \tag{1.1}$$

where the constant factor $\left(\frac{-p}{|r-1|}\right)^p$ is the best possible.

The inequality above was generalized in [1]. In 2014, Banyat Sroysang [3] made a direct generalization of the original Hardy inequalities:

Theorem 1.2. *Let $f \geq 0$, $g > 0$, $0 < r < 1$, $p > 1$, $q > p - r(p-1)$ and $F(x) = \int_0^x f(t)dt$. If $\frac{x}{g(x)}$ is non-increasing, then*

$$\int_0^\infty \frac{F^p(x)}{g^q(x)}dx \leq \frac{1}{((r-1)(1-p)+q-1)(1-r)^{p-1}} \int_0^\infty \frac{(xf(x))^p}{g^q(x)}dx. \tag{1.2}$$

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Theorem 1.3. *Let* $f \geq 0,\ g > 0,\ r > 0,\ 0 < p < 1,\ q > p + r(p-1)$ *and* $F(x) = \int_0^x f(t)dt$. *If* $\frac{x}{g(x)}$ *is non-decreasing, then*

$$\int_0^\infty \frac{F^p(x)}{g^q(x)}dx \geq \frac{1}{((1+r)(1-p)+q-1)(1+r)^{p-1}} \int_0^\infty \frac{(xf(x))^p}{g^q(x)}dx. \tag{1.3}$$

The aim of this paper is to give a new some Hardy-type integral inequalities which are an extension of the inequalities cited in theorems above. The structure of this study takes the form of three sections with an introduction. In the second section we give the proofs of our main results for parameters negatives $p; q < 0$, in the third section we present some applications to the theorems. In [2], the following corollary was proved and it is useful in the proofs of main theorems.

Corollary 1.1. *(Reverse Hölder's inequality.)*
Let $\varepsilon \subset \mathbb{R}^n$ *be a measurable set and* $p < 0$, *we suppose that* u, v *are measurable on* ε. *If* $u \in L_p(\varepsilon)$ *and* $v \in L_{p'}(\varepsilon)$ *(*p' *is the conjugate parameter), then*

$$\int_\varepsilon |u(x)v(x)|\, dx \geq \left(\int_\varepsilon |u(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_\varepsilon |v(x)|^{p'}\, dx\right)^{\frac{1}{p'}}. \tag{1.4}$$

2. Main Results

In this section, we present our results, the Theorem 2.1 and Theorem 2.2.

Theorem 2.1. *Let* $f, g > 0$ *measurable functions on* $(0, +\infty)$, r, p, $q < 0$, $1-(1-r)(1-p)-q < 0$ *and* $F(x) = \int_0^x f(t)dt$. *If* $\frac{x}{g(x)}$ *is non-decreasing on* $(0, +\infty)$, *then*

$$\int_0^\infty \frac{F^p(x)}{g^q(x)}dx \leq \frac{1}{((1-r)(1-p)+q-1)(1-r)^{p-1}} \int_0^\infty \frac{(xf(x))^p}{g^q(x)}dx \tag{2.1}$$

Proof. By the reverse Hölder inequality, we obtain

$$\begin{aligned} F(x) &= \int_0^x t^{\frac{-r}{p'}} t^{\frac{r}{p'}} f(t)dt \\ &\geq \left(\int_0^x t^{-r}dt\right)^{\frac{1}{p'}} \left(\int_0^x t^{\frac{rp}{p'}} f^p(t)dt\right)^{\frac{1}{p}}, \end{aligned}$$

then

$$F^p(x) \leq \left(\int_0^x t^{-r}dt\right)^{p-1} \int_0^x t^{r(p-1)} f^p(t)dt.$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{F^p(x)}{g^q(x)}dx &\leq \frac{1}{(1-r)^{p-1}}\int_0^\infty g^{-q}(x)x^{(1-r)(p-1)}\left(\int_0^x t^{r(p-1)}f^p(t)dt\right)dx\\ &= \frac{1}{(1-r)^{p-1}}\int_0^\infty t^{r(p-1)}f^p(t)\left(\int_t^\infty g(x)^{-q}x^{(1-r)(p-1)}dx\right)dt.\end{aligned}$$

By the assumption $\left(\frac{x}{g(x)}\right)^q$ is non-increasing on (t,∞), we get

$$\begin{aligned}\int_t^\infty g(x)^{-q}x^{(1-r)(p-1)}dx &= \int_t^\infty x^{(1-r)(p-1)-q}(\frac{x}{g(x)})^q dx\\ &\leq \left(\frac{t}{g(t)}\right)^q\int_t^\infty x^{(1-r)(p-1)-q}dx\\ &= \left(\frac{1}{(1-r)(1-p)+q-1}\right)\frac{t^{(r-1)(1-p)+1}}{g^q(t)}.\end{aligned}$$

We obtain that

$$\int_0^\infty \frac{F^p(x)}{g^q(x)}dx \leq \frac{1}{((1-r)(1-p)+q-1)(1-r)^{p-1}}\int_0^\infty \frac{t^p f^p(t)}{g^q(t)}dt. \qquad \diamondsuit$$

Theorem 2.2. *Let $f,g>0$ measurable functions on $(0,+\infty)$, $p,q<0,\ r>1$ and $F(x)=\int_x^\infty f(t)dt$.*
If $\frac{x}{g(x)}$ is non-increasing on $(0,+\infty)$, then

$$\int_0^\infty \frac{F^p(x)}{g^q(x)}dx \leq \frac{1}{((r-1)(1-p)-q+1)(r-1)^{p-1}}\int_0^\infty \frac{(xf(x))^p}{g^q(x)}dx. \tag{2.2}$$

Proof. By using the reverse Hölder inequality for $\frac{1}{p}+\frac{1}{p'}=1$, we get

$$\begin{aligned}F(x) &= \int_x^\infty t^{\frac{-r}{p'}}t^{\frac{r}{p'}}f(t)dt\\ &\geq \left(\int_x^\infty t^{-r}dt\right)^{\frac{1}{p'}}\left(\int_x^\infty t^{\frac{rp}{p'}}f^p(t)dt\right)^{\frac{1}{p}},\end{aligned}$$

and then

$$F^p(x) \leq \frac{1}{(r-1)^{p-1}}x^{(r-1)(1-p)}\int_x^\infty t^{r(p-1)}f^p(t)dt.$$

We deduce that

$$\begin{aligned}\int_0^\infty \frac{F^p(x)}{g^q(x)}dx &\leq \frac{1}{(r-1)^{p-1}}\int_0^\infty g^{-q}(x)x^{(r-1)(1-p)}\left(\int_x^\infty t^{r(p-1)}f^p(t)dt\right)dx\\ &= \frac{1}{(r-1)^{p-1}}\int_0^\infty t^{r(p-1)}f^p(t)\left(\int_0^t g^{-q}(x)x^{(r-1)(1-p)}dx\right)dt.\end{aligned}$$

Since $\left(\frac{x}{g(x)}\right)^q$ is non-decreasing on $(0,t)$, it follows that

$$\begin{aligned}\int_0^t g^{-q}(x)x^{(r-1)(1-p)}dx &= \int_0^t x^{(r-1)(1-p)-q}\left(\frac{x}{g(x)}\right)^q dx\\ &\leq \left(\frac{t}{g(t)}\right)^q \int_0^t x^{(r-1)(1-p)-q}dx\\ &= \left(\frac{1}{(r-1)(1-p)-q+1}\right)\frac{t^{(r-1)(1-p)+1}}{g^q(t)},\end{aligned}$$

this gives us that

$$\int_0^\infty \frac{F^p(x)}{g^q(x)}dx \leq \frac{1}{((r-1)(1-p)-q+1)(r-1)^{p-1}}\int_0^\infty \frac{t^p f^p(t)}{g^q(t)}dt. \qquad \diamondsuit$$

3. Applications

If we put $q=p$ in Theorem 2.1, we get

Corollary 3.1. *Let $f,g>0,\ r<0,\ p<0$ and $F(x)=\int_0^x f(t)dt$, if $\frac{x}{g(x)}$ is non-decreasing, then*

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \leq \frac{1}{r(p-1)(1-r)^{p-1}}\int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx. \tag{3.1}$$

In particular,

(i) for $g(x)=x^m,\ m<1$, one has

$$\int_0^\infty \left(\frac{F(x)}{x^m}\right)^p dx \leq \frac{1}{r(p-1)(1-r)^{p-1}}\int_0^\infty \left(\frac{f(x)}{x^{m-1}}\right)^p dx \tag{3.2}$$

(ii) for $g(x)=x$, one has

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leq \frac{1}{r(p-1)(1-r)^{p-1}}\int_0^\infty f^p(x)dx. \tag{3.3}$$

Let $Hf(x)=\frac{1}{x}\int_0^x f(t)dt$ (Hardy operator),

Remark 3.1. if we take $r=\frac{1}{p},\ m=0$ in (3.2) and (3.3), we get

$$\int_0^\infty (x\,Hf(x))^p\,dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty (xf(x))^p\,dx \tag{3.4}$$

$$\int_0^\infty (Hf)^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx. \tag{3.5}$$

Remark 3.2. we take $r=\frac{1}{p}$ and $m=1+\frac{1}{p}$ in (3.2), then

$$\int_0^\infty x^{-1}(Hf)^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty x^{-1}f^p(x)dx. \tag{3.6}$$

If we put $q = p$ in the theorem (2.2), we get

Corollary 3.2. *Let $f, g > 0,\ p < 0,\ r > 1$ and $F(x) = \int_x^\infty f(t)dt$.*
If $\frac{x}{g(x)}$ is non-increasing, then

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \leq \frac{1}{r(1-p)(r-1)^{p-1}} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx. \tag{3.7}$$

In particular,

(i) for $g(x) = x^r$, one has

$$\int_0^\infty \left(x^{-r}F(x)\right)^p dx \leq \frac{1}{r(1-p)(r-1)^{p-1}} \int_0^\infty \left(x^{-r}(xf(x))\right)^p dx, \tag{3.8}$$

(ii) for $g(x) = x$, one has

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leq \frac{1}{r(1-p)(r-1)^{p-1}} \int_0^\infty f^p(x)dx. \tag{3.9}$$

Let $H_1 f(x) = \frac{1}{x}\int_x^\infty f(t)dt$ (the dual Hardy operator),

Remark 3.3. we choose $r = 2$ in (3.8) and (3.9), we get

$$\int_0^\infty \left(x^{-1}H_1 f(x)\right)^p dx \leq \frac{1}{2(1-p)} \int_0^\infty \left(x^{-1}f(x)\right)^p dx, \tag{3.10}$$

$$\int_0^\infty (H_1 f)^p(x)dx \leq \frac{1}{2(1-p)} \int_0^\infty f^p(x)dx. \tag{3.11}$$

Remark 3.4. we take $r = 1 - \frac{1}{p}$ in (3.8) and (3.9), then

$$\int_0^\infty x(H_1 f)^p(x)dx \leq \frac{1}{(1-p)^2(-p)^{-p}} \int_0^\infty x f^p(x)dx, \tag{3.12}$$

$$\int_0^\infty (H_1 f)^p(x)dx \leq \frac{1}{(1-p)^2(-p)^{-p}} \int_0^\infty f^p(x)dx. \tag{3.13}$$

Remark 3.5. For $p < 0$ we have

$$\left(1 + \frac{p}{p-1}\right)(1-p)\left(\frac{p}{p-1}\right)^{p-1} \geq (1-p)\left(\frac{p}{p-1}\right)^p \geq \left(\frac{p}{p-1}\right)^p.$$

Taking $r = 1 + \frac{p}{p-1}$ in (3.8) and (3.9), we get

$$\int_0^\infty x^{\frac{p^2}{1-p}}(H_1 f)^p(x)dx \leq \left(\frac{p-1}{p}\right)^p \int_0^\infty x^{\frac{p^2}{1-p}} f^p(x)dx, \tag{3.14}$$

$$\int_0^\infty (H_1 f)^p(x)dx \leq \left(\frac{p-1}{p}\right)^p \int_0^\infty f^p(x)dx. \tag{3.15}$$

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[1]Faculty of Material Sciences, University of Tiaret,
Laboratory of Informatics and Mathematics, University of Tiaret-Algeria
E-mail address: bouharket.benaissa@univ-tiaret.dz

[2] Department of Mathematics, Faculty of Science and Arts,
Düzce University, Konuralp Campus, Düzce, Turkey
E-mail address: hsyn.budak@gmail.com