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ON NEW WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR CO-ORDINATED s-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we obtain some new weighted Ostrowski type inequalities for co-ordinated s-convex functions. Furthermore we present some weighted Midpoint type inequalities as special cases of main results. We also show that our results generalize the results obtained earlier studies.

1. INTRODUCTION

In the history of development calculus, integral inequalities has been thought of as a key factor in the theory of differential and integral equations. The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of scientists, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of A. M. Ostrowski [26] is the following classical integral inequality associated with the differentiable mappings:

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, the inequality holds:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Over the years, many variations of Ostrowski type inequalities have been studied for various function classes, such as convex functions, bounded functions, functions of bounded variation, and so on. Specifically, since convexity theory is an effective and powerful way to solve a large number of problems from different branches of pure and applied mathematics, many papers have been dedicated to Ostrowski inequality for convex functions. For instance,

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Alomari et al. proved some Ostrowski type inequalities for s-convex functions in [5]. Moreover, Ostrowski type inequalities for other kinds of convexities were studied in [17,25,29,40]. In [39], Set first obtained the Riemann-Liouville fractional version of Ostrowski inequality for s-convex functions. In addition to this, many researchers focused on establishing Ostrowski type inequalities for certain fractional integral operators, such as k-Riemann-Liouville fractional integrals, local fractional integrals, Raina fractional integrals, etc. For more information and unexplained subjects, we refer the reader to [1,9,11,14–16,23,37] and the references therein. On the other hand, several Ostrowski inequalities for co-ordinated convex mapping in involving double Riemann integrals and double Riemann-Liouville fractional integrals are introduced in the papers [22] and [19], respectively.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.1. A function $f : \Delta := [a, b] \times [c, d] \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x, u), (y, v) \in \Delta$ and $t, \lambda \in [0, 1]$, if it satisfies the following inequality:

$$f(tx + (1 - t)\ y, su + (1 - \lambda)\ v) \tag{1.2}$$
$$\leq \quad t\lambda\ f(x, u) + t(1 - \lambda)f(x, v) + \lambda(1 - t)f(y, u) + (1 - t)(1 - \lambda)f(y, v).$$

The mapping f is a co-ordinated concave on Δ if the inequality (1.2) holds in reversed direction for all $t, \lambda \in [0, 1]$ and $(x, u), (y, v) \in \Delta$.

Barnet and Dragomir gave the following Ostrowski type inequalities for double integrals in [7].

Theorem 1.1. *Let $f : \Delta := [a, b] \times [c, d] \to \mathbb{R}$ be continuous on Δ, $\frac{\partial^2 f}{\partial x \partial y}$ exists on $(a, b) \times (c, d)$ and is bounded, i. e.,*

$$\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)} \left| \frac{\partial^2 f(x, y)}{\partial x \partial y} \right| < \infty.$$

Then, we have the inequality:

$$\left| \int_a^b \int_c^d f(t, \lambda) d\lambda dt - (b - a)\,(d - c)\, f(x, y) - \left[(b - a) \int_c^d f(x, \lambda) d\lambda + (d - c) \int_a^b f(t, y) dt \right] \right|$$
$$\leq \left[\frac{1}{4}(b - a)^2 + \left(x - \frac{a + b}{2} \right)^2 \right] \left[\frac{1}{4}(d - c)^2 + \left(y - \frac{c + d}{2} \right)^2 \right] \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty \tag{1.3}$$

for all $(x, y) \in \Delta$.

In [10], Dragomir proved the following inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1.2. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex, then we have the following inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \qquad (1.4)\\ &\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx\right.\\ &+\left.\frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy\right]\\ &\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp. The inequalities in (1.4) hold in reverse direction if the mapping f is a co-ordinated concave mapping.

Over the years, many papers are dedicated on the generalizations and new versions of the inequalities (1.3) and (1.4) using the different type convex functions. For more inequalities obtained by using co-ordinated convex functions, please refer to ([2–6, 12–22, 27, 28, 30–43]).

We will frequently use the following lemma in our main results:

Lemma 1.1. [8, 44]*Let $w:\Delta:=[a,b]\times[c,d]\to[0,\infty)$ be an integrable function on Δ and let $f:\Delta\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order $\frac{\partial^2 f(t,\lambda)}{\partial t\partial\lambda}$ exist for al $(t,\lambda)\in\Delta$. Then we have the equality*

$$\Theta(a,b,c,d;f,w) \qquad (1.5)$$

$$\begin{aligned} = \;& \frac{(x-a)(y-c)}{m(a,b;c,d)}\int_0^1\int_0^1\left[\int_a^{U_1(t)}\int_c^{V_1(\lambda)} w(u,v)dvdu\right]\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_1(t),V_1(\lambda)\right)d\lambda dt\\ &+\frac{(x-a)(d-y)}{m(a,b;c,d)}\int_0^1\int_0^1\left[\int_a^{U_1(t)}\int_d^{V_2(\lambda)} w(u,v)dvdu\right]\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_1(t),V_2(\lambda)\right)d\lambda dt\\ &+\frac{(b-x)(y-c)}{m(a,b;c,d)}\int_0^1\int_0^1\left[\int_b^{U_2(t)}\int_c^{V_1(\lambda)} w(u,v)dvdu\right]\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_2(t),V_1(\lambda)\right)d\lambda dt\\ &+\frac{(b-x)(d-y)}{m(a,b;c,d)}\int_0^1\int_0^1\left[\int_b^{U_2(t)}\int_d^{V_2(\lambda)} w(u,v)dvdu\right]\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_2(t),V_2(\lambda)\right)d\lambda dt \end{aligned}$$

where $U_1(t) = tx+(1-t)a$, $U_2(t) = tx+(1-t)b$, $V_1(\lambda) = \lambda y+(1-\lambda)c$, $V_2(\lambda) = \lambda y+(1-\lambda)d$,

$$m(a,b;c,d) = \int_a^b \int_c^d w(u,v)dvdu$$

and

$$\begin{aligned} & \Theta(a,b,c,d;f,w) \qquad (1.6) \\ = \;& f(x,y) - \frac{1}{m(a,b;c,d)} \int_a^b \int_c^d w(u,v)f(u,y)dvdu \\ & -\frac{1}{m(a,b;c,d)} \int_a^b \int_c^d w(u,v)f(x,v)dvdu \\ & +\frac{1}{m(a,b;c,d)} \int_a^b \int_c^d w(u,v)f(u,v)dvdu. \end{aligned}$$

This paper aims to establish some weighed generalizations of Ostrowski type integral inequalities for co-ordinated s-convex functions. The results presented in this paper provide extensions of those given in [23].

2. Some New Weighted Ostrowski Type Inequalities

In this section, we present some weighted Ostrowski type inequalities for co-ordinated s-convex functions.

Theorem 2.1. *Suppose that the mapping w is as in Lemma 1.1. Moreover, let w is bounded on Δ, i.e. $\|w\|_\infty := \sup_{(x,y)\in\Delta} |w(x,y)|$. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|$ is a co-ordinated s-convex function on Δ, then for all $(x,y) \in \Delta$ we have the following weighted Ostrowski inequality*

$$\begin{aligned} & |\Theta(a,b,c,d;f,w)| \qquad (2.1) \\ \leq \;& \frac{\|w\|_\infty}{m(a,b;c,d)} \frac{1}{(s_1+2)(s_2+2)} \\ & \times \left\{ (x-a)^2 (y-c)^2 \left[\left|\frac{\partial^2 f}{\partial t \partial \lambda}(x,y)\right| + \frac{1}{s_2+1} \left|\frac{\partial^2 f}{\partial t \partial \lambda}(x,c)\right| \right.\right. \\ & \left. + \frac{1}{s_1+1} \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,y)\right| + \frac{1}{(s_1+1)(s_2+1)} \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right| \right] \\ & + (x-a)^2 (d-y)^2 \left[\left|\frac{\partial^2 f}{\partial t \partial \lambda}(x,y)\right| + \frac{1}{s_2+1} \left|\frac{\partial^2 f}{\partial t \partial \lambda}(x,d)\right| \right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{s_1+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|+\frac{1}{(s_1+1)(s_2+1)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\Bigg]\\
&+(b-x)^2(y-c)^2\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|+\frac{1}{s_2+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|\right.\\
&\left.+\frac{1}{s_1+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|+\frac{1}{(s_1+1)(s_2+1)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|\right]\\
&+(b-x)^2(d-y)^2\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|+\frac{1}{s_2+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|\right.\\
&\left.+\frac{1}{s_1+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|+\frac{1}{(s_1+1)(s_2+1)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right]\Bigg\}
\end{aligned}
$$

where the mapping Θ *is defined as in (1.6).*

Proof. By taking the modulus of inequality (1.5), we have

$$
\begin{aligned}
&|\Theta(a,b,c,d;f,w)| \qquad (2.2)\\
=\;&\frac{(x-a)(y-c)}{m(a,b;c,d)}\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_c^{V_1(\lambda)}w(u,v)dvdu\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_1(\lambda))\right|d\lambda dt\\
&+\frac{(x-a)(d-y)}{m(a,b;c,d)}\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_d^{V_2(\lambda)}w(u,v)dvdu\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_2(\lambda))\right|d\lambda dt\\
&+\frac{(b-x)(y-c)}{m(a,b;c,d)}\int_0^1\int_0^1\left|\int_b^{U_2(t)}\int_c^{V_1(\lambda)}w(u,v)dvdu\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_2(t),V_1(\lambda))\right|d\lambda dt\\
&+\frac{(b-x)(d-y)}{m(a,b;c,d)}\int_0^1\int_0^1\left|\int_b^{U_2(t)}\int_d^{V_2(\lambda)}w(u,v)dvdu\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_2(t),V_2(\lambda))\right|d\lambda dt.
\end{aligned}
$$

Since $w(x,y)$ is bounded on Δ and $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$ is co-ordinated s-convex on Δ, we obtain

$$
\begin{aligned}
&\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_c^{V_1(\lambda)}w(u,v)dvdu\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_1(\lambda))\right|d\lambda dt \qquad (2.3)\\
\le\;&\|w\|_\infty\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_c^{V_1(\lambda)}dvdu\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_1(\lambda))\right|d\lambda dt
\end{aligned}
$$

$$\leq (x-a)(y-c)\|w\|_\infty \int_0^1\int_0^1 t\lambda\left[t^{s_1}\lambda^{s_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| + t^{s_1}(1-\lambda)^{s_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|\right.$$
$$\left. +(1-t)^{s_1}\lambda^{s_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right| + (1-t)^{s_1}(1-\lambda)^{s_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\right] d\lambda dt$$
$$= (x-a)(y-c)\|w\|_\infty \left[\frac{1}{(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| + \frac{1}{(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|\right.$$
$$\left. +\frac{1}{(s_1+1)(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right| + \frac{1}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\right].$$

Similarly, we have

$$\int_0^1\int_0^1 \left|\int_a^{U_1(t)}\int_d^{V_2(\lambda)} w(u,v)dvdu\right| \left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_2(\lambda))\right| d\lambda dt \tag{2.4}$$
$$\leq (x-a)(d-y)\|w\|_\infty \left[\frac{1}{(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| + \frac{1}{(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|\right.$$
$$\left. +\frac{1}{(s_1+1)(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right| + \frac{1}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right],$$

$$\int_0^1\int_0^1 \left|\int_b^{U_2(t)}\int_c^{V_1(\lambda)} w(u,v)dvdu\right| \left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_2(t),V_1(\lambda))\right| d\lambda dt \tag{2.5}$$
$$\leq (b-x)(y-c)\|w\|_\infty \left[\frac{1}{(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| + \frac{1}{(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|\right.$$
$$\left. +\frac{1}{(s_1+1)(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right| + \frac{1}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|\right]$$

and

$$\int_0^1\int_0^1 \left|\int_b^{U_2(t)}\int_d^{V_2(\lambda)} w(u,v)dvdu\right| \left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_2(t),V_2(\lambda))\right| d\lambda dt \tag{2.6}$$
$$\leq (b-x)(d-y)\|w\|_\infty \left[\frac{1}{(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| + \frac{1}{(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|\right.$$
$$\left. +\frac{1}{(s_1+1)(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right| + \frac{1}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right].$$

If we substitute the inequalities (2.3)-(2.6) in (2.2), then we obtain the required result (2.1). This completes the proof. □

Remark 2.1. If we choose $s_1 = s_2 = 1$ in Theorem 2.1 , then we obtain the inequality

$$\begin{aligned}
&|\Theta(a,b,c,d;f,w)| \qquad (2.7)\\
\leq\ & \frac{\|w\|_\infty}{36 \times m(a,b;c,d)}\\
&\times\left\{(x-a)^2(y-c)^2\left[4\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| + 2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right| + 2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right| + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\right]\right.\\
&+(x-a)^2(d-y)^2\left[4\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| + 2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right| + 2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right| + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right]\\
&+(b-x)^2(y-c)^2\left[4\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| + 2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right| + 2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right| + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|\right]\\
&\left.+(b-x)^2(d-y)^2\left[4\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| + 2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right| + 2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right| + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right]\right\}
\end{aligned}$$

which is given by Budak in [8].

Corollary 2.1. *Under assumption of Theorem 2.1 with* $\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right| \leq M$, $(x,y)\in\Delta$, *then we have the following weighted Ostorowski type inequality*

$$|\Theta(a,b,c,d;f,w)| \leq \frac{4M(b-a)^2(d-c)^2\|w\|_\infty}{m(a,b;c,d)(s_1+1)(s_2+1)}\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\left[\frac{1}{4}+\frac{\left(y-\frac{c+d}{2}\right)^2}{(d-c)^2}\right].$$

Remark 2.2. If we choose $w(x,y) = 1$ and $s_1 = s_2 = s$ in Corollary 2.1, then Corollary 2.1 reduces to inequality proved by Latif and Dragomir in [23, Theorem 2.2].

Corollary 2.2. *Under assumption of Theorem 2.1 with* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$, *then we have following weighted midpoint type inequality*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right) + \frac{1}{m(a,b;c,d)}\int_a^b\int_c^d w(u,v)f(u,v)dvdu\right.\\
&\left.-\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d w(u,v)f\left(u,\frac{c+d}{2}\right)dvdu - \frac{1}{m(a,b;c,d)}\int_a^b\int_c^d w(u,v)f\left(\frac{a+b}{2},v\right)dvdu\right|\\
\leq\ & \frac{(b-a)^2(d-c)^2}{16(s_1+2)(s_2+2)}\frac{\|w\|_\infty}{m(a,b;c,d)}\left[2^{1-s_2}+\frac{1}{s_2+1}\right]\left[2^{1-s_1}+\frac{1}{s_1+1}\right]\\
&\times\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right| + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right| + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right].
\end{aligned}$$

Proof. By choosing $x = \frac{a+b}{2}$, $y = \frac{c+d}{2}$ and by using co-ordinated s-convexity of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$, then one can obtain the desired result. □

Remark 2.3. If we choose $w(x,y)=1$ and $s_1=s_2=1$ in Corollary 2.2, then Corollary 2.2 reduces to inequality proved by Latif and Dragomir in [24, Theorem 2.2].

Theorem 2.2. *Let w be as in Theorem 2.1. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is a co-ordinated s-convex function on Δ, then for all $(x,y)\in\Delta$ we have the following weighted Ostrowski inequality*

$$\begin{aligned}
&|\Theta(a,b,c,d;f,w)| \qquad (2.8)\\
\leq\ & \frac{\|w\|_\infty}{m(a,b;c,d)(p+1)^{\frac{2}{p}}}\left(\frac{4}{(s_1+1)(s_2+1)}\right)^{\frac{1}{q}}\\
&\times\left\{(x-a)^2(y-c)^2\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right]^{\frac{1}{q}}\right.\\
&+(x-a)^2(d-y)^2\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right]^{\frac{1}{q}}\\
&+(b-x)^2(y-c)^2\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right]^{\frac{1}{q}}\\
&\left.+(b-x)^2(d-y)^2\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

where the mapping Θ is defined as in (1.6) and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the well known Hölder inequality in (2.2), we obtain

$$\begin{aligned}
&|\Theta(a,b,c,d;f,w)| \qquad (2.9)\\
\leq\ & \frac{(x-a)(y-c)}{m(a,b;c,d)}\left(\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_c^{V_1(\lambda)}w(u,v)dvdu\right|^p d\lambda dt\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_1(\lambda))\right|^q d\lambda dt\right)^{\frac{1}{q}}\\
&+\frac{(x-a)(d-y)}{m(a,b;c,d)}\left(\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_d^{V_2(\lambda)}w(u,v)dvdu\right|^p d\lambda dt\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_2(\lambda))\right|^q d\lambda dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)(y-c)}{m(a,b;c,d)}\left(\int_0^1\int_0^1\left|\int_b^{U_2(t)}\int_c^{V_1(\lambda)}w(u,v)dvdu\right|^p d\lambda dt\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_2(t),V_1(\lambda))\right|^q d\lambda dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)(d-y)}{m(a,b;c,d)}\left(\int_0^1\int_0^1\left|\int_b^{U_2(t)}\int_d^{V_2(\lambda)}w(u,v)dvdu\right|^p d\lambda dt\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_2(t),V_2(\lambda))\right|^q d\lambda dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since w is bounded on Δ, we have

$$\begin{aligned}\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_c^{V_1(\lambda)} w(u,v)dvdu\right|^p d\lambda dt &\leq \|w\|_\infty^p \int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_c^{V_1(\lambda)} dvdu\right|^p d\lambda dt \\ &= \|w\|_\infty^p (x-a)^p (y-c)^p \int_0^1\int_0^1 \lambda^p t^p d\lambda dt \\ &= \frac{(x-a)^p (y-c)^p}{(p+1)^2}\|w\|_\infty^p .\end{aligned} \tag{2.10}$$

Similarly, we get

$$\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_d^{V_2(\lambda)} w(u,v)dvdu\right|^p d\lambda dt \leq \frac{(x-a)^p (d-y)^p}{(p+1)^2}\|w\|_\infty^p , \tag{2.11}$$

$$\int_0^1\int_0^1\left|\int_b^{U_2(t)}\int_c^{V_1(\lambda)} w(u,v)dvdu\right|^p d\lambda dt \leq \frac{(b-x)^p (y-c)^p}{(p+1)^2}\|w\|_\infty^p \tag{2.12}$$

and

$$\int_0^1\int_0^1\left|\int_b^{U_2(t)}\int_d^{V_2(\lambda)} w(u,v)dvdu\right|^p d\lambda dt \leq \frac{(b-x)^p (d-y)^p}{(p+1)^2}\|w\|_\infty^p . \tag{2.13}$$

On the other hand, as $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is a co-ordinated s-convex function on Δ, we obtain

$$\begin{aligned}&\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_1(\lambda))\right|^q d\lambda dt \qquad (2.14)\\ &\leq \frac{1}{(s_1+1)(s_2+1)}\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right],\end{aligned}$$

$$\begin{aligned}&\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_2(\lambda))\right|^q d\lambda dt \qquad (2.15)\\ &\leq \frac{1}{(s_1+1)(s_2+1)}\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right],\end{aligned}$$

$$\begin{aligned}&\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_2(t),V_1(\lambda))\right|^q d\lambda dt \qquad (2.16)\\ &\leq \frac{1}{(s_1+1)(s_2+1)}\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right],\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_2(t),V_2(\lambda)\right)\right|^q d\lambda dt \qquad (2.17)\\
\leq\ & \frac{1}{(s_1+1)(s_2+1)}\left[\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right].
\end{aligned}$$

If we substitute the inequalities (2.10)-(2.17) in (2.9), then we obtain the required inequality (2.8). □

Remark 2.4. If we choose $s_1=s_2=1$ in Theorem 2.1 , then we have the inequality

$$\begin{aligned}
&|\Theta(a,b,c,d;f,w)|\\
\leq\ & \frac{\|w\|_\infty}{2^{\frac{2}{q}}m(a,b;c,d)(p+1)^{\frac{2}{p}}}\\
&\times\Bigg\{(x-a)^2(y-c)^2\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right)^{\frac{1}{q}}\\
&+(x-a)^2(d-y)^2\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right)^{\frac{1}{q}}\\
&+(b-x)^2(y-c)^2\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)^{\frac{1}{q}}\\
&+(b-x)^2(d-y)^2\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}\Bigg\}
\end{aligned}$$

which is given by Budak in [8].

Corollary 2.3. *Under assumption of Theorem 2.2 with* $\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|\leq M$, $(x,y)\in\Delta$, *then we have the following weighted Ostrowski type inequality*

$$\begin{aligned}
|\Theta(a,b,c,d;f,w)| \leq\ & \frac{4M(b-a)^2(d-c)^2}{m(a,b;c,d)(p+1)^{\frac{2}{p}}}\left(\frac{4}{(s_1+1)(s_2+1)}\right)^{\frac{1}{q}}\\
&\times\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\left[\frac{1}{4}+\frac{\left(y-\frac{c+d}{2}\right)^2}{(d-c)^2}\right]\|w\|_\infty .
\end{aligned}$$

Remark 2.5. If we choose $w(x,y)=1$ and $s_1=s_2=s$ in Corollary 2.3, then Corollary 2.3 reduces to inequality proved by Latif and Dragomir in [23, Theorem 2.3].

Corollary 2.4. *Under assumption of Theorem 2.2 with $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, then we have the following weighted midpoint type inequality*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{1}{m(a,b;c,d)} \int_a^b \int_c^d w(u,v) f(u,v) dv du \right. \\
&\left. - \frac{1}{m(a,b;c,d)} \int_a^b \int_c^d w(u,v) f\left(u, \frac{c+d}{2}\right) dv du - \frac{1}{m(a,b;c,d)} \int_a^b \int_c^d w(u,v) f\left(\frac{a+b}{2}, v\right) dv du \right| \\
\leq\; & \frac{\|w\|_\infty (b-a)^2 (d-c)^2}{16 m(a,b;c,d)(p+1)^{\frac{2}{p}}} \left(\frac{1}{(s_1+1)(s_2+1)}\right)^{\frac{1}{q}} \\
&\times \left\{ \left[\left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, c\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(a, \frac{c+d}{2}\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q \right]^{\frac{1}{q}} \right. \\
&+ \left[\left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, d\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(a, \frac{c+d}{2}\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q \right]^{\frac{1}{q}} \\
&+ \left[\left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, c\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(b, \frac{c+d}{2}\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q \right]^{\frac{1}{q}} \\
&\left. + \left[\left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, d\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(b, \frac{c+d}{2}\right)\right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Theorem 2.3. *Let w be as in Theorem 2.1. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$, $q \geq 1$, is a co-ordinated s-convex function on Δ, then for all $(x,y) \in \Delta$ we have the following weighted Ostrowski inequality*

$$\begin{aligned}
&|\Theta(a,b,c,d;f,w)| \\
\leq\; & \frac{\|w\|_\infty}{2^{2-\frac{2}{q}} \times m(a,b;c,d)} \left(\frac{1}{(s_1+2)(s_2+2)}\right)^{\frac{1}{q}} \\
&\times \left\{ (x-a)^2 (y-c)^2 \left(\left|\frac{\partial^2 f}{\partial t \partial \lambda}(x,y)\right|^q + \frac{1}{s_2+1}\left|\frac{\partial^2 f}{\partial t \partial \lambda}(x,c)\right|^q \right. \right. \\
&\left. + \frac{1}{s_1+1}\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,y)\right|^q + \frac{1}{(s_1+1)(s_2+1)}\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q \right)^{\frac{1}{q}} \\
&+ (x-a)^2 (d-y)^2 \left(\left|\frac{\partial^2 f}{\partial t \partial \lambda}(x,y)\right|^q + \frac{1}{s_2+1}\left|\frac{\partial^2 f}{\partial t \partial \lambda}(x,d)\right|^q \right. \\
&\left. + \frac{1}{s_1+1}\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,y)\right|^q + \frac{1}{(s_1+1)(s_2+1)}\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q \right)^{\frac{1}{q}}
\end{aligned}$$

$$+(b-x)^2(y-c)^2\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\frac{1}{s_2+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q\right.$$
$$\left.+\frac{1}{s_1+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\frac{1}{(s_1+1)(s_2+1)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)^{\frac{1}{q}}$$
$$+(b-x)^2(d-y)^2\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\frac{1}{s_2+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q\right.$$
$$\left.\left.+\frac{1}{s_1+1}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\frac{1}{(s_1+1)(s_2+1)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}\right\}$$

where the mapping Θ *is defined as in* (1.6).

Proof. By utilizing the power mean inequality in (2.2), we obtain

$$|\Theta(a,b,c,d;f,w)| \tag{2.18}$$
$$\leq \frac{(x-a)(y-c)}{m(a,b;c,d)}\left(\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_c^{V_1(\lambda)} w(u,v)dvdu\right|d\lambda dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_c^{V_1(\lambda)} w(u,v)dvdu\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_1(\lambda))\right|^q d\lambda dt\right)^{\frac{1}{q}}$$
$$+\frac{(x-a)(d-y)}{m(a,b;c,d)}\left(\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_d^{V_2(\lambda)} w(u,v)dvdu\right|d\lambda dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\left|\int_a^{U_1(t)}\int_d^{V_2(\lambda)} w(u,v)dvdu\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_1(t),V_2(\lambda))\right|^q d\lambda dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)(y-c)}{m(a,b;c,d)}\left(\int_0^1\int_0^1\left|\int_b^{U_2(t)}\int_c^{V_1(\lambda)} w(u,v)dvdu\right|d\lambda dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\left|\int_b^{U_2(t)}\int_c^{V_1(\lambda)} w(u,v)dvdu\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_2(t),V_1(\lambda))\right|^q d\lambda dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)(d-y)}{m(a,b;c,d)}\left(\int_0^1\int_0^1\left|\int_b^{U_2(t)}\int_d^{V_2(\lambda)} w(u,v)dvdu\right|d\lambda dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1\left|\int_b^{U_2(t)}\int_d^{V_2(\lambda)}w(u,v)dvdu\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_2(t),V_2(\lambda)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(x-a)^2(y-c)^2\|w\|_\infty}{4^{1-\frac{1}{q}}\times m(a,b;c,d)}\left(\int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_1(t),V_1(\lambda)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$
$$+\frac{(x-a)^2(d-y)^2\|w\|_\infty}{4^{1-\frac{1}{q}}\times m(a,b;c,d)}\left(\int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_1(t),V_2(\lambda)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2(y-c)^2\|w\|_\infty}{4^{1-\frac{1}{q}}\times m(a,b;c,d)}\left(\int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_2(t),V_1(\lambda)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2(d-y)^2\|w\|_\infty}{4^{1-\frac{1}{q}}\times m(a,b;c,d)}\left(\int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_2(t),V_2(\lambda)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is a co-ordinated s-convex function on Δ, we have the following inequalities

$$\left(\int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_1(t),V_1(\lambda)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}} \tag{2.19}$$
$$\leq \left[\frac{1}{(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\frac{1}{(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q\right.$$
$$\left.+\frac{1}{(s_1+1)(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\frac{1}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right]^{\frac{1}{q}},$$

$$\left(\int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_1(t),V_2(\lambda)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}} \tag{2.20}$$
$$\leq \left[\frac{1}{(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\frac{1}{(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q\right.$$
$$\left.+\frac{1}{(s_1+1)(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\frac{1}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right]^{\frac{1}{q}},$$

$$\left(\int_0^1\int_0^1 t^{s_1}\lambda^{s_2}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(U_2(t),V_1(\lambda)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}} \tag{2.21}$$
$$\leq \left[\frac{1}{(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\frac{1}{(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q\right.$$
$$\left.+\frac{1}{(s_1+1)(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\frac{1}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right]^{\frac{1}{q}},$$

and

$$\left(\int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}(U_2(t),V_2(\lambda))\right|^q d\lambda dt\right)^{\frac{1}{q}} \tag{2.22}$$
$$\leq \left[\frac{1}{(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+\frac{1}{(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q\right.$$
$$\left.+\frac{1}{(s_1+1)(s_1+2)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\frac{1}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right]^{\frac{1}{q}}.$$

□

Remark 2.6. If we choose $s_1=s_2=1$ in Theorem 2.1 , then we have the inequality

$$|\Theta(a,b,c,d;f,w)|$$
$$\leq \frac{\|w\|_\infty}{4\cdot 3^{\frac{2}{q}}\cdot m(a,b;c,d)}$$
$$\left\{(x-a)^2(y-c)^2\left(4\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q+2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+(x-a)^2(d-y)^2\left(4\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q+2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right)^{\frac{1}{q}}$$
$$+(b-x)^2(y-c)^2\left(4\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,c)\right|^q+2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)^{\frac{1}{q}}$$
$$\left.+(b-x)^2(d-y)^2\left(4\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|^q+2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,d)\right|^q+2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,y)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{\frac{1}{q}}\right\}$$

which is given by Budak in [8].

Corollary 2.5. *Under assumption of Theorem 2.3 with* $\left|\frac{\partial^2 f}{\partial t\partial\lambda}(x,y)\right|\leq M$, $(x,y)\in\Delta$, *then we have the following weighted Ostrowski type inequality*

$$|\Theta(a,b,c,d;f,w)| \leq \frac{M(b-a)^2(d-c)^2}{m(a,b;c,d)}\left(\frac{4}{(s_1+1)(s_2+1)}\right)^{\frac{1}{q}}$$
$$\times\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\left[\frac{1}{4}+\frac{\left(y-\frac{c+d}{2}\right)^2}{(d-c)^2}\right]\|w\|_\infty .$$

Remark 2.7. If we choose $w(x,y)=1$ and $s_1=s_2=s$ in Corollary 2.5, then Corollary 2.5 reduces to inequality proved by Latif and Dragomir in [23, Theorem 2.4].

Corollary 2.6. *Under assumption of Theorem 2.3 with* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$, *then we have the following weighted midpoint type inequality*

$$\begin{aligned}
&\Bigg| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{1}{m(a,b;c,d)} \int_a^b \int_c^d w(u,v) f(u,v)\,dvdu \\
&- \frac{1}{m(a,b;c,d)} \int_a^b \int_c^d w(u,v) f\left(u, \frac{c+d}{2}\right) dvdu - \frac{1}{m(a,b;c,d)} \int_a^b \int_c^d w(u,v) f\left(\frac{a+b}{2}, v\right) dvdu \Bigg| \\
\leq\ & \frac{(b-a)^2 (d-c)^2 \|w\|_\infty}{4^{1-\frac{1}{q}} \times m(a,b;c,d)} \left(\frac{1}{(s_1+2)(s_2+2)}\right)^{\frac{1}{q}} \\
&\times \Bigg\{ \left(\left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right|^q + \frac{1}{s_2+1} \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, c\right) \right|^q \right. \\
&\left. + \frac{1}{s_1+1} \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(a, \frac{c+d}{2}\right) \right|^q + \frac{1}{(s_1+1)(s_2+1)} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a,c) \right|^q \right)^{\frac{1}{q}} \\
&+ \left(\left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right|^q + \frac{1}{s_2+1} \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, d\right) \right|^q \right. \\
&\left. + \frac{1}{s_1+1} \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(a, \frac{c+d}{2}\right) \right|^q + \frac{1}{(s_1+1)(s_2+1)} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a,d) \right|^q \right)^{\frac{1}{q}} \\
&+ \left(\left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right|^q + \frac{1}{s_2+1} \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, c\right) \right|^q \right. \\
&\left. + \frac{1}{s_1+1} \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(b, \frac{c+d}{2}\right) \right|^q + \frac{1}{(s_1+1)(s_2+1)} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b,c) \right|^q \right)^{\frac{1}{q}} \\
&+ \left(\left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right|^q + \frac{1}{s_2+1} \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(\frac{a+b}{2}, d\right) \right|^q \right. \\
&\left. + \frac{1}{s_1+1} \left| \frac{\partial^2 f}{\partial t \partial \lambda}\left(b, \frac{c+d}{2}\right) \right|^q + \frac{1}{(s_1+1)(s_2+1)} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b,d) \right|^q \right)^{\frac{1}{q}} \Bigg\}.
\end{aligned}$$

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