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EXPLICIT SOLUTIONS OF POWERS OF THREE AS SUMS OF THREE PELL NUMBERS BASED ON BAKER'S TYPE INEQUALITIES

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ABSTRACT. In this paper, we consider the Diophantine equation $P_n + P_m + P_t = 3^a$ and obtain all solutions for this equation. In the proof of the main theorem, we use lower bounds for the absolute value of linear combinations of logarithms and a version of the Baker-Davenport reduction method.

1. INTRODUCTION

Recently, many authors are investigating the solutions of Diophantine equations involving linear recurrence sequences. For example, Bravo and Luca [1,2] solved the equation $u_n+u_m = 2^a$ for the cases, where u_n is the Fibonacci sequence and the Lucas sequence respectively. These are pioneer publications in this field. Later, many researchers made an effort to expand and generalize such publications by increasing the number of terms in both the numerical part of the equation and the recurrence sequence part. One can see the publications [4, 5, 7, 11, 12].

In this paper, our aim is to completely solve the Diophantine equation

$$P_n + P_m + P_t = 3^a \tag{1.1}$$

where P_n is the Pell sequence and n, m, t and a are nonnegative integers such that $n \geq m \geq t$.

The main argument used for the solution of such problems is Baker's theory (lower bound for the absolute value of linear combinations of logarithms of algebraic numbers) and a version of the Baker-Davenport reduction method.

2. PRELIMINARIES

A linear recurrence sequence of order k is a sequence whose general term is

$$(a_n) = L(a_{n-1}, a_{n-2}, \ldots, a_{n-k}) \tag{2.1}$$

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where k is a fixed positive integer and L is a linear function. A linear recurrence sequence of order 2 is called *a binary recurrence sequence*. Pell sequence, one of the most familiar binary recurrence sequence, is defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$. Some of the terms of the Pell sequence are given by $0, 1, 2, 5, 12, 29, 70, \ldots$. Its characteristic polynomial is of the form $x^2 - 2x - 1 = 0$ whose roots are $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Binet's formula enables us to rewrite the Pell sequence by using the roots α and β as

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}. \tag{2.2}$$

Also, it is known that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1}. \tag{2.3}$$

We give the definition of the logarithmic height of an algebraic number and its some properties.

Definition 2.1. Let ξ be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \cdot \prod_{i=1}^{d} (x - \xi_i)$$

where a_i's are relatively prime integers with $a_0 > 0$ and ξ_i's are conjugates of ξ. Then

$$h(\xi) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\xi_i|, 1\}\right)\right)$$

is called *the logarithmic height of* ξ. The following proposition gives some properties of logarithmic height that can be found in [14].

Proposition 2.1. *Let $\xi, \xi_1, \xi_2, \ldots, \xi_t$ be elements of an algebraic closure of $\mathbb{Q}$ and $m \in \mathbb{Z}$. Then*

(a) $h(\xi_1 \cdots \xi_t) \leq \sum_{i=1}^{t} h(\xi_i)$
(b) $h(\xi_1 + \cdots + \xi_t) \leq \log t + \sum_{i=1}^{t} h(\xi_i)$
(c) $h(\xi^m) = |m|\, h(\xi)$.

We will use the following theorem (see [10] or Theorem 9.4 in [6]) and lemma (see [3] which is a variation of the result due to [9]) for proving our results.

Theorem 2.1. *Let $\gamma_1, \gamma_2, \ldots, \gamma_s$ be nonzero elements of a real algebraic number field $\mathbb{F}$ of degree D, $b_1, b_2, \ldots, b_s$ rational integers. Set*

$$B := \max\{|b_1|, \ldots, |b_s|\}$$

and

$$\Lambda := \gamma_1^{b_1} \ldots \gamma_s^{b_s} - 1.$$

If Λ is nonzero, then

$$\log|\Lambda| > -3 \cdot 30^{s+4} \cdot (s+1)^{5.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log(sB)) \cdot A_1 \cdots A_s$$

where

$$A_i \geq \max\{D \cdot h(\gamma_i), |\log \gamma_i|, 0.16\}$$

for all $1 \le i \le s$. *If* $\mathbb{F} = \mathbb{R}$, *then*

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

Lemma 2.1. *Let* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$ *and let* γ *be an irrational number and* M *be a positive integer. Take* p/q *as a convergent of the continued fraction of* γ *such that* $q > 6M$. *Set* $\varepsilon := \|\mu q\| - M\|\gamma q\| > 0$ *where* $\|\cdot\|$ *denotes the distance from the nearest integer. Then there is no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v *and* w *with*

$$u \le M \text{ and } w \ge \frac{\log \frac{Aq}{\varepsilon}}{\log B}.$$

3. Main Result

Theorem 3.1. *The only nonnegative integer quads* n, m, t, a *with* $n \ge m \ge t$ *satisfying the Diophantine equation (1.1) as follows*

$$(n, m, t, a) \in \{(1, 0, 0, 0), (1, 1, 1, 1), (2, 1, 0, 1), (3, 2, 2, 2)\}.$$

Proof. First, let us examine some solutions of the equation case by case according to the states of n, m and t. In the case that $t = 0$, if also $m = 0$ then equation becomes

$$P_n = 3^a.$$

So, by Carmichael's primitive divisor theorem [8] n must be less than or equal to 12. This implies only the existence of the solution $(1, 0, 0, 0)$. For the same case, if $m > 0$ then we have the equation

$$P_n + P_m = 3^a.$$

In [13], they gave an upper bound for the solutions of the general case of the above equation. Taking into account this upper bound, a brute force search with Mathematica reveals that $(2, 1, 1)$ and $(1, 0, 0)$ are only solutions of above equation. Thus, the equation (1.1), in this case, has solution $(2, 1, 0, 1)$ as distinct from the previous case.
In the other case when $n = m = t$, the equation (1.1) takes the form

$$P_n = 3^{a-1}$$

and so has solution $(1, 1, 1, 1)$.
From now on, we will assume that $n \ge m \ge t \ge 1$ except for the last case above. A computer search with Mathematica for $n \le 200$ found out that there are no other solutions to the equation (1.1) than those stated in Theorem 3.1.
Assume that $n > 200$. Let us try to find a relation between a and n. Using the right hand

side of the inequality (2.3) we get

$$\begin{aligned}3^a &= P_n + P_m + P_t \\ &\leq \alpha^{n-1} + \alpha^{m-1} + \alpha^{t-1} \\ &< 3^{n-1}\left(1 + 3^{m-n} + 3^{t-n}\right) \\ &< 3^{n-1} \cdot 3 \\ &= 3^n.\end{aligned}$$

Thus, we have that $a < n$. When we replace P_n in the equation (1.1) with its closed form, we obtain

$$\frac{\alpha^n}{2\sqrt{2}} - 3^a = \frac{\beta^n}{2\sqrt{2}}\left(P_m + P_t\right).$$

By taking the absolute value of both sides of the above relation and using the upper bound in relation (2.3), it is yielded that

$$\left|\frac{\alpha^n}{2\sqrt{2}} - 3^a\right| \leq \frac{|\beta^n|}{2\sqrt{2}} + P_m + P_t < \frac{1}{6} + \left(\alpha^m + \alpha^t\right).$$

When we divide both sides of the above expression by $\frac{\alpha^n}{2\sqrt{2}}$ to apply Matveev's result in Theorem 2.1, we have

$$\begin{aligned}\left|1 - 3^a \cdot \alpha^{-n} \cdot 2\sqrt{2}\right| &< \frac{2\sqrt{2}}{\alpha^n}\left(\frac{1}{6} + \alpha^m + \alpha^t\right) \\ &= 2\sqrt{2}\alpha^{m-n}\left(\frac{1}{6}\alpha^{-m} + 1 + \alpha^{t-m}\right) \\ &< \frac{7}{\alpha^{n-m}}.\end{aligned} \tag{3.1}$$

The first application of the Matveev's result:

Set $\Delta_1 := 1 - 3^a \cdot \alpha^{-n} \cdot 2\sqrt{2}$. Δ_1 is not zero. If it were zero, it would be $\alpha^{2n} \in \mathbb{Z}$, which is impossible. Let us take $s := 3$, $(\gamma_1, \gamma_2, \gamma_3) := \left(3, \alpha, 2\sqrt{2}\right)$ and $(b_1, b_2, b_3) := (a, -n, 1)$. We have $D := 2$ since each γ_i belongs to $\mathbb{Q}\left(\sqrt{2}\right)$.

A_1, A_2, A_3 and B can be chosen as follows:

$$\begin{aligned}A_1 &:= 2.2 > 2.1972 \simeq 2 \cdot \log 3 = D \cdot h(\gamma_1) \\ A_2 &:= 0.9 > 0.8813 \simeq \log \alpha = D \cdot h(\gamma_2) \\ A_3 &:= 2.1 > 2.079 \simeq 2 \cdot \log\left(2\sqrt{2}\right) = D \cdot h(\gamma_3) \\ B &:= n.\end{aligned}$$

From Theorem 2.1, it is obtained that

$$\begin{aligned} |\Delta_1| &> \exp\left(-C_1 \cdot (1+\log n) \cdot 2.2 \cdot 0.9 \cdot 2.1\right) \\ \frac{7}{\alpha^{n-m}} &> \exp\left(-C_1 \cdot (1+\log n) \cdot 2.2 \cdot 0.9 \cdot 2.1\right) \qquad \text{from (3.1)} \end{aligned}$$

where $C_1 = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2)$. Proceeding to appropriate operations, we have

$$\begin{aligned} \frac{7}{\alpha^{n-m}} &> \exp\left(-C_1 \cdot (1+\log n) \cdot 2.2 \cdot 0.9 \cdot 2.1\right) \\ (n-m)\log\alpha - \log 7 &< C_1 \cdot (1+\log n) \cdot 2.2 \cdot 0.9 \cdot 2.1. \end{aligned}$$

Since $C_1 < 9.7 \cdot 10^{11}$ and $1+\log n < 2\log n$ for $n \geq 3$, we get

$$\begin{aligned} (n-m)\log\alpha - \log 7 &< 9.7 \cdot 10^{11} \cdot (1+\log n) \cdot 2.2 \cdot 0.9 \cdot 2.1 \\ (n-m)\log\alpha &< 8.2 \cdot 10^{12} \cdot \log n. \end{aligned} \tag{3.2}$$

To find an upper bound on $n-t$, let's rewrite the equation (1.1) as a second linear form in logarithms as follows

$$\frac{\alpha^n}{2\sqrt{2}} + \frac{\alpha^m}{2\sqrt{2}} - 3^a = \frac{\beta^n}{2\sqrt{2}} + \frac{\beta^m}{2\sqrt{2}} - P_t.$$

If we take the absolute value of both sides and apply the triangle inequality, we get

$$\begin{aligned} \left|\frac{\alpha^n}{2\sqrt{2}}\left(1+\alpha^{m-n}\right) - 3^a\right| &\leq \frac{|\beta|^n + |\beta|^m}{2\sqrt{2}} + P_t \\ &< \frac{1}{3} + \alpha^t. \end{aligned}$$

Dividing both sides by $\frac{\alpha^n}{2\sqrt{2}}\left(1+\alpha^{m-n}\right)$, we obtain

$$\begin{aligned} \left|1 - 3^a\alpha^{-n}2\sqrt{2}\left(1+\alpha^{m-n}\right)^{-1}\right| &< \frac{2\sqrt{2}}{\alpha^n \cdot \left(1+\alpha^{m-n}\right)}\left(\frac{1}{3} + \alpha^t\right) \\ &= \frac{2\sqrt{2}}{\left(1+\alpha^{m-n}\right)}\alpha^{t-n}\left(\frac{1}{3}\alpha^{-t} + 1\right) \\ &< \frac{8\sqrt{2}}{3 \cdot \left(1+\frac{1}{\alpha}\right)}\alpha^{t-n} \\ &< \frac{3}{\alpha^{n-t}}. \end{aligned} \tag{3.3}$$

Second application of the Matveev's result:

Set $\Delta_2 := 1 - 3^a\alpha^{-n}2\sqrt{2}\left(1+\alpha^{m-n}\right)^{-1}$. Here Δ_2 is different from zero. In fact, if Δ_2 is zero then

$$3^a 2\sqrt{2} = \alpha^n\left(1+\alpha^{m-n}\right) = \alpha^n + \alpha^m.$$

By conjugating in $\mathbb{Q}\left(\sqrt{2}\right)$ we get

$$-3^a 2\sqrt{2} = \beta^n + \beta^m.$$

So, considering last two equation we obtain

$$\begin{aligned} 2\cdot 3^a &= \frac{\alpha^n-\beta^n}{2\sqrt{2}}+\frac{\alpha^m-\beta^m}{2\sqrt{2}} \\ &= P_n+P_m. \end{aligned}$$

If the last equation is replaced in equation 1.1, $P_t=-3^a$ is derived and this is a contradiction. Now we are ready to apply the Theorem 2.1 for the second time.

We take $s:=3$, $(\gamma_1,\gamma_2,\gamma_3):=\left(3,\alpha,2\sqrt{2}\left(1+\alpha^{m-n}\right)^{-1}\right)$ and $(b_1,b_2,b_3):=(a,-n,1)$. We have $D:=2$ since each γ_i belongs to $\mathbb{Q}\left(\sqrt{2}\right)$.

A_1, A_2 and B can be chosen as follows:

$$\begin{aligned} A_1 &:= 2.2 > 2.1972 \simeq 2\cdot\log 3 = D\cdot h(\gamma_1) \\ A_2 &:= 0.9 > 0.8813 \simeq \log\alpha = D\cdot h(\gamma_2) \\ B &:= n. \end{aligned}$$

Now, let us compare the $h(\gamma_3)$ and $\log\gamma_3$ to find an appropriate value for A_3:

$$\begin{aligned} h(\gamma_3) &= h\left(\frac{2\sqrt{2}}{1+\alpha^{m-n}}\right) \\ &\le h\left(2\sqrt{2}\right)+h\left(1+\alpha^{m-n}\right) && \text{from Proposition 2.1(a)} \\ &\le \log\left(2\sqrt{2}\right)+h(1)+h\left(\alpha^{m-n}\right)+\log 2 && \text{from Proposition 2.1(b)} \\ &= \log\left(4\sqrt{2}\right)+|m-n|\cdot h(\alpha) && \text{from Proposition 2.1(c)} \\ &= \log\left(4\sqrt{2}\right)+(n-m)\frac{\log\alpha}{2}. \end{aligned}$$

Considering

$$\gamma_3 = 2\sqrt{2}\left(1+\alpha^{m-n}\right)^{-1} < \sqrt{2}$$

and

$$\gamma_3^{-1} = \frac{\left(1+\alpha^{m-n}\right)}{2\sqrt{2}} < \frac{2}{2\sqrt{2}} < \sqrt{2}$$

we get $|\log\gamma_3|<1$. Thus,

$$A_3 := 3.47+(n-m)\cdot\log\alpha > \log 32+(n-m)\cdot\log\alpha = \max\left\{2h(\gamma_3),|\log\gamma_3|,0.16\right\}.$$

Now Theorem 2.1 implies that

$$\begin{aligned} \frac{3}{\alpha^{n-t}} &> \left|1-3^a\alpha^{-n}2\sqrt{2}\left(1+\alpha^{m-n}\right)^{-1}\right| \\ &> \exp\left(-C_2\cdot(1+\log n)\cdot 2.2\cdot 0.9\cdot(3.47+(n-m)\log\alpha)\right) \end{aligned}$$

where $C_2 := 1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2) < 9.7\cdot 10^{11}$. Taking the logarithm of both sides in the last inequality, considering that $1+\log n<2\log n$ for $n\ge 3$ and using the inequality (3.2), one can see that

$$(n-t)\log\alpha < 3.2\cdot 10^{25}\cdot\log^2 n. \tag{3.4}$$

Third application of the Matveev's result for bounding n:

Now, we will make the equation (1.1) a little more explicit with the help of Binet formula and get a new linear form as follows:

$$\frac{\alpha^n}{2\sqrt{2}}+\frac{\alpha^m}{2\sqrt{2}}+\frac{\alpha^t}{2\sqrt{2}}-3^a=\frac{\beta^n}{2\sqrt{2}}+\frac{\beta^m}{2\sqrt{2}}+\frac{\beta^t}{2\sqrt{2}}.$$

By taking absolute value and using triangle inequality we get

$$\left|\frac{\alpha^n}{2\sqrt{2}}\left(1+\alpha^{m-n}+\alpha^{t-n}\right)-3^a\right|\leq\frac{|\beta|^n}{2\sqrt{2}}+\frac{|\beta|^m}{2\sqrt{2}}+\frac{|\beta|^t}{2\sqrt{2}}<\frac{1}{2}.$$

We obtain below inequality by a simple division with the first term of the left hand side of the above inequality.

$$\left|1-3^a\alpha^{-n}2\sqrt{2}\left(1+\alpha^{m-n}+\alpha^{t-n}\right)^{-1}\right|<\frac{1}{\alpha^n}. \tag{3.5}$$

Set $\Delta_3:=1-3^a\alpha^{-n}2\sqrt{2}\left(1+\alpha^{m-n}+\alpha^{t-n}\right)^{-1}$. First of all, it must be shown that Δ_3 is not zero. For this, assume that $\Delta_3=0$. So, we have

$$2\sqrt{2}\cdot 3^a=\alpha^n\left(1+\alpha^{m-n}+\alpha^{t-n}\right)=\alpha^n+\alpha^m+\alpha^t \tag{3.6}$$

and by conjugating this equation we get

$$-2\sqrt{2}\cdot 3^a=\beta^n+\beta^m+\beta^t. \tag{3.7}$$

Substracting the equation (3.7) from the equation (3.6) we obtain

$$2\cdot 3^a=\frac{\alpha^n-\beta^n}{2\sqrt{2}}+\frac{\alpha^m-\beta^m}{2\sqrt{2}}+\frac{\alpha^t-\beta^t}{2\sqrt{2}}=P_n+P_m+P_t$$

which contradicts to the equation (1.1). Like the previous ones, we are ready to implement the Matveev's result with $s=3$, $(\gamma_1,\gamma_2,\gamma_3):=\left(3,\alpha,2\sqrt{2}\left(1+\alpha^{m-n}+\alpha^{t-n}\right)^{-1}\right)$ and $(b_1,b_2,b_3):=(a,-n,1)$. We can take $A_1:=2.2$, $A_2:=0.9$ and $B:=n$. Now, let us again compare the $h(\gamma_3)$ and $\log\gamma_3$ to find the value of A_3:

$$\gamma_3=2\sqrt{2}\left(1+\alpha^{m-n}+\alpha^{t-n}\right)^{-1}<\sqrt{2}$$

and

$$\gamma_3^{-1}=\frac{\left(1+\alpha^{m-n}+\alpha^{t-n}\right)}{2\sqrt{2}}<\frac{1+\frac{1}{2}+\frac{1}{2}}{2\sqrt{2}}<\sqrt{2}.$$

So, these two inequalities imply that $|\log \gamma_3| < 1$. Let us find the value of $h(\gamma_3)$.

$$\begin{aligned} h(\gamma_3) &= h\left(2\sqrt{2}\left(1+\alpha^{m-n}+\alpha^{t-n}\right)^{-1}\right) \\ &\leq h\left(2\sqrt{2}\right)+h\left(\left(1+\alpha^{m-n}+\alpha^{t-n}\right)^{-1}\right) \\ &= h\left(2\sqrt{2}\right)+h\left(1+\alpha^{m-n}+\alpha^{t-n}\right) \\ &\leq h\left(2\sqrt{2}\right)+|m-n|\,h(\alpha)+|t-n|\,h(\alpha)+2\log 2 \\ &= \log 2\sqrt{2}+(n-m)\frac{\log\alpha}{2}+(n-t)\frac{\log\alpha}{2}+2\log 2 \\ &= \log 8\sqrt{2}+(n-m)\frac{\log\alpha}{2}+(n-t)\frac{\log\alpha}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} 2\cdot h(\gamma_3) &\leq \log 128+(n-m)\log\alpha+(n-t)\log\alpha \\ &< 5+(n-m)\log\alpha+(n-t)\log\alpha. \end{aligned}$$

Considering $|\log\gamma_3| < 1$ and $2\cdot h(\gamma_3) < 5+(n-m)\log\alpha+(n-t)\log\alpha$ we can take

$$A_3 := 5+(n-m)\log\alpha+(n-t)\log\alpha.$$

As a result, if we apply Matveev's result with the inequality (3.5) we find

$$n\log\alpha < 1.84\times 10^{12}\times\log n\times\left(5+8.2\times 10^{12}\times\log n+3.2\times 10^{25}\times\log^2 n\right)$$

and so

$$n < 6.9\times 10^{43}.$$

Now let us improve this upper bound on n a little bit more. Set

$$z_1 := a\log 3-n\log\alpha+\log\left(2\sqrt{2}\right).$$

The inequality (3.1) can be also written as

$$|1-e^{z_1}| < \frac{7}{\alpha^{n-m}}.$$

By using (1.1) and (2.2) we get

$$\frac{\alpha^n}{2\sqrt{2}} = P_n+\frac{\beta^n}{2\sqrt{2}} < P_n+1 \leq P_n+P_m+P_t = 3^a.$$

Therefore, we have

$$1 < 3^a\alpha^{-n}2\sqrt{2}$$

and so

$$z_1 > 0.$$

In this case, we get

$$0 < z_1 < e^{z_1}-1 < \frac{7}{\alpha^{n-m}}.$$

If we divide both sides of the above inequality by $\log\alpha$ and make some necessary operations we obtain

$$0 < (2a+1)\frac{\log 3}{\log\alpha} - 2n < \frac{16}{\alpha^{n-m}}. \tag{3.8}$$

To find an upper bound for $n-m$ we will use the above inequality which of type $|x\gamma - y|$. Recall that $a < n < 6.9\times 10^{43}$. So, this implies $2a+1 < 14\times 10^{43}$. If we take $\frac{p_k}{q_k}$ as kth convergent of γ, one can see with Mathematica that $q_{70} < 14\times 10^{43} < q_{71}$ and maximum of continued fractions of γ up to 71, say $x_{\max}$, is 181. We obtain

$$\frac{1}{(x_{\max}+2)(2a+1)} < (2a+1)\gamma - 2n < \frac{16}{\alpha^{n-m}}$$

from the known properties of continued fractions. Thus, above inequality yields

$$\alpha^{n-m} < 16\times 183\times 14\times 10^{43}$$

which means that

$$n-m < 125.$$

Let us find an upper bound for $n-t$ using the inequality (3.3). Set

$$z_2 := a\log 3 - n\log\alpha + \log\left(2\sqrt{2}\left(1+\alpha^{m-n}\right)^{-1}\right).$$

From (3.3) we have

$$|1-e^{z_2}| < \frac{3}{\alpha^{n-t}}.$$

The inequality

$$\frac{\alpha^n}{2\sqrt{2}} + \frac{\alpha^m}{2\sqrt{2}} < P_n + P_m + 1 \leq P_n + P_m + P_t = 3^a$$

means that

$$3^a\alpha^{-n}2\sqrt{2}\left(1+\alpha^{m-n}\right) > 1$$

and hence $z_2 > 0$. Therefore, the inequality

$$0 < z_2 < \frac{3}{\alpha^{n-t}}$$

is obtained. In this inequality, if we write z_2 in its open form and divide both sides with $\log\alpha$, the following inequality emerges immediately:

$$0 < a\frac{\log 3}{\log\alpha} - n + \frac{\log\left(2\sqrt{2}\left(1+\alpha^{m-n}\right)^{-1}\right)}{\log\alpha} < \frac{3.5}{\alpha^{n-t}}. \tag{3.9}$$

Now, we can operate Lemma 2.1 with the parameters

$$\gamma = \frac{\log 3}{\log\alpha}, \mu = \frac{\log\left(2\sqrt{2}\left(1+\alpha^{m-n}\right)^{-1}\right)}{\log\alpha}, A = 3.5, B = \alpha, w = n-t.$$

We can set $M := 6.9\times 10^{43}$and if we take the denominator of the 85th convergence of γ, then we get $q > 6M$ but in this case $\varepsilon < 0$ appears for many values of $n-m$. So, we take the denominator of the 86th convergence except for $n-m=2$. Performing the Lemma 2.1 for these values of $n-m$, we find that solutions of the equation (1.1) are available for

$n-t \in [1,120]$. Now, let us take the special case $n-m=2$. Since $\mu=1$, the inequality (3.9) turns into the following:

$$0 < a\frac{\log 3}{\log \alpha} - (n-1) < \frac{3.5}{\alpha^{n-t}}.$$

Using the properties of continued fractions, this inequality can be rewritten as

$$\frac{1}{(a_{\max}+2)\cdot a} < a\frac{\log 3}{\log \alpha} - (n-1) < \frac{3.5}{\alpha^{n-t}} \tag{3.10}$$

where $a_{\max}$ is the maximum value of the continued fractions of the γ up to 71. From the inequality (3.10), if we use $a < 6.9 \times 10^{43}$ then we get $n-t < 122$. Therefore, $n-t < 122$ is a valid inequality in all cases.

Now we can proceed to improve the upper bound we find for n. Set

$$z_3 := a\log 3 - n\log\alpha + \log\left(2\sqrt{2}\left(1+\alpha^{m-n}+\alpha^{t-n}\right)^{-1}\right).$$

In this case, we can write

$$|1-e^{z_3}| < \frac{1}{\alpha^n}$$

from the inequality (3.5). It can be shown easily that $z_3 \neq 0$ as in the previous cases. Now let us examine $z_3 < 0$ and $z_3 > 0$ cases separately. If $z_3 > 0$ then

$$0 < z_3 < e^{z_3} - 1 < \frac{1}{\alpha^n}.$$

Now suppose $z_3 < 0$. Using $|e^{z_3}-1| < 1/2$ we get $e^{|z_3|} < 2$. So,

$$0 < |z_3| \le e^{|z_3|} - 1 < e^{|z_3|}\left(e^{z_3}-1\right) < \frac{2}{\alpha^n}.$$

Considering both cases, the inequality

$$0 < |z_3| < \frac{2}{\alpha^n}$$

can be considered as a general case. Substituting the value of z_3 in the above inequality and dividing both sides by $\log\alpha$ we obtain

$$0 < \left| a\left(\frac{\log 3}{\log\alpha}\right) - n + \frac{\log\left(2\sqrt{2}\left(1+\alpha^{m-n}+\alpha^{t-n}\right)^{-1}\right)}{\log\alpha}\right| < \frac{3}{\alpha^n}.$$

If we take $M := 6.9\cdot 10^{43}$ and apply Lemma 2.1to the above inequality for all values of $n-m$ and $n-t$ in the ranges $[0,124]$and $[0,121]$ respectively, we always get $\varepsilon > 0$ in the 90th convergence of $\frac{\log 3}{\log\alpha}$. So, from the result of Lemma 2.1 it is obtained that if there is a solution of the equation (1.1) then it must be $n < 128$ which contradicts our assumption $n > 200$. This completes our proof. □

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