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SOLVABILITY OF A SEQUENTIAL PROBLEM OF DUFFING RAYLEIGH TYPE

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Abstract. In this work, we shall be concerned with a new fractional sequential problem of Duffing Rayleigh type. The considered problem allows us to obtain the classical Duffing Rayleigh equation as a special case under some particular values on its input data. An existence and uniqueness result is proved by application of Banach contraction principle. Then, we prove the existence of, at least, one solution for the problem. At the end, we present some examples to show the applicability of the two main results.

1. Introduction

Fractional calculus theory is a mathematical analysis where the interaction of integrals and derivatives of arbitrary order can be found. In recent years, considerable interest has been observed and this theory finds its importance in numerical analysis, applied mathematics and other areas of physics and engineering, see [1, 3, 5, 6, 8, 9, 17, 26]. The study of existence and uniqueness of solutions for of non-linear fractional differential equations is very important to understand the behavior of complex nonlinear physical phenomena. To present a contribution, in this work, we shall be concerned with the study of a problem of nonlinear differential equations of Duffing Rayleigh type. To be able to do this, we begin by recalling that the Duffing Rayleigh equation (DRE for short) has played a very important role in applied sciences. For more details, see the research works [4, 11, 19, 22, 27, 28]. The DRE equation has the following forme:

$$X'' + (\beta_1 + \beta_2 {X'}^2)X' + \gamma X' + w_0^2 X + kX^3 = w(t),$$

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where β_1 and β_2 denote some coefficients of linear and nonlinear damping, respectively, the function $h(X) = \omega_0^2 X + kX^3$ is the strongly nonlinear function that represents restoring force, with ω_0 being the natural frequency and k the intensity of nonlinearity, $w(t)$ is a Gaussian noise with intensity $2D$, while $D^\alpha X(t)$ represents the fractional derivative. Progress has been made in the studies of nonlinear Duffing Rayleigh oscillators. With fractional derivative approaches, we begin by the paper [24], where Min Xiao et al. have used the collocation method to study the following Rayleigh oscillator with a small fractional damping problem:

$$\begin{cases} X''(t) - \varepsilon\left(1 - \left(D^\alpha X(t)\right)^2\right) D^\alpha X(t) + X(t) = 0 \\ 0 < \alpha \leq 1, 0 < \varepsilon \leq 1, \end{cases}$$

with

$$X(0) = A, \quad X'(0) = B,$$

where the derivative $D^\alpha X(t)$ is of Caputo.

Fractional Caputo modified DR oscillator is also discussed by the authors in [27]:

$$\begin{cases} D^\alpha x(t) = y \\ D^\alpha y = ax - bx^3 + \varepsilon\left[\mu y(1 - |y|) + F\cos(\omega t)\right] \\ t \in [0, T], 0 < \alpha \leq 1, \end{cases}$$

Then, A.G.M. Selvam et al. [21] have been concerned with Ulam stability of the following discrete forced fractional DRE equation:

$$\begin{cases} D^\alpha X(t) + \delta X(t+\alpha) + \eta\left(X(t+\alpha)\right)^3 + h(t+\alpha) = 0 \\ t \in [0, T] \cap \mathbb{N}_{2-\alpha}, 1 < \alpha \leq 2 \\ X(0) = A, \quad X'(0) = B, \end{cases}$$

where D^α is of Caputo, δ and η are used to control the stiffness, h : $\mathbb{Q} \to \mathbb{R}$ is the driving force with $A, B \in \mathbb{R}_+$. Other papers dealing with Duffing oscillators can be found in the recent papers [14, 18, 23]. In this paper, we continue with the study of DRE problem. So, we consider the following problem:

$$\begin{cases} D^{\alpha_1}\left(D^{\alpha_2} + \beta_1\right) y(t) + \beta_2 f\left(t, y(t), D^{\alpha_3} y(t)\right) + \gamma g\left(t, y(t), D^{\alpha_4} y(t)\right) \\ +h(t, y(t)) = w(t), t \in [0, 1] \\ 0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1, 0 < \alpha_3 \leq 1, 0 < \alpha_4 \leq 1 \\ \gamma > 0, \beta_1 > 0, \beta_2 > 0 \\ y(0) = a_0, y(1) = b_0, \end{cases} \tag{1.1}$$

where $D^{\alpha_1}, D^{\alpha_2}, D^{\alpha_3}$ and D^{α_4} are the Caputo derivatives of orders $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $f, g \in C\left([0, 1], \mathbb{R}^2\right), h \in C\left([0, 1], \mathbb{R}\right), w \in C\left([0, 1]\right)$

The equation (1.1) is important since it is sequential and, on the other hand, it allows us to obtain the above DRE equation as a special case under some particular values on its input data like for instance when we take $\alpha_1 = \alpha_2 = 1, \alpha_3 = 1, f(t, x, y) = y^3, g(t, x, y) = 0, h(t, x) = ax + bx^3$. It is to note that this equation has no periodic conditions nor non locale ones but it has two simple initial conditions and four Caputo derivatives; these two special cases allow us to obtain a new class of fractional DRE equations. More details about classical and fractional DRE systems and Duffing oscillators can be found in [2, 10, 12, 13, 16, 20, 25].

2. Definitions and lemmas

To study (1.1), we need some definitions and lemmas. For more details, see [15, 17].

Definition 2.1. The Riemann Liouville integral of order $\alpha \geq 0$, for a continuous function $f : [0, \infty[\to \mathbb{R}$ is defined as:

$$\begin{aligned} J^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha} f(u)\, du, \alpha > 0, t \geq 0 \\ J^0 f(t) &= f(t), t \geq 0. \end{aligned} \tag{2.1}$$

Definition 2.2. The Caputo derivative of order α for a function $y : [0, \infty[\to \mathbb{R}$, which is at least n-times differentiable is the following:

$$\begin{aligned} D^{\alpha} y(t) &= J^{n-\alpha} D^n y(t), \alpha > 0 \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)\, ds, \end{aligned} \tag{2.2}$$

for $n := [\alpha] + 1$.

For the following two lemmas, one can consult the references [17].

Lemma 2.1. *Let us take $\alpha > 0$. Hence, the general solution $D^{\alpha} y(t) = 0$ is given by*

$$y(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, \tag{2.3}$$

for $c_i \in \mathbb{R}, i = 0 \ldots n-1$, $n = [\alpha] + 1$.

Lemma 2.2. *Taking $n \in N^*$, and $n - 1 < \alpha \leq n$, hence,*

$$J^{\alpha} D^{\alpha} y(t) = y(t) + \sum_{i=0}^{n-1} c_i t^i, \tag{2.4}$$

c_i are constants.

Now, we pass to prove the first result for the integral representation.

Lemma 2.3. *Taking a function $F \in C([0, 1])$. Therefore, the problem*

$$\begin{cases} D^{\alpha_1} (D^{\alpha_2} + \beta_1) y(t) = F(t) \\ t \in [0, 1] \\ 0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1 \\ \beta_1 > 0 \\ y(0) = a_0, y(1) = b_0 \end{cases} \tag{2.5}$$

has

$$
\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}F(\tau)\,d\tau\right)-\beta_1 y(s))ds+(a_0-b_0)\,t^{\alpha_2} \\
&+\frac{t^{\alpha_2}}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}F(\tau)\,d\tau\right)-\beta_1 y(s))ds+a_0
\end{aligned}
$$

as solution.

Proof. We have
$y(t)=J^{\alpha_2}J^{\alpha_1}F(t)-\beta_1 J^{\alpha_2}y(t)-J^{\alpha_2}c_0-c_1,$

where c_0 and c_1 are arbitrary constants to be determined.
By (2.5), we have

$$
\begin{aligned}
&c_1=-a_0 \\
&c_0=\Gamma(\alpha_2+1)\left(a_0-b_0+J^{\alpha_2}J^{\alpha_1}F(1)-\beta_1 J^{\alpha_2}y(1)\right).
\end{aligned}
$$

The above lemma is proved. □

3. A Criterion for Existence and Uniqueness of Solutions

We will use the fixed point theorem to study this problem. To do this, we introduce the following set:

$$
\begin{aligned}
&X:=\left\{y\in C([0,1],\mathbb{R}),\,D^{\alpha_3}y\in C([0,1],\mathbb{R}),\,D^{\alpha_4}y\in C([0,1],\mathbb{R})\right\}, \\
&||.||_X=Max\left(||y||_\infty,||D^{\alpha_3}y||_\infty,||D^{\alpha_4}y||_\infty\right), \\
&||y||_\infty=\sup_{t\in[0,1]}(|y(t)|), \\
&||D^{\alpha_3}y||_\infty=\sup_{t\in[0,1]}(|D^{\alpha_3}y(t)|), \\
&||D^{\alpha_4}y||_\infty=\sup_{t\in[0,1]}(|D^{\alpha_4}y(t)|).
\end{aligned}
$$

it is a simple task to prove that it is a Banach space.
Then, over the above space, we define the nonlinear operator H by:

$$
\begin{aligned}
Hy(t) &= \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^t (t-s)^{\alpha_1+\alpha_2-1}F(s)\,ds-\frac{\beta_1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}y(s)ds \\
&+\frac{t^{\alpha_2}}{\Gamma(\alpha_1+\alpha_2)}\int_0^1 (1-s)^{\alpha_1+\alpha_2-1}F(s)\,ds-\beta_1\frac{t^{\alpha_2}}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1}y(s)ds \\
&+(a_0-b_0)\,t^{\alpha_2}+a_0.
\end{aligned}
$$

We use also the following expression:

$$
\begin{aligned}
& D^{\delta} Hy\,(t) \\
= \; & \frac{1}{\Gamma\left(\alpha_1+\alpha_2-\delta\right)} \int_0^t (t-s)^{\alpha_1+\alpha_2-\delta-1} F\,(s)\,ds - \frac{\beta_1}{\Gamma\left(\alpha_2-\delta\right)} \int_0^t (t-s)^{\alpha_2-\delta-1} y(s)ds \\
& + \frac{t^{\alpha_2-\delta}\Gamma\left(\alpha_2+1\right)}{\Gamma\left(\alpha_1+\alpha_2\right)\Gamma\left(\alpha_2-\delta+1\right)} \int_0^1 (1-s)^{\alpha_1+\alpha_2-1} F\,(s)\,ds - \beta_1 \frac{t^{\alpha_2-\delta}\Gamma\left(\alpha_2+1\right)}{\Gamma\left(\alpha_2\right)\Gamma\left(\alpha_2-\delta+1\right)} \\
& \times \left[\int_0^1 (1-s)^{\alpha_2-1} y(s)ds\right] + (a_0-b_0)\,\frac{t^{\alpha_2-\delta}\Gamma\left(\alpha_2+1\right)}{\Gamma\left(\alpha_2-\delta+1\right)}.
\end{aligned}
$$

We consider the following hypotheses which are only sufficient:

(A1): The functions f and g defined on $[0,1]\times\mathbb{R}^2$ are continuous,
and h defined on $[0,1]\times\mathbb{R}$ is also continuous.

(A2): There exist constants $L_f, L_g, L_h > 0$, such that for, $t\in[0,1]\,; y_1, y_1^*, y_2, y_2^* \in \mathbb{R}$

$$
\begin{aligned}
\left|f\left(t,y_1,y_1^*\right)-f\left(t,y_2,y_2^*\right)\right| &\leq L_f\left(\left|y_1-y_2\right|+\left|y_1^*-y_2^*\right|\right); \\
\left|g\left(t,y_1,y_1^*\right)-g\left(t,y_2,y_2^*\right)\right| &\leq L_g\left(\left|y_1-y_2\right|+\left|y_1^*-y_2^*\right|\right); \\
\left|h\left(t,y_1\right)-h\left(t,y_2\right)\right| &\leq L_h\left|y_1-y_2\right|.
\end{aligned}
$$

(A3): There exist constants $M_{f1}, M_{f2}, M_{g1}, M_{g2}, M_h$ nonnegative, such that for any $t\in[0,1]\,; x,y\in\mathbb{R}$ we have:

$$
\begin{aligned}
|f(t,x,y)| &\leq M_{f1}\,|x| + M_{f2}\,|y| \\
|g(t,x,y)| &\leq M_{g1}\,|x| + M_{g2}\,|y| \\
|h(t,y)| &\leq M_h\,|y|.
\end{aligned}
$$

(A4): The function w is such that $||w||_\infty = M_w$.

We note that the above hypotheses are just sufficient to prove the theorem. It is to note also that we can take Caratheodory functions/ times Lypshitz functions/ or exponential Lypshitz functions...

Let us now consider the values:

$$
D_1 = \frac{2\upsilon_4}{\Gamma\left(\alpha_1+\alpha_2+1\right)} + \frac{2\beta_1}{\Gamma\left(\alpha_2+1\right)},
$$

$$
D_2 = \upsilon_4\left(\frac{1}{\Gamma\left(\alpha_1+\alpha_2-\delta+1\right)} + \frac{\Gamma\left(\alpha_2+1\right)}{\Gamma\left(\alpha_2-\delta+1\right)\Gamma\left(\alpha_1+\alpha_2+1\right)}\right) + \beta_1\left(\frac{1}{\Gamma\left(\alpha_2-\delta+1\right)} + \frac{1}{\Gamma\left(\alpha_2+1\right)}\right),
$$

where,

$$\begin{aligned}
\upsilon_1 &= \beta_2 M_{f1} + \gamma M_{g1} + M_h; \upsilon_2 = \beta_2 M_{f2}; \upsilon_3 = \gamma M_{g2}; \upsilon_4 = \left(\beta_2 L_f + \gamma L_g + L_h\right);\\
\Lambda_1 &= \frac{2\upsilon_1}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{2\beta_1}{\Gamma(\alpha_2+1)}; \Lambda_2 = \frac{2\upsilon_2}{\Gamma(\alpha_1+\alpha_2+1)}; \Lambda_3 = \frac{2\upsilon_3}{\Gamma(\alpha_1+\alpha_2+1)};\\
\Lambda_4 &= \frac{\upsilon_1}{\Gamma(\alpha_1+\alpha_2-\delta+1)} + \frac{\upsilon_1\Gamma(\alpha_2+1)}{\Gamma(\alpha_2-\delta+1)\Gamma(\alpha_1+\alpha_2+1)} + \beta_1\left(\frac{1}{\Gamma(\alpha_2-\delta+1)} + \frac{1}{\Gamma(\alpha_2+1)}\right);\\
\Lambda_5 &= \frac{\upsilon_2}{\Gamma(\alpha_1+\alpha_2-\delta+1)} + \frac{\upsilon_2\Gamma(\alpha_2+1)}{\Gamma(\alpha_2-\delta+1)\Gamma(\alpha_1+\alpha_2+1)};\\
\Lambda_6 &= \frac{\upsilon_3}{\Gamma(\alpha_1+\alpha_2-\delta+1)} + \frac{\upsilon_3\Gamma(\alpha_2+1)}{\Gamma(\alpha_2-\delta+1)\Gamma(\alpha_1+\alpha_2+1)};\\
\varepsilon_1 &= \frac{2M_w}{\Gamma(\alpha_1+\alpha_2+1)} + |b_0| + 2|a_0|;\\
\varepsilon_2 &= \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2-\delta+1)}|b_0+a_0| + M_w\left(\frac{1}{\Gamma(\alpha_1+\alpha_2-\delta+1)} + \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2-\delta+1)\Gamma(\alpha_1+\alpha_2+1)}\right).
\end{aligned}$$

Theorem 3.1. *The problem* (1.1) *has a unique solution on* $[0,1]$ *if (A2), (A4) and* $Max\{D_1, D_2\} < 1$.

Proof. In the following we not that δ takes the order derivative values α_3 and α_4. For $(x,y) \in X \times X$, we can write:

$$\begin{aligned}
\sup_{t\in[0,1]} |Hx(t) - Hy(t)| &\leq \left(\frac{2\upsilon_4}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{2\beta_1}{\Gamma(\alpha_2+1)}\right)|x(t)-y(t)|\\
&\leq D_1|x(t)-y(t)|
\end{aligned}$$

Then, one can see that the following inequality is valid:

$$||Hx(t) - Hy(t)||_\infty \leq D_1 ||x(t)-y(t)||_X.$$

We have also

$$\begin{aligned}
\sup_{t\in[0,1]} \left|D^\delta Hx(t) - D^\delta Hy(t)\right| &\leq \upsilon_4\left(\frac{1}{\Gamma(\alpha_1+\alpha_2-\delta+1)} + \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2-\delta+1)\Gamma(\alpha_1+\alpha_2+1)}\right)|x(t)-y(t)|\\
&\quad +\beta_1\left(\frac{1}{\Gamma(\alpha_2-\delta+1)} + \frac{1}{\Gamma(\alpha_2+1)}\right)|x(t)-y(t)|\\
&\leq D_2|x(t)-y(t)|.
\end{aligned}$$

So, we obtain

$$\left|\left|D^\delta Hx(t) - D^\delta Hy(t)\right|\right|_\infty \leq D_2 ||x(t)-y(t)||_X.$$

Therefore, we can write

$$||Hx(t) - Hy(t)||_X \leq D ||x(t)-y(t)||_X.$$

According to Banach theorem, there is a unique solution for (1.1). □

Now, we present to the reader the following second main result.

Theorem 3.2. *The validation of hypotheses (A1), (A3) and (A4) allows us to state that* (1.1) *has at least a solution.*

Proof. We can use Schaefer theorem.

Continuity:
We start by showing that H is continuous over X.
We have:

$$||Hy_n(t) - Hy(t)||_\infty \leq D_1 ||y_n(t) - y(t)||_X, \tag{3.1}$$

and

$$\left|\left|D^\delta Hy_n(t) - D^\delta Hy(t)\right|\right|_\infty \leq D_2 ||y_n(t) - y(t)||_X. \tag{3.2}$$

By (3.1) and (3.2), we can obtain:

$$||Hy_n(t) - Hy(t)||_X \leq D ||y_n(t) - y(t)||_X.$$

Therefore, we conclude that:

$$||Hy_n(t) - Hy(t)||_X \to 0, n \to \infty.$$

Then, we confirm that H is continuous over X.

Boundedness:
Using the sup norm, we can write

$$||Hy(t)||_\infty \leq \Lambda_1 ||y(t)||_X + \Lambda_2 ||D^{\alpha_3}y(t)||_X + \Lambda_3 ||D^{\alpha_4}y(t)||_X + \varepsilon_1 < +\infty.$$

Also, we have

$$\underset{t\in[0,1]}{sup}\left|D^\delta Hy(t)\right| \leq \Lambda_4 |y(t)| + \Lambda_5 |D^{\alpha_3}y(t)| + |D^{\alpha_4}y(t)| \Lambda_6 + \varepsilon_2.$$

Which implies that

$$\left|\left|D^\delta Hy(t)\right|\right|_\infty \leq \Lambda_4 ||y(t)||_X + \Lambda_5 ||D^{\alpha_3}y(t)||_X + \Lambda_6 ||D^{\alpha_4}y(t)||_X + \varepsilon_2 < +\infty,$$

Putting $r = \max\{\varepsilon_1 + \varepsilon_2, \Lambda_1 + \Lambda_4, \Lambda_2 + \Lambda_5, \Lambda_3 + \Lambda_6\}$,
then by the $X-$norm definition, it yields that

$$||Hy||_X < r < +\infty, \tag{3.3}$$

where, y is supposed to in the ball $X_r := \{y \in X, \|y\|_X < r\}$.
Therefore, H is uniformly bounded over X_r.

Equi-continuity:
Let $t_1, t_2 \in [0,1]$ that $t_1 < t_2$. Then, we have

$$\begin{aligned}
& |Hy(t_1) - Hy(t_2)| \\
= \; & |\frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^{t_1} (t_1-s)^{\alpha_1+\alpha_2-1} F(s)\,ds - \frac{\beta_1}{\Gamma(\alpha_2)} \int_0^{t_1} (t_1-s)^{\alpha_2-1} y(s)ds \\
& + \frac{t_1^{\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} \int_0^1 (1-s)^{\alpha_1+\alpha_2-1} F(s)\,ds - \beta_1 \frac{t_1^{\alpha_2}}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} y(s)ds \\
& + (a_0-b_0)\,t_1^{\alpha_2} - \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^{t_2} (t_2-s)^{\alpha_1+\alpha_2-1} F(s)\,ds \\
& + \frac{\beta_1}{\Gamma(\alpha_2)} \int_0^{t_2} (t_2-s)^{\alpha_2-1} y(s)ds - \frac{t_2^{\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} \int_0^1 (1-s)^{\alpha_1+\alpha_2-1} F(s)\,ds \\
& + \beta_1 \frac{t_2^{\alpha_2}}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} y(s)ds - (a_0-b_0)\,t_2^{\alpha_2}| \\
= \; & |\frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^{t_1} (t_1-s)^{\alpha_1+\alpha_2-1} F(s)\,ds - \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_{t_1}^{t_2} (t_2-s)^{\alpha_1+\alpha_2-1} F(s)\,ds \\
& - \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^{t_2} (t_2-s)^{\alpha_1+\alpha_2-1} F(s)\,ds + \frac{(t_1^{\alpha_2}-t_2^{\alpha_2})}{\Gamma(\alpha_1+\alpha_2)} \int_0^1 (1-s)^{\alpha_1+\alpha_2-1} F(s)\,ds \\
& + \beta_1 \frac{(t_2^{\alpha_2}-t_1^{\alpha_2})}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} y(s)ds - \frac{\beta_1}{\Gamma(\alpha_2)} \int_0^{t_1} (t_1-s)^{\alpha_2-1} y(s)ds \\
& + \frac{\beta_1}{\Gamma(\alpha_2)} \int_{t_1}^{t_2} (t_2-s)^{\alpha_2-1} y(s)ds + \frac{\beta_1}{\Gamma(\alpha_2)} \int_0^{t_2} (t_2-s)^{\alpha_2-1} y(s)ds \\
& + (a_0-b_0)\,(t_1^{\alpha_2}-t_2^{\alpha_2})|.
\end{aligned}$$

Thanks to (A1), we can write

$$||Hy(t_1) - Hy(t_2)||_\infty \to 0, \text{as } t_1 \to t_2.$$

Also, we can observe that:

$$\begin{aligned}
& \left|D^\delta Hy(t_1) - D^\delta Hy(t_2)\right| \\
= \; & |\frac{1}{\Gamma(\alpha_1+\alpha_2-\delta)} \int_0^{t_1} (t_1-s)^{\alpha_1+\alpha_2-\delta-1} F(s)\,ds \\
& - \frac{1}{\Gamma(\alpha_1+\alpha_2-\delta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha_1+\alpha_2-\delta-1} F(s)\,ds - \frac{1}{\Gamma(\alpha_1+\alpha_2-\delta)} \int_0^{t_2} (t_2-s)^{\alpha_1+\alpha_2-\delta-1} F(s)\,ds \\
& - \frac{\beta_1}{\Gamma(\alpha_2-\delta)} \int_0^{t_1} (t_1-s)^{\alpha_2-\delta-1} y(s)ds + \frac{\beta_1}{\Gamma(\alpha_2-\delta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha_2-\delta-1} y(s)ds \\
& + \frac{\beta_1}{\Gamma(\alpha_2-\delta)} \int_0^{t_2} (t_2-s)^{\alpha_2-\delta-1} y(s)ds + \frac{(t_1^{\alpha_2-\delta}-t_2^{\alpha_2-\delta})\Gamma(\alpha_2+1)}{\Gamma(\alpha_1+\alpha_2)\,\Gamma(\alpha_2-\delta+1)} \int_0^1 (1-s)^{\alpha_1+\alpha_2-1} F(s)\,ds \\
& + \beta_1 \frac{(t_2^{\alpha_2-\delta}-t_1^{\alpha_2-\delta})\Gamma(\alpha_2+1)}{\Gamma(\alpha_2)\,\Gamma(\alpha_2-\delta+1)} \int_0^1 (1-s)^{\alpha_2-1} y(s)ds \\
& - (a_0-b_0)\,\frac{(t_2^{\alpha_2-\delta}-t_1^{\alpha_2-\delta})\Gamma(\alpha_2+1)}{\Gamma(\alpha_2-\delta+1)}|.
\end{aligned}$$

Consequently,

$$\left|\left|D^\delta Hy(t_1) - D^\delta Hy(t_2)\right|\right|_\infty \to 0, \text{as } t_1 \to t_2$$

As a consequence of the above three steps and thanks to Arzela-Ascoli theorem, we conclude that H is completely continuous.

A Priori Boundedness:
Let $y \in S$, such that $S := \{x \in X, x = \lambda Hx; 0 < \lambda < 1\}$. Then, $y = \lambda Hy$ for a certain $0 < \lambda < 1$. So, we have for any $t \in [0,1]$:

$$\begin{aligned} \|y\|_\infty &= \lambda \left|\left|Hy(t)\right|\right|_\infty \\ &\leq \lambda \left(\Lambda_1 \left|\left|y(t)\right|\right|_X + \Lambda_2 \left|\left|D^{\alpha_3} y(t)\right|\right|_X + \Lambda_3 \left|\left|D^{\alpha_4} y(t)\right|\right|_X + \varepsilon_1\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \left|\left|D^\delta y(t)\right|\right|_\infty &= \lambda \left|\left|D^\delta Hy(t)\right|\right|_\infty \\ &\leq \lambda(\Lambda_4 \left|\left|y(t)\right|\right|_X + \Lambda_5 \left|\left|D^{\alpha_3} y(t)\right|\right|_X + \Lambda_6 \left|\left|D^{\alpha_4} y(t)\right|\right|_X + \varepsilon_2). \end{aligned}$$

Consequently, S is bounded.
Using the Schaefer fixed point theorem, H has a fixed point which is a solution of (1.1). $\square$

4. Two Illustrative Examples

To be sequential differential examples, it is important to note that following two problems need to satisfy that the sum of their order derivatives of the left hand sides are not in the interval $[0,1]$.

Example 4.1. We consider the following problem:

$$\begin{cases} D^{0.99}\left(D^{0.35}+0.25\right) y(t) + 0.005\left(\frac{2|y(t)|}{(300+t)(1+|y(t)|)} + \frac{|D^{\alpha_3} y(t)|}{(150+t)(1+|D^{\alpha_3} y(t)|)}\right) \\ +\frac{\sin(2y(t)-D^{\alpha_4} y(t))}{36(t^2+t+1)} + \frac{2|y(t)|}{(15+t^2)(1+|y(t)|)} = \cos 2t \\ t \in [0,1] \\ y(0) = a_0, y(1) = b_0, \end{cases}$$

where

$$\begin{aligned} f(t,u,v) &= \left(\frac{2|u|}{(300+t)(1+|u|)} + \frac{|v|}{(150+t)(1+|v|)}\right) \\ g(t,u,v) &= \frac{\sin(2u-v)}{36(t^2+t+1)} \\ h(t,u) &= \frac{2|u|}{(15+t^2)(1+|u|)} \\ w(t) &= \cos 2t, \end{aligned}$$

and, also we take

$$\begin{aligned} &\delta = 0.12, \quad D_1 = 0.79353, \quad D_2 = 0.79350 \\ &D := Max\{D_1, D_2\} = 0.79353 \end{aligned}$$

The conditions of Theorem 3.1 are valid. Thus, problem (1.1) has a unique solution on $[0,1]$.

Example 4.2. As a second illustrative example, we consider the problem:

$$\begin{cases} D^{0.85}\left(D^{0.97}+\frac{1}{5}\right)y\left(t\right)+\frac{1}{300}\left(\frac{|y(t)|}{(300+t^2|y(t)|)}+\frac{|D^{\alpha_3}y(t)|}{(9+|D^{\alpha_3}y(t)|)}\right) \\ +\frac{2}{7}\left(\frac{\cos(y(t)+2)}{280(t+e^{-2t})}+\frac{t^3+|D^{\alpha_4}y(t)|}{50}+\ln(t+1)\right)+\frac{\sin(y(t))}{(450+t)e^{t^2+1}}=\ln(t+2) \\ t\in[0,1] \\ y\left(0\right)=a_0, y\left(1\right)=b_0, \end{cases}$$

such that,

$$\begin{aligned} f(t,u,v) &= \frac{|u|}{(300+t^2\,|u|)}+\frac{|v|}{(9+|v|)}, \\ g(t,u,v) &= \frac{\cos(u+2)}{280\,(t+e^{-2t})}+\frac{t^3+|v|}{50}+\ln(t+1), \\ h(t,u) &= \frac{\sin(u)}{(450+t)\,e^{t^2+1}}, \\ w(t) &= \ln(t+2) \end{aligned}$$

and

$$\delta=0.004,\qquad D_1=0.653\,61,\qquad D_2=0.654\,56$$
$$D:=Max\left\{D_1,D_2\right\}=0.654\,56.$$

Thanks to Theorem 3.1, we can state that (1) has a unique solution on $[0,1]$.

5. Conclusion

In this work, a DRE equation with four Caputo derivatives and two initial conditions has been considered. First, we have established an integral equivalence to the considered differential problem. Then, by means of fixed point theory, we have proved a uniqueness result. Then, based on Schaefer theorem, some sufficient conditions have been established to ensure the existence of solutions for the problem. At the end, some examples are discussed to illustrate the applicability of the first main result. In the future, we continue to work on the above problem and to study the stability of its solutions; the Ulam-Hyers and Ulam Hyers Rassias stabilities will be considered. For the interested reader, there is another way to work on the above problem; it can be generalized by introducing some convergent series on the right hand side of (1). The existence and uniqueness of solutions for the resulting problem can be investigated...

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References

[1] B. Ahmad, S.K. Ntouyas, R.P. Agarwal and A. Alsaedi, *Existence results for sequential fractional integro-differential equations with nonlocal multi-point and strip conditions*, Boundary Value Problems, **2016**, Article number: 205 (2016), 1–16.

[2] M. Bezziou, I. Jebril and Z. Dahmani, *A new nonlinear duffing system with sequential fractional derivatives,* Chaos, Solitons & Fractals **151**(2021), 111247.

[3] M. Bengrine, Z. Dahmani, *Boundary Value Problems For Fractional Differential Equations,* International Journal of Open Problems in Computer Science & Mathematics, **5**(4)(2012), 7–15.

[4] S. Chatterjee, S. Dey, *Nonlinear dynamics of two harmonic oscillators coupled by Rayleigh type self-exciting force,* Nonlinear Dynamics, **72**(1) (2013), 113–128.

[5] D. Delbosco, L. Rodino, *Existence and uniqueness for a nonlinear fractional differential equation,* Journal of Mathematical Analysis & Applications, **204**(2) (1996), 609–625.

[6] K. Diethelm, G. Walz, *Numerical solution of fractional order differential equations by extrapolation,* Numerical Algorithms, **16**(3) (1997), 231–253.

[7] RH. Enns, GG. McGuire, *Nonlinear physics with Mathematica for scientists and engineers,* Springer Science & Business Media, 2012.

[8] A. Feddaoui, J. Llibre and Amar Makhlouf, *Periodic solutions for a class of Duffing differential equations,* Moroccan J. Pure and Appl. Anal., **5**(1) (2019), 86–103.

[9] R. Gorenflo, *Fractional calculus: some numerical methods,* Courses and lectures-international centre for mechanical sciences, 277–290, 1997.

[10] Y. Gouari, Z. Dahmani and I. Jebril, *Application of fractional calculus on a new differential problem of Duffing type,* Advances in Mathematics: Scientific Journal, **9**(12) (2020), 10989–11002.

[11] B.Z. Kaplan, Y. Horen, *Switching-mode counterparts of the Rayleigh and Van-der-Pol oscillators,* International Journal of Circuit Theory & Applications, **28**(1)(2000), 31–49.

[12] CK. Kwuimy, BN. Nbendjo, *Active control of horseshoes chaos in a driven Rayleigh oscillator with fractional order deflection,* Physics Letters A, **375**(39) (2012), 3442–3449.

[13] L. Lai, Y.D. Ji, S.C. Zhong and L. Zhang, *Sequential parameter identification of fractional-order duffing system based on differential evolution algorithm,* Mathematical Problems in Engineering, 2017.

[14] C. Liu, C. Kuo and J. Chang, *Solving the Optimal Control Problems of Nonlinear Duffing Oscillators By Using an Iterative Shape Functions Method,* Computer Modeling in Engineering & Sciences, **122**(1) (2020), 33–48.

[15] KS. Miller, B. Ross,*An Introduction to The Fractional Calculus and Fractional Differential Equations,* John-Wily and Sons. Inc. New York, 1993.

[16] A. Ndiaye, M. Kaid and Z. Dahmani, *Solvability for differential systems of Duffing type involving sequential Caputo derivatives,* Annals Of Pure and Applied Mathematical Sciences, **1**(1) (2021), 1–13.

[17] K. Oldham, J. Spanier: *The fractional calculus theory and applications of differentiation and integration to arbitrary order,* Elsevier, 1974.

[18] P. Pirmohabbati, A.H. Refahi Sheikhani, H. Saberi Najafi and A. Abdolahzadeh Ziabari, *Numerical solution of full fractional Duffing equations with Cubic-Quintic-Heptic nonlinearities,* AIMS Mathematics, **5**(2) (2020), 1621–1641.

[19] Z. Ran-Ran, X. Wei, Y. Gui-Dong and H. Qun, *Response of a Duffing Rayleigh system with a fractional derivative under Gaussian white noise excitation,* Chinese Physics B, **24**(2) (2015), 20–24.

[20] M. Rostami, M. Haeri, *Undamped oscillations in fractional-order Duffing oscillator,* Signal processing, **107** (2015), 361–367.

[21] A.G.M. Selvam, D. Baleanu, J. Alzabut, D. Vignesh and S. Abbas, *On Hyers–Ulam Mittag-Leffler stability of discrete fractional Duffing equation with application on inverted pendulum,* Advances in Difference Equations, **7** (2020), Article number: 456, 1–15.

[22] M.S. Siewe, H. Cao, M.A. Sanjuán, *Effect of nonlinear dissipation on the basin boundaries of a driven two-well Rayleigh–Duffing oscillator,* Chaos, Solitons & Fractals, **39**(3) (2009), 1092–1099.

[23] W. Wawrzynski, *Duffing-type oscillator under harmonic excitation with a variable value of excitation amplitude and time-dependent external disturbances,* Scientific Reports, **11** (2021), Article number: 2889, 1–15.

[24] M. Xiao, G. Jiang and J. Cao, *Asymptotic Solutions and Circuit Implementations of a Rayleigh Oscillator Including Cubic Fractional Damping Terms,* Circuits, Systems, Signal Processing, **35**(6) (2016), 2041–2053.

[25] J.H. Yang, H. Zhu, *Bifurcation and resonance induced by fractional-order damping and time delay feedback in a Duffing system,* Communications in Nonlinear Science and Numerical Simulation, **18**(5) (2013), 1316–1326.

[26] X. Yue, W. Xu and Y. Zhang, *Global bifurcation analysis of Rayleigh–Duffing oscillator through the composite cell coordinate system method,* Nonlinear Dynamics, **69**(1) (2013), 437–457.

[27] Y.L. Zhang, C.K. Li, *Fractional modified Duffing–Rayleigh system and its synchronization,* Nonlinear Dynamics, **88**(4) (2017), 3023–3041.

[28] L. Zhou, S. Liu and F. Chen, *Chaotic dynamics of a Rayleigh-Duffing oscillator with periodically external and parametric excitations,* In 6th International Conference on Mechatronics, Materials, Biotechnology & Environment, 286–292, 2016.

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