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THE NEW ITERATION PROCESS FOR MULTIVALUED NONEXPANSIVE MAPPING IN KOHLENBACH HYPERBOLIC SPACE

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ABSTRACT. In this paper, we introduce an iteration scheme for multivalued mappings in Kohlenbach hyperbolic spaces and establish the strong and Δ-convergence theorems for approximating a fixed point of nonexpansive multivalued mapping with this iterative process under appropriate condition in Kohlenbach hyperbolic space. Our results generalize some previous works results in literature.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory contributes significantly to the theory of nonlinear functional analysis. The theory of iterative construction of fixed points of a nonlinear mapping under suitable set of control conditions is coined as metric fixed point theory. So, it has been study fixed point problems associated with a class of mappings in a suitable nonlinear structure. The term nonlinear structure in the fixed point theory is referred as a metric space embedded with a "convex structure". The metric spaces don't have a such structure. Hence, there is need to introduce convex structure in the metric space. The notion of convex metric spaces was first studied by Takahashi [24]. Shimizu and Takahashi [22] generalized results of Lim [15] given above from uniformly convex Banach spaces to convex metric spaces. Many authors have studied to a great extent the Banach spaces with convex structures.

We work in the setting of hyperbolic spaces introduced by Kohlenbach [13], the hyperbolic space is an example of a metric space with convex structure. The hyperbolic space introduced by Kohlenbach is more restrictive than the type by Goebel and Kirk [5] but more general than the type by Reich and Shafrir [19]. Non-positively curved hyperbolic space introduced by Kohlenbach provides rich geometrical structures suitable for metric fixed point theory of various classes of mappings.

Definition 1.1. [13] A space (X, d) coupled with $W : X^2 \times [0, 1] \to X$ fulfilling the following conditions:

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(i) $d(\nu, W(\varkappa, \omega, \beta)) \leq (1-\beta) d(\nu, \varkappa) + \beta d(\nu, \omega)$;
(ii) $d(W(\varkappa, \omega, \beta), W(\varkappa, \omega, \gamma)) = |\beta - \gamma| d(\varkappa, \omega)$;
(iii) $W(\varkappa, \omega, \beta) = W(\omega, \varkappa, 1-\beta)$;
(iv) $d(W(\varkappa, \nu, \beta), W(\omega, w, \beta)) \leq \beta d(\varkappa, \omega) + (1-\beta) d(\nu, w)$;
for all $\varkappa, \omega, \nu, w \in X$ and $\beta, \gamma \in [0,1]$ *is called a hyperbolic space.*

Definition 1.2. Let X be a hyperbolic space with a mapping $W : X^2 \times [0,1] \to X$.

i: A nonempty subset $E \subseteq X$ is convex if $W(\varkappa, \omega, \beta) \in E$ for all $\varkappa, \omega \in E$ and $\beta \in [0,1]$.

ii: A hyperbolic space (X, d, W) is uniformly convex ([22]) if for any $k > 0$ and $\varepsilon \in (0,2]$,there exists a $\delta \in (0,1]$ such that for all $t, \varkappa, \omega \in X$,

$$d\left(W\left(\varkappa, \omega, \frac{1}{2}\right), t\right) \leq (1-\delta) k,$$

whenever $d(\varkappa, t) \leq k$, $d(\omega, t) \leq k$ and $d(\varkappa, \omega) \geq \varepsilon k$.

iii: A map $\eta : (0, \infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(k, \varepsilon)$ for given $k > 0$ and $\varepsilon \in (0,2]$, is known as the modulus of uniform convexity.

Throughout this paper, we will represent a complete uniformly convex hyperbolic space unless otherwise stated.

Definition 1.3. [15] Let X be a metric space. For a bounded sequence $\{\varkappa_n\} \subseteq X$ and $\varkappa \in X$, define $r(., \{\varkappa_n\}) : X \to [0, \infty)$ by

$$r(\varkappa, \{\varkappa_n\}) = \limsup_{n \to \infty} d(\varkappa_n, \varkappa).$$

Then;

(a): The asymptotic radius of $\{\varkappa_n\}$ relative to $E \subset X$ is $r(E, \{\varkappa_n\}) = \inf\{r(\varkappa, \{\varkappa_n\}) : \varkappa \in X\}$.

(b): For any $\omega \in E \subset X$, the asymptotic center of $\{\varkappa_n\}$in relation to E is the set $A_E(\{\varkappa_n\}) = \{\varkappa \in X : r(\varkappa, \{\varkappa_n\}) \leq r(\omega, \{\varkappa_n\})\}$.

Definition 1.4. [14] If every subsequence $\{\varkappa_{n_i}\}$ of $\{\varkappa_n\} \subseteq X$ has a unique asymptotic center $\varkappa \in X$, then we say $\{\varkappa_n\}$ $\Delta-$converges to $\varkappa$. It be written $\Delta - \lim \varkappa_n = \varkappa$.

Let X be a metric space. A $E \subset X$ is a proximal set if there is a point $\omega \in E$ such that

$$d(\varkappa, \omega) = dist(\varkappa, E) := \{\inf d(\varkappa, z) : z \in E\} \text{ for all } \varkappa \in X.$$

It is denoted by $P(E)$ the family of nonempty proximal bounded subsets of E. The Hausdorff metric H on $P(E)$ is defined by $H\left(\hat{A}, \hat{C}\right) := \max\left\{\sup_{\varkappa \in \hat{A}} d\left(\varkappa, \hat{C}\right), \sup_{\omega \in \hat{C}} d(\omega, \hat{A})\right\}$ for all $\hat{A}$,$\hat{C} \in P(E)$. A multivalued map $T : E \to P(E)$ is nonexpansive if

$$H(T\varkappa, T\omega) \leq d(\varkappa, \omega)$$

for all $\varkappa, \omega \in E$. A point $\varkappa \in E$ is a fixed point of a map T if $\varkappa \in T\varkappa$. Denote the set of all fixed points of T by $F(T)$ or F and $P_T(\varkappa) = \{\omega \in T\varkappa : d(\varkappa, \omega) = d(\varkappa, T\varkappa)\}$.

Lemma 1.1. [12] *Let E be a nonempty closed convex subset of X. The asymptotic center of every bounded sequence $\{\varkappa_n\}$ in X is unique. Suppose $A(E,\{\varkappa_n\}) = \{\varkappa\}$ and $\{\omega_n\}$ is a subsequence of $\{\varkappa_n\}$ with $A(E,\{\omega_n\}) = \{\omega\}$. If $\{d(\varkappa_n,\omega)\}$ is convergent, then $x = \omega$.*

Lemma 1.2. [12] *Let (X,d,W) be a uniformly convex hyperbolic space and $\varkappa \in X$. Let $\{\alpha_n\} \in [b,c]$ for some $b,c \in (0,1)$ and $\{\varkappa_n\}$,$\{\omega_n\} \subseteq X$. If for some $l > 0$, $\limsup_{n\to\infty} d(\varkappa_n,\varkappa) \leq l$, $\limsup_{n\to\infty} d(\omega_n,\varkappa) \leq l$ and $\lim_{n\to\infty} d(W(\varkappa_n,\omega_n,\alpha_n),\varkappa) = l$. Then $\lim_{n\to\infty} d(\varkappa_n,\omega_n) = 0$.*

Lemma 1.3. [2] *Let a mapping $T : E \to P(E)$ be multivalued and $P_T(\varkappa) = \{\omega \in T(\varkappa) : d(\varkappa,\omega) = d(\varkappa,T(\varkappa))\}$.Then the following are hold:*

- **i:** $F(T) = F(P_T)$,
- **ii:** $P_T(\varkappa) = \{\varkappa\}$ *for each* $\varkappa \in F(T)$,
- **iii:** *For each* $\varkappa \in E$, $P_T(\varkappa)$ *is closed subset of* $T(\varkappa)$ *and so it is compact,*
- **iv:** $d(\varkappa,T(\varkappa)) = d(\varkappa,P_T(\varkappa))$ *for each* $\varkappa \in E$.

The normal Mann iteration scheme [16] have played a very helpful role in approximating the fixed point of a nonexpansive mapping in Banach space. Ishikawa [9] introduced a new iterative process which performs better than the Mann iteration for approximating the fixed points of same mapping in Hilbert space. Sastry and Babu [21] restate the Ishikawa iteration for multivalued mappings in Hilbert spaces. Phuengrattana and Suantai [18] introduced SP-iteration as a generalization of the Mann, Ishikawa and Noor iterations. Glowinski and Le Tallec [4] showed the three steps iteration yield better numerical results than the one or two steps iterations. Haubruge et al. [3] showed that three steps iteration process lead to highly parallel iterations in certain situations. Atalan and Karakaya [1] have investigated of some fixed point theorems in hyperbolic spaces for a three step iteration process.

In this work, we introduce an iteration scheme for multivalued mappings in Kohlenbach hyperbolic spaces and use $P_T(\varkappa) = \{y \in T\varkappa : \|\varkappa - y\| = d(\varkappa,T\varkappa)\}$ instead of a stronger condition $T\varkappa = \{\varkappa\}$ for any $\varkappa \in F(T)$ to approximate fixed point of multivaled nonexpansive mapping for proposed process under some conditions hyperbolic space. Our algortihm is defined as follows:

Let E be a nonempty convex subset of a hyperbolic space X. Let $T : E \to P(E)$ multivalued mapping and $P_T(\varkappa) = \{y \in T\varkappa : \|\varkappa - y\| = d(\varkappa,T\varkappa)\}$. Select $\varkappa_0 \in E$ and define $\{\varkappa_n\}$ as follows:

$$\begin{aligned} \varkappa_{n+1} &= u_n \\ y_n &= W\left(v_n, z_n, \frac{\beta_n}{1-\alpha_n}\right) \\ z_n &= W(x_n, w_n, \alpha_n) \end{aligned} \tag{1.1}$$

where $w_n \in P_T(\varkappa_n)$, $v_n \in P_T(z_n) = P_T(W(x_n,w_n,\alpha_n))$,
$u_n \in P_T(y_n) = P_T\left(W\left(v_n,z_n,\frac{\beta_n}{1-\alpha_n}\right)\right)$ and α_n, $\beta_n \in (0,1)$ such that $0 < \alpha_n + \beta_n < 1$.

2. Main Results

Lemma 2.1. *Let E be a nonempty closed convex subset of a uniformly convex hyperbolic space X and $T : E \to P(E)$ be a multivalued mapping such that P_T is nonexpansive mapping and with $F \neq \emptyset$. Let $\{x_n\}$ be the sequence define by algorithm (1.1). Then $\lim_{n\to\infty} d(\varkappa_n, \varkappa)$ exists for each $\varkappa \in F$.*

Proof. Let $\varkappa \in F$. Then $\varkappa \in P_T(\varkappa) = \{\varkappa\}$ by Lemma 1.3. Using (1.1), we have

$$\begin{aligned} d(y_n, \varkappa) &= d\left(W\left(v_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), \varkappa\right) \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(v_n, \varkappa) + \frac{\beta_n}{1-\alpha_n} d(z_n, \varkappa) \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) H(P_T(z_n), P_T(\varkappa)) + \frac{\beta_n}{1-\alpha_n} d(W(\varkappa_n, w_n, \alpha_n), \varkappa) \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(z_n, \varkappa) + \frac{\beta_n}{1-\alpha_n} [(1-\alpha_n) d(\varkappa_n, \varkappa) + \alpha_n d(w_n, \varkappa)] \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(W(\varkappa_n, w_n, \alpha_n), \varkappa) + \frac{\beta_n}{1-\alpha_n} [(1-\alpha_n) d(\varkappa_n, \varkappa) + \alpha_n d(w_n, \varkappa)] \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) [(1-\alpha_n) d(x_n, \varkappa) \\ &\quad + \alpha_n d(w_n, \varkappa)] + \frac{\beta_n}{1-\alpha_n} [(1-\alpha_n) d(\varkappa_n, \varkappa) + \alpha_n d(w_n, \varkappa)] \\ &= (1-\alpha_n) d(\varkappa_n, \varkappa) + \alpha_n d(w_n, \varkappa) \\ &\leq (1-\alpha_n) d(\varkappa_n, \varkappa) + \alpha_n H(P_T(\varkappa_n), P_T(\varkappa)) \\ &\leq (1-\alpha_n) d(\varkappa_n, \varkappa) + \alpha_n d(\varkappa_n, \varkappa) = d(\varkappa_n, \varkappa) \end{aligned}$$

that is,

$$d(y_n, \varkappa) \leq d(\varkappa_n, \varkappa). \tag{2.1}$$

Also,

$$\begin{aligned} d(\varkappa_{n+1}, \varkappa) &= d(u_n, \varkappa) \leq d(u_n, P_T(\varkappa)) \leq H(P_T(y_n), P_T(\varkappa)) \\ &\leq d(y_n, \varkappa) \\ &\leq d(\varkappa_n, \varkappa). \end{aligned} \tag{2.2}$$

Thus $d(\varkappa_{n+1}, \varkappa) \leq d(\varkappa_n, \varkappa)$. This means that $\lim_{n\to\infty} d(x_n, \varkappa)$ exists for each $\varkappa \in F$. □

Lemma 2.2. *Let E be a nonempty closed convex subset of a uniformly convex hyperbolic space X and $T : E \to P(E)$ be a multivalued mapping such that P_T is nonexpansive mapping and with $F \neq \emptyset$. Let $\{x_n\}$ be the sequence define by algorithm (1.1). Let $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$ satisfy $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$. For sequence $\{\varkappa_n\}$ in (1.1), then we have* $\lim_{n\to\infty} d(\varkappa_n, P_T(\varkappa_n)) = 0$.

Proof. By Lemma 2.1, $\lim\limits_{n\to\infty} d(\varkappa_n, \varkappa)$ exists for each $\varkappa \in F$. Presume that $\lim\limits_{n\to\infty} d(\varkappa_n, \varkappa) = c$ for some $c \geq 0$. The case $c = 0$ is trivial. Let's show that $c > 0$.

Now $\lim_{n\to\infty} d(\varkappa_{n+1}, \varkappa) = c$ can be rewritten as $\lim_{n\to\infty} d(u_n, \varkappa) = c$. As P_T is nonexpansive, we have

$$\begin{aligned} d(w_n, \varkappa) &\leq d(w_n, P_T(\varkappa)) \\ &\leq H(P_T(\varkappa_n), P_T(\varkappa)) \\ &\leq d(\varkappa_n, \varkappa). \end{aligned}$$

Taking $\limsup\limits_{n\to\infty}$ to the both sides, we get

$$\limsup_{n\to\infty} d(w_n, \varkappa) \leq \text{c}. \tag{2.3}$$

Next,

$$\begin{aligned} d(z_n, \varkappa) &= d(W(\varkappa_n, w_n, \alpha_n), \varkappa) \\ &\leq (1-\alpha_n) d(x_n, \varkappa) + \alpha_n d(w_n, \varkappa) \\ &\leq (1-\alpha_n) d(\varkappa_n, \varkappa) + \alpha_n H(P_T(\varkappa_n), P_T(\varkappa)) \\ &\leq (1-\alpha_n) d(\varkappa_n, \varkappa) + \alpha_n d(\varkappa_n, \varkappa) \\ &= d(\varkappa_n, \varkappa). \end{aligned}$$

Taking $\limsup\limits_{n\to\infty}$ to the both sides, we obtain

$$\limsup_{n\to\infty} d(z_n, \varkappa) \leq c \text{ and } \limsup_{n\to\infty} d(W(\varkappa_n, w_n, \alpha_n), \varkappa) \leq c. \tag{2.4}$$

Also,

$$\begin{aligned} d(v_n, \varkappa) &\leq d(v_n, P_T(\varkappa)) \\ &\leq H(P_T(z_n), P_T(\varkappa)) \\ &\leq d(z_n, \varkappa), \end{aligned}$$

hence

$$\limsup_{n\to\infty} d(v_n, \varkappa) \leq c.$$

Now, (1.1) can be rewritten as

$$\begin{aligned} d(\varkappa_{n+1}, \varkappa) &= d(u_n, \varkappa) \leq d(u_n, P_T(\varkappa)) \\ &\leq H(P_T(y_n), P_T(\varkappa)) \\ &\leq d(y_n, \varkappa) \\ &= d\left(W\left(v_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), \varkappa\right) \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(v_n, \varkappa) + \frac{\beta_n}{1-\alpha_n} d(z_n, \varkappa) \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) H(P_T(z_n), P_T(\varkappa)) + \frac{\beta_n}{1-\alpha_n} d(z_n, \varkappa) \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(z_n, \varkappa) + \frac{\beta_n}{1-\alpha_n} d(z_n, \varkappa) \\ &= d(z_n, \varkappa). \end{aligned}$$

Taking $\liminf\limits_{n\to\infty}$ to the both sides

$$c \leq \liminf_{n\to\infty} d\left(z_n, \varkappa\right). \tag{2.5}$$

From (2.4) and (2.5), we have

$$\lim_{n\to\infty} d\left(z_n, \varkappa\right) = c = \lim_{n\to\infty} d\left(W\left(\varkappa_n, w_n, \alpha_n\right), \varkappa\right). \tag{2.6}$$

Moreover, we obtain from $\lim_{n\to\infty} d\left(\varkappa_n, \varkappa\right) = c$,(2.3), (2.6) and Lemma 1.2 that

$$\lim_{n\to\infty} d\left(\varkappa_n, w_n\right) = 0$$

Also, from (1.1) and (2.2),we can write

$$\begin{aligned} d\left(\varkappa_n, u_n\right) &\leq d\left(\varkappa_n, \varkappa_{n+1}\right) + d\left(\varkappa_{n+1}, u_n\right) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $d\left(\varkappa, P_T\left(\varkappa\right)\right) = \inf_{z\in P_T(\varkappa)} d\left(\varkappa, z\right)$, therefore

$$d\left(\varkappa_n, P_T\left(\varkappa_n\right)\right) \leq d\left(\varkappa_n, w_n\right) \to 0 \quad \text{as } n \to \infty, \tag{2.7}$$

and

$$d\left(\varkappa_n, P_T\left(y_n\right)\right) \leq d\left(\varkappa_n, u_n\right) \to 0 \quad \text{as } n \to \infty.$$

This completes the proof. □

Theorem 2.1. *Let E be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and T, P_T and $\{\varkappa_n\}$ be as in Lemma 2.2. Then $\{\varkappa_n\}$ $\Delta-$converges to a fixed point of T.*

Proof. Let $\varkappa \in F(T) = F(P_T)$. By the proof of Lemma 2.1, $\{\varkappa_n\}$ is bounded and therefore $A\left(\{\varkappa_n\}\right) = \{\varkappa\}$. Let $\{\varkappa_{n_k}\}$ be any subsequence of $\{\varkappa_n\}$ such that $A\left(\{\varkappa_{n_k}\}\right) = \{\varkappa^*\}$. By Lemma 2.2, $\lim_{n\to\infty} d\left(\varkappa_n, P_T\left(\varkappa_n\right)\right) = 0$. We will show that $\varkappa^*$ is a fixed point of P_T. For this, take $\{w_m\}$ in $P_T\left(\varkappa^*\right)$. Then

$$\begin{aligned} r\left(w_m, \{\varkappa_{n_k}\}\right) &= \limsup_{k\to\infty} d\left(w_m, \varkappa_{n_k}\right) \\ &\leq \limsup_{k\to\infty} \left\{d\left(w_m, P_T\left(\varkappa_{n_k}\right)\right) + d\left(P_T\left(\varkappa_{n_k}\right), \varkappa_{n_k}\right)\right\} \\ &\leq \limsup_{k\to\infty} H\left(P_T\left(\varkappa^*\right), P_T\left(x_{n_k}\right)\right) \\ &\leq \limsup_{k\to\infty} d\left(\varkappa^*, \varkappa_{n_k}\right) \\ &= r\left(\varkappa^*, \{\varkappa_{n_k}\}\right). \end{aligned}$$

This yields $\left|r\left(w_m, \{\varkappa_{n_k}\}\right) - r\left(\varkappa^*, \{\varkappa_{n_k}\}\right)\right| \to 0$ as $m \to \infty$. From Lemma 1.1, we get $\lim_{m\to\infty} w_m = \varkappa^*$. Note that $T\varkappa^* \in P(E)$ being proximal is closed, hence $P_T\left(\varkappa^*\right)$ is closed and bounded. Hence $\lim_{m\to\infty} w_m = \varkappa^* \in P_T\left(\varkappa^*\right)$. Consequently $\varkappa^* \in F\left(P_T\right)$. From the uniqueness of the asymptotic center, we get

$$\begin{aligned}\limsup_{k\to\infty} d(\varkappa_{n_k},\varkappa^*) &< \limsup_{k\to\infty} d(\varkappa_{n_k},\varkappa)\\ &\leq \limsup_{n\to\infty} d(\varkappa_n,\varkappa)\\ &< \limsup_{n\to\infty} d(x_n,\varkappa^*)\\ &= \limsup_{k\to\infty} d(x_{n_k},\varkappa^*).\end{aligned}$$

This is a contradiction and hence, $\varkappa=\varkappa^*$. Therefore $A\left(\{\varkappa_{n_k}\}\right)=\{\varkappa^*\}$. Hence this shows that $\{\varkappa_n\}$ $\Delta-$converges to a fixed point of T. $\square$

A map $T:E\to P\left(E\right)$ is semicompact if any bounded sequence $\{\varkappa_n\}$ satisfying $\lim_{n\to\infty} d\left(\varkappa_n, T(\varkappa_n)\right)=0$ has a convergent subsequence.

Let $h:[0,\infty)\to[0,\infty)$ be a nondecreasing function with $f\left(0\right)=0$, $f\left(r\right)>0$ for $r\in(0,\infty)$ and $T:E\to P\left(E\right)$ be a multivalued map. Then a map T is said to satisfy condition (I) if $d\left(\varkappa,T\varkappa\right)\geq f(d(\varkappa,F)$ for all $\varkappa\in E$.

Theorem 2.2. *Let E be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and T, P_T and $\{\varkappa_n\}$ be as in Lemma 2.1. Then $\{\varkappa_n\}$ converges strongly to a fixed point $\varkappa\in F$ if and only if $\lim\inf_{n\to\infty} d\left(\varkappa_n,F\right)=0$.*

Proof. If $\{\varkappa_n\}$ converges to $\varkappa\in F$, then $\lim_{n\to\infty} d\left(\varkappa_n,\varkappa\right)=0$. Since $0\leq d\left(\varkappa_n,F\right)\leq d\left(\varkappa_n,\varkappa\right)$, it follows that $\lim\inf_{n\to\infty} d\left(\varkappa_n,F\right)=0$. Conversely, suppose that $\lim\inf_{n\to\infty} d\left(\varkappa_n,F\right)=0$. From Lemma 2.1, we write

$$d\left(\varkappa_{n+1},\varkappa\right)\leq d\left(\varkappa_n,F\right),$$

which implies that

$$d\left(\varkappa_{n+1},F\right)\leq d\left(\varkappa_n,F\right).$$

So, $\lim_{n\to\infty} d\left(\varkappa_n,F\right)$ exists. By hypothesis $\liminf_{n\to\infty} d\left(\varkappa_n,F\right)=0$, thus $\lim_{n\to\infty} d\left(\varkappa_n,F\right)=0$. Next,we show that $\{\varkappa_n\}$ is a Cauchy sequence in E. For $k,n\in N$ and $k>n$, we can write

$$d\left(\varkappa_k,\varkappa_n\right)\leq d\left(\varkappa_k,\varkappa\right)+d\left(\varkappa,\varkappa_n\right)\leq 2d\left(\varkappa_n,\varkappa\right).$$

Taking inf on the set F, we get $d\left(\varkappa_k,\varkappa_n\right)\leq d\left(\varkappa_n,F\right)$. Letting $m,n\to\infty$ in the inequality $d\left(\varkappa_k,\varkappa_n\right)\leq d\left(\varkappa_n,F\right)$ shows that$\{\varkappa_n\}$is a Cauchy sequence in E and therefore $\{\varkappa_n\}\to\varkappa^*\in E$. Next, we prove that $\varkappa^*\in F$. By $d\left(\varkappa_n,F(P_T)\right)=\inf_{\varkappa^*\in F(P_T)} d\left(\varkappa_n,\varkappa^*\right)$ and for each $\epsilon>0$, there exists $z_n^{(\epsilon)}\in F(P_T)$ such that,

$$d\left(\varkappa_n,z_n^{(\epsilon)}\right)<d\left(\varkappa_n,F(T)\right)+\frac{\epsilon}{2}.$$

This means that $d\left(\varkappa_n,z_n^{(\epsilon)}\right)\leq\frac{\epsilon}{2}$. From $d\left(z_n^{(\epsilon)},\varkappa^*\right)\leq d\left(\varkappa_n,z_n^{(\epsilon)}\right)+d\left(\varkappa_n,\varkappa^*\right)$, we obtain

$$\lim_{n\to\infty} d\left(z_n^{(\epsilon)},\varkappa^*\right)\leq\frac{\epsilon}{2}.$$

Finally,

$$\begin{aligned} d\left(P_T\left(\varkappa^*\right), \varkappa^*\right) &\leq d\left(\varkappa^*, z_n^{(\epsilon)}\right) + d\left(z_n^{(\epsilon)}, P_T\left(\varkappa^*\right)\right) \\ &\leq d\left(\varkappa^*, z_n^{(\epsilon)}\right) + H\left(P_T\left(z_n^{(\epsilon)}\right), P_T\left(\varkappa^*\right)\right) \\ &\leq 2d\left(\varkappa^*, z_n^{(\epsilon)}\right) \end{aligned}$$

which yields that $d\left(P_T\left(\varkappa^*\right), \varkappa^*\right) < \varepsilon$. Since ε is arbitrary, so $d\left(P_T\left(\varkappa^*\right), \varkappa^*\right) = 0$. F is closed, then $\varkappa^* \in F$. □

Theorem 2.3. *Let E be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and T, P_T, F and $\{\varkappa_n\}$ be as in Lemma 2.2. Suppose P_T satisfy condition (I), then the sequence $\{\varkappa_n\}$ converges strongly to $\varkappa \in F$.*

Proof. For all $\varkappa \in F$, $\lim_{n\to\infty} d\left(\varkappa_n, \varkappa\right)$ exists. We call it c for some $c \geq 0$. If $c = 0$, then results follows directly. Assume that $c > 0$. By (2.2), we can write

$$\inf_{\varkappa\in F(T)} d\left(\varkappa_{n+1}, \varkappa\right) \leq \inf_{\varkappa\in F(T)} d\left(\varkappa_n, \varkappa\right),$$

implies that $d\left(\varkappa_{n+1}, F(T)\right) \leq d\left(\varkappa_n, F(T)\right)$. Therefore $\lim_{n\to\infty} d\left(\varkappa_n, F\right)$ exists. With the help of Lemma 2.2 and condition (I), we can write follows that

$$\lim_{n\to\infty} f\left(d\left(\varkappa_n, F(T)\right)\right) \leq \lim_{n\to\infty} d\left(\varkappa_n, T(\varkappa_n)\right) = 0.$$

Thus

$$\lim_{n\to\infty} f\left(d\left(\varkappa_n, F\right)\right) = 0.$$

By definition of f, it follows that $\lim_{n\to\infty} d\left(\varkappa_n, F\right) = 0$. From proof of Theorem 2.2, we get the desired results. □

Theorem 2.4. *Let E be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and T, P_T, F,$\{\alpha_n\}$ and $\{\varkappa_n\}$ be as in Lemma 2.2. Suppose that P_T is semicompact, then sequence $\{\varkappa_n\}$ converges strongly to $\varkappa \in F$.*

Proof. From Lemma 2.1, $\{\varkappa_n\}$ is bounded. By Lemma 2.2, $\lim_{n\to\infty} d\left(\varkappa_n, P_T\left(\varkappa_n\right)\right) = 0$. Since P_T is semi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{\varkappa_n\}$ which converges to $\varkappa$. It follows from (2.7) and Lemma 2.1 that $\varkappa \in F$. Since Lemma 2.1, $\lim_{n\to\infty} d\left(\varkappa_n, \varkappa\right)$ exists and therefore $\varkappa_n \to \varkappa$. □

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