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ON INEQUALITIES FOR $\alpha(x)$-CONVEX FUNCTIONS

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ABSTRACT. In this paper we give inequalities for $\alpha(x)$-convex function. We obtain Slater's inequality and refinement of Jensen's inequality for $\alpha(x)$-convex function. We establish mean value theorems and construct generalized Cauchy type means. Also we give improvement and reversion of Slater's inequality for $\alpha(x)$-convex functions.

1. INTRODUCTION

First, let us recall the definition of convex function.

Definition 1.1. Let I be an interval in $\mathbb{R}$. A function $\psi : I \to \mathbb{R}$ is called convex if

$$\psi(\lambda x + (1-\lambda)y) \leq \lambda\psi(x) + (1-\lambda)\psi(y) \tag{1.1}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. It is called strictly convex if the inequality (1.1) holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$. If $-\psi$ is convex (respectively, strictly convex), we say that ψ is concave (respectively, strictly concave). If ψ is both convex and concave, ψ is said to be affine.

We give the well- known Jensen's inequality for convex function:

Theorem 1.1. *Let $\psi : I \to \mathbb{R}$ be a convex function on interval $I \subseteq \mathbb{R}$ and p_i be non negative real numbers and $x_i \in I$ $(i = 1, 2, \ldots, n)$, while $P_n = \sum_{i=1}^{n} p_i > 0$. Then following inequality holds*

$$\psi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi(x_i). \tag{1.2}$$

If ψ is strictly convex then inequality (1.2) is strict unless $x_1 = x_2 = \cdots = x_n$.

The following converse of Jensen's inequality has been proved by Dragomir and Goh in [3].

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Theorem 1.2. *Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable convex function defined on interval I. If $x_i \in I, i = 1, 2, .., n (n \geq 2)$ are arbitrary members and $p_i \geq 0$ $(i = 1, 2, .., n)$ with $P_n = \sum_{i=1}^{n} p_i > 0$ and let*

$$\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \qquad \overline{y} = \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi(x_i).$$

Then the inequalities

$$0 \leq \overline{y} - \psi(\overline{x}) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi'(x_i)(x_i - \overline{x}) \tag{1.3}$$

hold.

In the case when ψ is strictly convex, we have equalities in (1.3) if and only if there is some $c \in I$ such that $x_i = c$ holds for all i with $p_i > 0$.

In [8] Pečarić gave general Slater's inequality:

Theorem 1.3 ([8])**.** *Suppose that $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex function on interval I, for $x_1, x_2, .., x_n \in I$ and for $p_1, p_2, .., p_n \geq 0$ with $P_n = \sum_{i=1}^{n} p_i > 0$. Let*

$$\sum_{i=1}^{n} p_i \psi'_+(x_i) \neq 0, \quad \frac{\sum_{i=1}^{n} p_i \psi'_+(x_i) x_i}{\sum_{i=1}^{n} p_i \psi'_+(x_i)} \in I,$$

then the following Slater's inequality holds

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \psi(x_i) \leq \psi\left(\frac{\sum_{i=1}^{n} p_i \psi'_+(x_i) x_i}{\sum_{i=1}^{n} p_i \psi'_+(x_i)}\right). \tag{1.4}$$

When ψ is strictly convex on I, inequality (1.4) becomes equality if and only if $x_i = c$ for some $c \in I$ and for all i with $p_i > 0$.

Now we quote some definitions and state some basic properties of $\alpha(x)$-convex functions established in [4].

Definition 1.2 ([4, Definition 2.1])**.** Let ψ, α be real functions defined on interval $I \subseteq \mathbb{R}$ such that ψ is differentiable and $\alpha\psi'$ integrable. Function ψ is called $\alpha(x)$-convex on interval I if for every $x, y \in I$

$$(y - x)(\psi'(y) - \psi'(x)) \geq (y - x)\int_x^y \alpha(t)\psi'(t)dt \tag{1.5}$$

holds. Function ψ is called $\alpha(x)$-concave if the inequality in (1.5) is reversed.

Notice that for $\alpha(x) = 0$, ψ is convex. $\alpha(x)$-convexity criteria given in the following theorems.

Theorem 1.4 ([4, Theorem 2.1])**.** *If ψ'' is a continuous function and $\alpha\psi'$ an integrable function on interval I, ψ is $\alpha(x)$-convex on interval I if and only if $\psi''(x) - \alpha(x)\psi'(x) \geq 0$.*

Theorem 1.5 ([4, Theorem 2.2])**.** *A function ψ is $\alpha(x)$-convex on interval I if and only if*

$$\psi(y) - \psi(x) - \psi'(x)(y - x) \geq \int_x^y (y - t)\alpha(t)\psi'(t)dt \tag{1.6}$$

for all $x, y \in I$.

Generalized Jensen's inequality for $\alpha(x)$-convex function is given in the following theorem.

Theorem 1.6 ([4, Theorem 2.3])**.** *Let $\psi : I \to \mathbb{R}$ be $\alpha(x)$-convex function, $x_i \in I$ and $p_i \in [0,1], i = 1,\dots,n$ such that $\sum_{i=1}^{n} p_i = 1$ and let $\overline{x} = \sum_{i=1}^{n} p_i x_i$. Then the inequality*

$$\sum_{i=1}^{n} p_i \psi(x_i) - \psi(\overline{x}) \geq \sum_{i=1}^{n} p_i \int_{\overline{x}}^{x_i} (x_i - t)\alpha(t)\psi'(t)dt \tag{1.7}$$

holds.

For more recent results related to convex functions and its application we recommend [9–18].

In this paper we give some general inequality for $\alpha(x)$-convex function which implies generalized Slater's inequality and refinement of Jensen's inequality. We prove mean value theorems and construct Cauchy type means. We give exponential convexity and log convexity for the parametric family associated with the general inequality. By using some log convexity criteria we establish improvement and reversion of Slater's inequality. At the end we give some determinantal inequalities which give us improvement and reversion of Slater's inequality.

2. Generalization of Matić-Pečarić inequality

The first theorem that we prove here is the more general inequality for $\alpha(x)$-convex function which is in fact the generalization of the inequality given in [6].

Theorem 2.1. *Let $\psi : I \to \mathbb{R}$ be $\alpha(x)$-convex function, $x_i \in I$ and nonnegative real numbers p_i such that $P_n := \sum_{i=1}^{n} p_i > 0$ and let $\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$, $\overline{y} = \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi(x_i)$. If $d \in I$ is arbitrarily chosen number, then we have*

$$\psi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi'(x_i)(x_i - d) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi(x_i) \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \int_{x_i}^{d} (d-t)\alpha(t)\psi'(t)dt \tag{2.1}$$

Proof. From (1.6) we have

$$\psi(y) - \psi(x) - \psi'(x)(y-x) \geq \int_{x}^{y} (y-t)\alpha(t)\psi'(t)dt$$

by replacing $y \to d$ and $x \to x_i$ we get

$$\psi(d) - \psi(x_i) - \psi'(x_i)(d - x_i) \geq \int_{x_i}^{d} (d-t)\alpha(t)\psi'(t)dt.$$

Multiplying both hand side by $\frac{p_i}{P_n}$ and summing over i we have

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \psi(d) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi'(x_i)(d - x_i) \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \int_{x_i}^{d} (d-t)\alpha(t)\psi'(t)dt$$

$$\psi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi'(x_i)(x_i - d) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi(x_i) \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \int_{x_i}^{d} (d-t)\alpha(t)\psi'(t)dt$$

$$\psi(d)P_n + \sum_{i=1}^{n} p_i \psi'(x_i)(x_i - d) - \sum_{i=1}^{n} p_i \psi(x_i) \geq \sum_{i=1}^{n} p_i \int_{x_i}^{d} (d-t)\alpha(t)\psi'(t)dt.$$

□

Integral version of the Theorem 2.1 can be stated as:

Theorem 2.2. *Let $\psi : I \to \mathbb{R}$ be $\alpha(x)$-convex function and $f : [a,b] \to I$, be a function such that $\psi(f)$, $\psi'(f)$ are integrable functions on I. Let $p : [a,b] \to \mathbb{R}$ be non negative integrable functions such that $\int_a^b p(x)dx > 0$, then for any $d \in I$, we have*

$$\psi(d) + \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\psi'(f(x))(f(x)-d)dx - \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\psi(f(x))dx \geq \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\int_{f(x)}^{d}(d-t)\alpha(t)\psi'(t)dtdx. \tag{2.2}$$

The following simple consequence of Theorem 2.1 is the refinement of Jensen's inequality for $\alpha(x)$-convex function.

Corollary 2.1. *Under the assumptions of Theorem 2.1 we have*

$$\begin{aligned}&\frac{1}{P_n}\sum_{i=1}^{n} p_i \int_{\overline{x}}^{x_i}(x_i-t)\alpha(t)\psi'(t)dt \leq \overline{y} - \psi(\overline{x})\\ &\leq \frac{1}{P_n}\sum_{i=1}^{n} p_i\psi'(x_i)(x_i-\overline{x}) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\int_{x_i}^{\overline{x}}(t-\overline{x})\alpha(t)\psi'(t)dt.\end{aligned} \tag{2.3}$$

Remark 2.1. If we put $p_i \in [0,1], P_n = 1$ in the first inequality in (2.3), then we deduce (1.7).

Integral version of the Corollary 2.1 can be stated as:

Corollary 2.2. *Let $\psi : I \to \mathbb{R}$ be $\alpha(x)$-convex function and $f : [a,b] \to I$, $\psi(f)$, $\psi'(f)$ are integrable functions on I. Let $p : [a,b] \to \mathbb{R}$ be non negative integrable function such that $\int_a^b p(x)dx > 0$ and let $\bar{f} = \frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}$, then we have*

$$\begin{aligned}&\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\int_{\bar{f}}^{f(x)}(f(x)-t)\alpha(t)\psi'(t)dtdx \leq \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\psi(f(x))dx - \psi(\bar{f})\\ &\leq \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\psi'(f(x))(f(x)-\bar{f})dx + \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\int_{f(x)}^{\bar{f}}(t-\bar{f})\alpha(t)\psi'(t)dtdx.\end{aligned} \tag{2.4}$$

Corollary 2.3. *Let $\psi : I \to \mathbb{R}$ be $\alpha(x)$-convex function, $x, y \in I$ and real numbers $p, q \in [0,1]$ such that $p + q = 1$ and let $\overline{x} = px + qy$. Then the inequality*

$$\begin{aligned}&p\int_{\overline{x}}^{x}(x-t)\alpha(t)\psi'(t)dt + q\int_{\overline{x}}^{y}(y-t)\alpha(t)\psi'(t)dt \leq p\psi(x) + q\psi(y) - \psi(px+qy)\\ &\leq p\psi'(x)(x-\overline{x}) + q\psi'(y)(y-\overline{x}) + p\int_{x}^{\overline{x}}(t-\overline{x})\alpha(t)\psi'(t)dt + q\int_{y}^{\overline{x}}(t-\overline{x})\alpha(t)\psi'(t)dt.\end{aligned} \tag{2.5}$$

Proof. Apply (2.5) with $n = 2, x_1 = x, x_2 = y, p_1 = p \in [0,1]$ and $p_2 = q \in [0,1]$. □

Remark 2.2. The first inequality in (2.5) has been given in [4].

Corollary 2.4. *Let $\psi : I \to \mathbb{R}$ be $\alpha(x)$-convex function, $x_1, x_2, x_3 \in I, x_1 \leq x_2 \leq x_3$. Then the inequality*

$$\begin{aligned}&(x_2 - x_1)\int_{x_2}^{x_3}(x_3 - t)\alpha(t)\psi'(t)dt - (x_3 - x_2)\int_{x_1}^{x_2}(x_1 - t)\alpha(t)\psi'(t)dt\\&\leq (x_3 - x_2)\psi(x_1) + (x_1 - x_3)\psi(x_2) + (x_2 - x_1)\psi(x_3)\\&\leq (x_2 - x_1)(x_3 - x_2)(\psi'(x_3) - \psi'(x_1))\\&\quad + (x_3 - x_2)\int_{x_1}^{x_2}(t - x_2)\alpha(t)\psi'(t)dt - (x_2 - x_1)\int_{x_2}^{x_3}(t - x_2)\alpha(t)\psi'(t)dt. \qquad (2.6)\end{aligned}$$

Proof. Use (2.5) for $x = x_1$, $px + qy = x_2$, $y = x_3, p + q = 1$. □

Remark 2.3. The first inequality in (2.6) has been given in [4].

The following generalization of Slater's inequality for $\alpha(x)$-convex function is valid:

Corollary 2.5. *Let $\psi : I \to \mathbb{R}$ be $\alpha(x)$-convex function, $x_i \in I$ and nonnegative real numbers p_i such that $P_n := \sum_{i=1}^{n} p_i > 0$ and let $\bar{y} = \frac{1}{P_n}\sum_{i=1}^{n} p_i\psi(x_i)$. If*

$$\sum_{i=1}^{n} p_i\psi'(x_i) \neq 0,\ \ \bar{\bar{x}} := \frac{\sum_{i=1}^{n} p_i\psi'(x_i)x_i}{\sum_{i=1}^{n} p_i\psi'(x_i)} \in I,$$

then

$$\bar{y} \leq \psi(\bar{\bar{x}}) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\int_{x_i}^{\bar{\bar{x}}}(t - \bar{\bar{x}})\alpha(t)\psi'(t)dt. \qquad (2.7)$$

Proof. Put $d = \bar{\bar{x}}$ in the inequality (2.1) we get (2.7). □

Integral version of the Corollary 2.5 can be stated as:

Theorem 2.3. *Let $\psi : I \to \mathbb{R}$ be $\alpha(x)$-convex function and $f : [a, b] \to I$, $\psi \circ f$, $\psi'(f)$ are integrable functions on I and $p : [a, b] \to \mathbb{R}$ be non negative integrable functions such that $\int_a^b p(x)dx > 0$ and let $\bar{f} = \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\psi(f(x))dx$. If*

$$\int_a^b p(x)\psi'(f(x))dx \neq 0,\ \ \bar{\bar{f}} := \frac{\int_a^b p(x)\psi'(f(x))f(x)dx}{\int_a^b p(x)\psi'(f(x))dx} \in I,$$

then

$$\bar{f} \leq \psi(\bar{\bar{f}}) + \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\int_{f(x)}^{\bar{\bar{f}}} p(x)(t - \bar{\bar{f}})\alpha(t)\psi'(t)dtdx. \qquad (2.8)$$

Now we are in the position to give mean value theorem for the generalized inequalities. In the proof of mean value theorems we will use the following Lemma.

Lemma 2.1 ([4])**.** *Let I be an open interval. Let α be an integrable function and $s \in C^2(I)$ be such that $s'' - \alpha s'$ is bounded by integrable functions M and m, that is $m(x) \leq s''(x) - \alpha s'(x) \leq M(x)$, for every $x \in I$. Then the functions ψ_1, ψ_2 are defined by*

$$\psi_1(x) = R_1(x) - s(x),$$
$$\psi_2(x) = s(x) - R_2(x)$$

where

$$R_1(x)=\int\left(e^{\int\alpha(x)dx}\int M(x)e^{-\int\alpha(x)dx}dx\right)dx,$$

$$R_2(x)=\int\left(e^{\int\alpha(x)dx}\int m(x)e^{-\int\alpha(x)dx}dx\right)dx$$

are $\alpha(x)$-convex.

Theorem 2.4. *Let α, $s^{''}$ be continuous and g be the positive and continuous functions on compact interval $I\subseteq\mathbb{R}$. Let $x_i, d\in I$ and p_i be the non negative real numbers such that $P_n=\sum_{i=1}^{n}p_i>0$. Then there exists $\eta\in I$ such that*

$$\begin{aligned}&s(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)s^{'}(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_is(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s^{'}(t)dt\\&=\frac{s^{''}(\eta)-\alpha(\eta)s^{'}(\eta)}{g(\eta)}\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)g(t)dt\right).\end{aligned}\tag{2.9}$$

Proof. As $\frac{s^{''}(x)-\alpha(x)s^{'}(x)}{g(x)}$ is continuous on compact interval I, therefore there exists

$$m=\min_{x\in I}\left(\frac{s^{''}(x)-\alpha(x)s^{'}(x)}{g(x)}\right)\text{ and }M=\max_{x\in I}\left(\frac{s^{''}(x)-\alpha(x)s^{'}(x)}{g(x)}\right).\tag{2.10}$$

Using (2.10) and Lemma 2.1, the functions ψ_1,ψ_2 defined by

$$\psi_1=R_1(x)-s(x),\psi_2=s(x)-R_2(x)$$

where

$$R_1(x)=\int\left(e^{\int\alpha(x)dx}\int Mg(x)e^{-\int\alpha(x)dx}dx\right)dx$$

$$R_2(x)=\int\left(e^{\int\alpha(x)dx}\int mg(x)e^{-\int\alpha(x)dx}dx\right)dx$$

are $\alpha(x)-convex$.

By applying (2.1) on functions ψ_1, we get the following inequality

$$\psi_1(d)P_n+\sum_{i=1}^{n}p_i\psi_1^{'}(x_i)(x_i-d)-\sum_{i=1}^{n}p_i\psi_1(x_i)\geq\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)\psi_1^{'}(t)dt$$

$$\begin{aligned}&\Rightarrow(R_1(d)-s(d))P_n+\sum_{i=1}^{n}p_i(R_1^{'}(x_i)-s^{'}(x_i))(x_i-d)-\sum_{i=1}^{n}p_i(R_1(x_i)-s(x_i))(x_i)\\&-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)(R_1^{'}(t)-s^{'}(t))dt\geq 0\end{aligned}$$

$$\begin{aligned}&\Rightarrow R_1(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)R_1^{'}(x_i)-\sum_{i=1}^{n}p_iR_1(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)R_1^{'}(t)dt\geq\\&s(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)s^{'}(x_i)-\sum_{i=1}^{n}p_is(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s^{'}(t)dt\end{aligned}$$

$$\Rightarrow M\left(R_3(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)R_3^{'}(x_i)-\sum_{i=1}^{n}p_iR_3(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)R_3^{'}(t)dt\right)$$
$$\geq s(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)s^{'}(x_i)-\sum_{i=1}^{n}p_is(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s^{'}(t)dt, \tag{2.11}$$

where

$$R_3(x)=\int\left(e^{\int\alpha(x)dx}\int g(x)e^{-\int\alpha(x)dx}dx\right)dx.$$

Similarly applying (2.1) on function ψ_2, we get the following inequality

$$s(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)s^{'}(x_i)-\sum_{i=1}^{n}p_is(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s^{'}(t)dt\geq$$
$$m\left(R_3(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)R_3^{'}(x_i)-\sum_{i=1}^{n}p_iR_3(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)R_3^{'}(t)dt\right) \tag{2.12}$$

from (2.11) and (2.12) it follows

$$m\left(R_3(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)R_3^{'}(x_i)-\sum_{i=1}^{n}p_iR_3(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)R_3^{'}(t)dt\right)$$
$$\leq s(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)s^{'}(x_i)-\sum_{i=1}^{n}p_is(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s^{'}(t)dt\leq \tag{2.13}$$
$$M\left(R_3(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)R_3^{'}(x_i)-\sum_{i=1}^{n}p_iR_3(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)R_3^{'}(t)dt\right)$$

If $R_3(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)R_3^{'}(x_i)-\sum_{i=1}^{n}p_iR_3(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)R_3^{'}(t)dt=0$

Then as

$$R_3(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)R_3^{'}(x_i)-\sum_{i=1}^{n}p_iR_3(x_i)-$$
$$\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)(R_3^{''}(t)-g(t))dt=\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)g(t)dt. \tag{2.14}$$

Therefore from (2.13) we have

$$s(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)s^{'}(x_i)-\sum_{i=1}^{n}p_is(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s^{'}(t)dt=0.$$

So in this case (2.16) holds for any $\eta\in I$.

If $R_3(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)R_3^{'}(x_i)-\sum_{i=1}^{n}p_iR_3(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)R_3^{'}(t)dt>0.$

Then from (2.13) we have

$$\begin{aligned}
&m \leq \\
&\frac{s(d)P_n + \sum_{i=1}^{n} p_i(x_i - d)s'(x_i) - \sum_{i=1}^{n} p_i s(x_i) - \sum_{i=1}^{n} p_i \int_{x_i}^{d}(d-t)\alpha(t)s'(t)dt}{\left(R_3(d)P_n + \sum_{i=1}^{n} p_i(x_i - d)R_3'(x_i) - \sum_{i=1}^{n} p_i R_3(x_i) - \sum_{i=1}^{n} p_i \int_{x_i}^{d}(d-t)\alpha(t)R_3'(t)dt\right)} \\
&\leq M
\end{aligned} \tag{2.15}$$

As $\frac{s''(x)-\alpha(x)s'(x)}{g(x)}$ is continuous on I, therefore by using (2.10) and intermediate value theorem we can find $\eta \in I$ such that

$$\begin{aligned}
&s(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - d)s'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i s(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \int_{x_i}^{d}(d-t)\alpha(t)s'(t)dt \\
&= \frac{s''(\eta) - \alpha(\eta)s'(\eta)}{g(\eta)}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i \int_{x_i}^{d}(d-t)g(t)dt\right].
\end{aligned}$$

□

We can give mean value theorem for the refinement of Jensen's inequality:

Corollary 2.6. *Let α, s'' be continuous and g be the positive and continuous functions on compact interval $I \subseteq \mathbb{R}$. Let $x_i \in I$ and p_i be the non negative real numbers such that $P_n = \sum_{i=1}^{n} p_i > 0$ and let $\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$. Then there exists $\eta \in I$ such that*

$$\begin{aligned}
&s(\overline{x}) - \frac{1}{P_n}\sum_{i=1}^{n} p_i s(x_i) + \frac{1}{P_n}\sum_{i=1}^{n} p_i(x_i - \overline{x})s'(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \int_{x_i}^{d}(\overline{x}-t)\alpha(t)s'(t)dt \\
&= \frac{s''(\eta) - \alpha(\eta)s'(\eta)}{g(\eta)}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \int_{x_i}^{\overline{x}}(\overline{x}-t)g(t)dt\right).
\end{aligned} \tag{2.16}$$

Integral version of the Theorem 2.4 can be stated as:

Theorem 2.5. *Let α, s'' be continuous and g be a positive and continuous function on $[a,b]$. Let $p : [a,b] \to \mathbb{R}$ be non negative function with $\int_a^b p(x)dx > 0$, $f : [a,b] \to \mathbb{R}$ be a function with $Imf \subseteq [a,b]$ and let $f, s(f), s'(f)$ be integrable functions and $d \in [a,b]$. Then there exists $\eta \in [a,b]$ such that*

$$\begin{aligned}
&s(d) + \frac{\int_a^b p(x)s'(f(x))(f(x)-d)dx}{\int_a^b p(x)dx} - \frac{\int_a^b p(x)s(f(x))dx}{\int_a^b p(x)dx} - \frac{\int_a^b p(x)\int_{f(x)}^{d}(d-t)\alpha(t)s'(t)dtdx}{\int_a^b p(x)dx} \\
&= \frac{s''(\eta) - \alpha(\eta)s'(\eta)}{g(\eta)}\left(\frac{\int_a^b p(x)\int_{f(x)}^{d}(d-t)f(t)dtdx}{\int_a^b p(x)dx}\right).
\end{aligned} \tag{2.17}$$

Theorem 2.6. *Let I be a compact interval in $\mathbb{R}$. Let $x_i \in I$ and p_i be the non negative real number such that $P_n = \sum_{i=1}^{n} p_i > 0$. Let $s_1, s_2 \in C^2(I)$ and α be the continuous function* *such that*

$$s_2(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)s_2^{'}(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_is_2(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s_2^{'}(t)dt\neq 0, \quad (2.18)$$

then there exists $\eta \in I$ *such that*

$$\begin{aligned}&\frac{s_1^{''}(\eta)-\alpha(\eta)s_1^{'}(\eta)}{s_2^{''}(\eta)-\alpha(\eta)s_2^{'}(\eta)}\\ =&\frac{s_1(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)s_1^{'}(x_i)-\sum_{i=1}^{n}p_is_1(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s_1^{'}(t)dt}{s_2(d)P_n+\sum_{i=1}^{n}p_i(x_i-d)s_2^{'}(x_i)-\sum_{i=1}^{n}p_is_2(x_i)-\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s_2^{'}(t)dt}\end{aligned} \quad (2.19)$$

Proof. Let

$$c_1=s_2(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)s_2^{'}(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_is_2(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s_2^{'}(t)dt$$

$$c_2=s_1(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)s_1^{'}(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_is_1(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s_1^{'}(t)dt$$

Now apply (2.16) for the function $c_1h_1-c_2h_2$ we have

$$\begin{aligned}&c_1\left[s_1(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)s_1^{'}(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_is_1(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s_1^{'}(t)dt\right]-\\ &c_2\left[s_2(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)s_2^{'}(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_is_2(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)\alpha(t)s_2^{'}(t)dt\right]\end{aligned} \quad (2.20)$$

$$=\frac{c_1s_1^{''}(\eta)-c_2s_2^{''}(\eta)-a(\eta)(c_1s_1^{'}(\eta)-c_2s_2^{'}(\eta))}{g(\eta)}\left[\frac{1}{P_n}\sum_{i=1}^{n}p_i\int_{x_i}^{d}(d-t)g(t)dt\right]$$

It is easy to see that the left-hand side of (2.20) is equal to 0, so the right-hand side should also be equal to 0. From (2.22) we get that the right-hand side in (2.16) is not equal to 0, so the part in square brackets on the right-hand of (2.20) is not equal to 0. For the right hand side in (2.20) to be equal to zero it follows that $c_1s_1^{''}(\eta)-c_2s_2^{''}(\eta)-a(\eta)(c_1s_1^{'}(\eta)-c_2s_2^{'}(\eta))=0$. After some calculation, it is easy to say that (2.23) follows from $c_1(s_1''(\eta)-\alpha(\eta)s_1'(\eta))-c_2(s_2''(\eta)-\alpha(\eta)s_2'(\eta))=0$, so the proof is complete. □

Integral version of the Theorem 2.6 can be stated as:

Theorem 2.7. *Let* $s_1,s_2\in C^2([a,b])$, α *be a continuous function on* $[a,b]$, $p:[a,b]\to\mathbb{R}$ *be non negative integrable function with* $\int_a^b p(x)dx>0$. *Let* $f:[a,b]\to\mathbb{R}$ *be a function such that* $Imf\subseteq[a,b]$ *and* $f,s_1(f),s_1'(f),s_2(f),s_2'(f)$ *are integrable functions and let* $d\in[a,b]$ *and*

$$\begin{aligned}s_2(d)+\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)s_2'(f(x))(f(x)-d)dx-\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)s_2(f(x))dx\\ -\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\int_{f(x)}^{d}(d-t)\alpha(t)s_2'(t)dtdx\neq 0.\end{aligned}$$

Then there exists $\eta \in [a,b]$ *such that*

$$\frac{s_1''(\eta)-a(\eta)s_1'(\eta)}{s_2''(\eta)-a(\eta)s_2'(\eta)}=$$

$$\frac{s_1(d)+\frac{\int_b^a p(x)s_1'(f(x))(f(x)-d)dx}{\int_b^a p(x)dx}-\frac{\int_b^a p(x)s_1(f(x))dx}{\int_b^a p(x)dx}-\frac{1}{\int_b^a p(x)dx}\int_b^a p(x)\int_{f(x)}^d (d-t)\alpha(x)s_1'(f(x))dtdx}{s_2(d)+\frac{\int_b^a p(x)s_2'(f(x))(f(x)-d)dx}{\int_b^a p(x)dx}-\frac{\int_b^a p(x)s_2(f(x))dx}{\int_b^a p(x)dx}-\frac{1}{\int_b^a p(x)dx}\int_b^a p(x)\int_{f(x)}^d (d-t)\alpha(x)s_2'(f(x))dtdx}. \tag{2.21}$$

Corollary 2.7. *Let I be a compact interval in $\mathbb{R}$. Let $x_i \in I$ and p_i be the non negative real number such that $P_n=\sum_{i=1}^n p_i>0$ and let $\overline{x}=\frac{1}{P_n}\sum_{i=1}^n p_i x_i$. Let $s_1,s_2\in C^2(I)$ and α be the continuous function on I such that*

$$s_2(\overline{x})+\frac{1}{P_n}\sum_{i=1}^n p_i(x_i-\overline{x})s_2'(x_i)-\frac{1}{P_n}\sum_{i=1}^n p_i s_2(x_i)-\frac{1}{P_n}\sum_{i=1}^n p_i\int_{x_i}^{\overline{x}}(\overline{x}-t)\alpha(t)s_2'(t)dt\neq 0, \tag{2.22}$$

then there exists $\eta\in I$ *such that*

$$\begin{aligned}&\frac{s_1''(\eta)-\alpha(\eta)s_1'(\eta)}{s_2''(\eta)-\alpha(\eta)s_2'(\eta)}\\ =&\frac{s_1(\overline{x})P_n+\sum_{i=1}^n p_i(x_i-\overline{x})s_1'(x_i)-\sum_{i=1}^n p_i s_1(x_i)-\sum_{i=1}^n p_i\int_{x_i}^{\overline{x}}(\overline{x}-t)\alpha(t)s_1'(t)dt}{s_2(\overline{x})P_n+\sum_{i=1}^n p_i(x_i-\overline{x})s_2'(x_i)-\sum_{i=1}^n p_i s_2(x_i)-\sum_{i=1}^n p_i\int_{x_i}^{\overline{x}}(\overline{x}-t)\alpha(t)s_2'(t)dt}.\end{aligned} \tag{2.23}$$

3. Improvement and reversion of generalized Slater's inequality

Definition 3.1 ([7])**.** A function $\psi: I\to\mathbb{R}$ is convex if

$$\psi(s_1)(s_3-s_2)+\psi(s_2)(s_1-s_3)+\psi(s_3)(s_2-s_1)\geq 0 \tag{3.1}$$

holds for every $s_1<s_2<s_3, s_1,s_2,s_3\in I$.

Definition 3.2 ([5])**.** A function $\psi: I\longrightarrow\mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{k,j=1}^n l_k l_j \psi(x_j+x_k)\geq 0$$

for all $n\in\mathbb{N}$, $l_k\in\mathbb{R}$ and $x_k\in I$, $k=1,\ldots,n$ such that $(x_j+x_k)\in I, 1\leq j,k\leq n$, or equivalently

$$\sum_{k,j=1}^n l_k l_j \psi\left(\frac{x_j+x_k}{2}\right)\geq 0.$$

Lemma 3.1 ([5])**.** *Let $\psi:(a,b)\to\mathbb{R}$. The following statements are equivalent:*

(i) *ψ is exponentially convex,*

(ii) *ψ is continuous and*

$$\sum_{j,k=1}^n l_j l_k \psi\left(\frac{x_j+x_k}{2}\right)\geq 0$$

for every $n\in\mathbb{N}$, $l_j\in\mathbb{R}$ and every $x_j\in(a,b)$, $1\leq j\leq n$.

Corollary 3.1 ([5]). *If ψ is exponentially convex function, then*

$$det\left[\psi\left(\frac{x_j+x_k}{2}\right)\right]_{k,j=1}^{n}\geq 0$$

for every $n\in\mathbb{N}$ $x_j\in I$, $k=1,\ldots,n$.

Corollary 3.2 ([5]). *If $\psi:I\longrightarrow(0,\infty)$ is exponentially convex function, then ψ is a log-convex function that is*

$$\psi(\lambda x+(1-\lambda)y)\leq\psi^{\lambda}(x)\psi^{1-\lambda}(y),$$

for all $x,y\in I$, $\lambda\in[0,1]$.

Lemma 3.2. *Let $p\in\mathbb{R}$. Then function ψ_p defined by*

$$\psi_p(x)=\int\left(e^{\int\alpha(x)dx}\int x^{p-2}e^{-\int\alpha(x)dx}dx\right)dx \tag{3.2}$$

is $\alpha(x)$-convex function for $x>0$.

Proof. Since $\psi_p''(x)-\alpha(x)\psi_p'(x)=x^{p-2}\geq 0$, $x>0$, therefore $\psi_p(x)$ is $\alpha(x)-convex$ function for $x>0$. □

For p $\in\mathbb{R}$, let the function $\Gamma(p)$ is defined as follows:

$$\Gamma(p)=\begin{cases}\frac{d^p}{p(p-1)}+\frac{1}{P_n}\sum_{i=1}^n p_i(x_i-d)\frac{x_i^{p-1}}{p-1}-\frac{1}{P_n}\sum_{i=1}^n p_i\frac{x_i^p}{p(p-1)}, & p\neq 0,1;\\ -\log d-1+\frac{d}{P_n}\sum_{i=1}^n\frac{p_i}{x_i}+\frac{1}{P_n}\sum_{i=1}^n p_i(\log x_i), & p=0;\\ d\log d+\frac{1}{P_n}\sum_{i=1}^n p_ix_i-\frac{d}{P_n}\sum_{i=1}^n p_i(1+\log x_i), & p=1.\end{cases} \tag{3.3}$$

where p_i be non negative real number and $P_n=\sum_{i=1}^n p_i>0$ and $\Gamma(p)>0$ for all $p\in\mathbb{R}$.

Lemma 3.3. *Let $p\in\mathbb{R}$, let the function ψ_p be defined by (3.2) for mutually different numbers $x_i>0$, $i=1,\ldots,n$ and let the function $\Gamma(p)$ be defined above. Then*

$$\Gamma(p)=\psi_p(d)+\frac{1}{P_n}\sum_{i=1}^n p_i\psi_p^{'}(x_i)(x_i-d)-\frac{1}{P_n}\sum_{i=1}^n p_i\psi_p(x_i)-\frac{1}{P_n}\sum_{i=1}^n p_i\int_{x_i}^{d}(d-t)\alpha(t)\psi_p'(t)dt \tag{3.4}$$

holds where, $i=1,\ldots,n$, and $P_n=\sum_{i=1}^n p_i>0$.

Proof. Since $\psi_p^{''}(x)-\alpha(x)\psi_p^{'}(x)=x^{p-2}$, we have $\alpha(x)\psi_p^{'}(x)=\psi_p^{''}(x)-x^{p-2}$, so

$$\int_{x_i}^{d}(d-t)\alpha(t)\psi_p^{'}(t)dt=\int_{x_i}^{d}(d-t)(\psi_p^{'}(t)+t^{p-2})dt$$

Hence

$$\psi_p(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)\psi_p'(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_i\psi_p(x_i)-\frac{1}{P_n}\int_{x_i}^{d}(d-t)\alpha(t)\psi_p'(t)dt$$

$$=\psi_p(d)+\frac{1}{P_n}\sum_{i=1}^{n}p_i(x_i-d)\psi_p'(x_i)-\frac{1}{P_n}\sum_{i=1}^{n}p_i\psi_p(x_i)$$

$$-\frac{1}{P_n}\sum_{i=1}^{n}p_i\begin{cases}(x_i-d)\psi_p'(x_i)+\psi_p(d)-\psi_p(x_i)-d\frac{d^{p-1}-x_i^{p-1}}{p-1}+\frac{d^p-x_i^p}{p}, & p\neq 0,1;\\ (x_i-d)\psi_p'(x_i)+\psi_p(d)-\psi_p(x_i)+d\left(\frac{1}{d}-\frac{1}{x_i}\right)+\ln d-\ln x_i, & p=0;\\ (x_i-d)\psi_p'(x_i)+\psi_p(d)-\psi_p(x_i)-d(\ln d-\ln x_i)+d-x_i & p=1;\end{cases}$$

$$=\Gamma(p).$$

□

Theorem 3.1. *Let $p\in\mathbb{R}$, let the function $\Gamma(p)$ be defined by (3.2) for mutually different numbers $x_i>0$, $i=1,\dots,n$. Let p_i be the non negative real number such that $P_n=\sum_{i=1}^{n}p_i>0$. Then*

(i) *the function $p\mapsto\Gamma(p)$ is continuous on $\mathbb{R}$,*

(ii) *for every $n\in\mathbb{N}$ and $\zeta_j\in\mathbb{R}$, $k=1,\dots,n$, the matrix $[\Gamma(\zeta_j+\zeta_k)/2)_{j,k=1}^{n}]$ is a positive semi definite matrix. Particularly*

$$det\left[\Gamma\left(\frac{\zeta_j+\zeta_k}{2}\right)\right]_{j,k=1}^{n}\geq 0$$

(iii) *the function $p\mapsto\Gamma(p)$ is exponentially convex,*

(iv) *if $\Gamma(p)>0$, then the function $p\mapsto\Gamma(p)$ is log convex, i.e for $-\infty<r<s<p<\infty$, we have*

$$(\Gamma(s))^{p-r}\leq(\Gamma(r))^{p-s}(\Gamma(p))^{s-r}. \tag{3.5}$$

Proof. (i) In order to prove that the function $p\mapsto\Gamma(p)$ is continuous on $\mathbb{R}$, we need to verify that $\lim_{p\to 0}\Gamma(p)=\Gamma(0)$ and $\lim_{p\to 1}\Gamma(p)=\Gamma(1)$. Both are obtained by simple a calculation. Hence, $\Gamma(p)$ is continues on $\mathbb{R}$.

(ii) Let $n\in\mathbb{N}$, $l_j\in\mathbb{R}$, $\zeta_j\in\mathbb{R}$ $j=1,2,\dots,n$. Denote $\zeta_{jk}=(\zeta_j+\zeta_k)/2$. Let ψ_p be defined by (3.2). Consider the function $y:\mathbb{R}^+\to\mathbb{R}$,

$$y(x)=\sum_{j,k=1}^{n}l_jl_k\psi_{\zeta_{jk}}(x).$$

Then

$$y''(x)-\alpha(x)y'(x)=\sum_{j,k=1}^{n}l_jl_k\psi''_{\zeta_{jk}}(x)-\alpha(x)\sum_{j,k=1}^{n}l_jl_k\psi'_{\zeta_{jk}}(x)$$

$$= \sum_{j,k=1}^{n} l_j l_k (\psi''_{\zeta_{jk}}(x) - \alpha(x)\psi'_{\zeta_{jk}}(x))$$

$$= \sum_{j,k=1}^{n} l_j l_k x^{\zeta_{jk}-2}$$

$$= \left(\sum_{j=1}^{n} l_j x^{(\zeta_j-2)/2}\right)^2 \geq 0$$

Hence, $y(x)$ is $\alpha(x)$-convex function.
Now we apply (2.1) to the function y defined above, and obtained

$$\sum_{j,k=1}^{n} l_j l_k \left(\psi_{\zeta_{jk}}(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi'_{\zeta_{jk}}(x)(x_i - d) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \psi_{\zeta_{jk}}(x) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \int_{x_i}^{d} (d-t)\alpha(t)\acute{\psi}_{\zeta_{jk}}(t)dt\right) \geq 0$$

Now, from (3.4) it follows that

$$\sum_{j,k=1}^{n} l_j l_k \Gamma(\zeta_{jk}) \geq 0$$

Therefore, the matrix $\left[\Gamma\left(\frac{\zeta_j+\zeta_k}{2}\right)\right]_{j,k=1}^{n}$ is positive semi-definite.

(iii) Follow from $(i), (ii)$ and Lemma 3.1.

(iv) Let $\Gamma(p) > 0$, then by Corollary 3.1 we have that $\Gamma(p)$ is log-convex i.e $p \to \log\Gamma(p)$ is convex and by (3.1) for $-\infty < r < s < p < \infty$ and taking $\psi(p) = \log\Gamma(p)$, we get

$$\log\Gamma(r)(p-s) + \log\Gamma(s)(r-p) + \log\Gamma(p)(s-r) \geq 0$$

After some calculation, it is equivalent to (3.5).

□

Theorem 2.6 enables us to define various types of means, because if the function $\dfrac{(s''_1 - \alpha s'_1)}{(s''_2 - \alpha s'_2)}$ has inverse, from (2.23) we have

$$\eta = \left(\frac{s''_1 - \alpha s'_1}{s''_2 - \alpha s'_2}\right)^{-1} \left(\frac{s_1(d)P_n + \sum_{i=1}^{n} p_i(x_i - d)s'_1(x_i) - \sum_{i=1}^{n} p_i s_1(x_i) - \sum_{i=1}^{n} p_i \int_{x_i}^{d}(d-t)\alpha(t)s'_1(t)dt}{s_2(d)P_n + \sum_{i=1}^{n} p_i(x_i - d)s'_2(x_i) - \sum_{i=1}^{n} p_i s_2(x_i) - \sum_{i=1}^{n} p_i \int_{x_i}^{d}(d-t)\alpha(t)s'_2(t)dt}\right), \ \eta \in I.$$

Let us observe differential equations $s_1''(\eta)-\alpha(\eta)s_1''(\eta)=\eta^{p-2}$ and $s_2''(\eta)-\alpha(\eta)s_2''(\eta)=\eta^{s-2}$. Then from (2.23) we have

$$\eta=\left(\frac{s_1(d)P_n+\sum_{i=1}^n p_i(x_i-d)s_1'(x_i)-\sum_{i=1}^n p_i s_1(x_i)-\sum_{i=1}^n p_i\int_{x_i}^d(d-t)\alpha(t)s_1'(t)dt}{s_2(d)P_n+\sum_{i=1}^n p_i(x_i-d)s_2'(x_i)-\sum_{i=1}^n p_i s_2(x_i)-\sum_{i=1}^n p_i\int_{x_i}^d(d-t)\alpha(t)s_2'(t)dt}\right)^{\frac{1}{p-s}}.$$

From (3.4) we have

$$\eta=\left(\frac{s(s-1)}{p(p-1)}\cdot\frac{d^pP_n+p\sum_{i=1}^n p_i(x_i-d)x_i^{p-1}-\sum_{i=1}^n p_ix_i^p}{d^sP_n+s\sum_{i=1}^n p_i(x_i-d)x_i^{s-1}-\sum_{i=1}^n p_ix_i^s}\right)^{\frac{1}{p-s}}.$$

Hence we have mean

$$M(\mathbf{x};p,s)=\left(\frac{s(s-1)}{p(p-1)}\cdot\frac{d^pP_n+p\sum_{i=1}^n p_i(x_i-d)x_i^{p-1}-\sum_{i=1}^n p_ix_i^p}{d^sP_n+s\sum_{i=1}^n p_i(x_i-d)x_i^{s-1}-\sum_{i=1}^n p_ix_i^s}\right)^{\frac{1}{p-s}} \tag{3.6}$$

where $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ is n-tuple of mutually different numbers greater than zero, $p\neq s$, $p,s\neq 0,1$. We have

$$M(\mathbf{x};p,s)=\left(\frac{\Gamma(p)}{\Gamma(s)}\right)^{\frac{1}{p-s}},$$

where Γ is defined by (3.3). All continuous extensions of (3.6) are now obvious but the case $p=s$:

$$M(x;p,s)=\exp\left(\frac{P_nd^p\ln p+\sum_{i=1}^n p_i(x_i-d)x_i^{p-1}+p\sum_{i=1}^n p_i(x_i-d)x_i^{p-1}\ln(p-1)-\sum_{i=1}^n p_ix_i^p\ln p}{d^pP_n+p\sum_{i=1}^n p_i(x_i-d)x_i^{p-1}-\sum_{i=1}^n p_ix_i^p}+\frac{1-2p}{p(p-1)}\right), \tag{3.7}$$

$$p\neq 0,1.$$

In the following theorem we give improvement and reversion of generalized Slater's inequality.

Theorem 3.2. *Let x_i, p_i, $d_p\in\mathbb{R}^+$ (i=1,…,n), $P_n=\sum_{i=1}^n p_i>0$, where $d_p=\frac{\sum_{i=1}^n p_ix_i\psi_p'(x_i)}{\sum_{i=1}^n p_i\psi_p'(x_i)}$. Let Z_p be defined by*

$$Z_p=\psi_p(d_p)-\frac{1}{P_n}\sum_{i=1}^n p_i\psi_p(x_i)-\int_{x_i}^{d_p}(d_p-t)\alpha(t)\psi_p'(t)dt.$$

Then

(i)

$$Z_p\geq[W(s;p)]^{\frac{p-r}{s-r}}[W(r;p)]^{\frac{s-p}{s-r}}, \tag{3.8}$$

for $-\infty<r<s<p<\infty$ and $-\infty<p<r<s<\infty$.

(ii)

$$Z_p \leq [W(s;p)]^{\frac{p-r}{s-r}}[W(r;p)]^{\frac{s-p}{s-r}}, \tag{3.9}$$

for $-\infty < r < p < s < \infty$.

$$\text{where } W(s;p) = \psi_s(d_p) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\psi_s^{'}(x_i)(x_i - d_p) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\psi_s(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\int_{x_i}^{d_p}(d_p - t)\alpha(t)\psi_s'(t)dt. \tag{3.10}$$

Proof. (i) By putting $d = d_p$ in (3.4), then $\Gamma(p)$ becomes Z_p and for $-\infty < r < s < p < \infty$, by putting $d = d_p$ in (3.5), we get

$$\left(\psi_s(d_p) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\psi_s^{'}(x_i)(x_i - d_p) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\psi_s(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\int_{x_i}^{d_p}(d_p - t)\alpha(t)\psi_s'(t)dt\right)^{p-r}$$

$$\leq \left(\psi_r(d_p) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\psi_r^{'}(x_i)(x_i - d_p) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\psi_r(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\int_{x_i}^{d_p}(d_p - t)\alpha(t)\psi_r'(t)dt\right)^{p-s} (Z_p)^{s-r}$$

$$\Rightarrow [W(s;p)]^{p-r} \leq [W(r;p)]^{p-s}[Z_p]^{s-r}$$

$$Z_p \geq [W(s;p)]^{\frac{p-r}{s-r}}[W(r;p)]^{\frac{p-s}{s-r}}.$$

Similarly for $-\infty < p < r < s < \infty$ (3.5) becomes

$$(\Gamma(r))^{s-p} \leq (\Gamma(p))^{s-r}(\Gamma(s))^{r-p}; \tag{3.11}$$

by putting $d_p = \frac{\sum_{i=1}^{n} p_i x_i \psi_p^{'}(x_i)}{\sum_{i=1}^{n} p_i\psi_p^{'}(x_i)}$ in (3.11), we have

$$[W(r;p)]^{s-p} \leq [W(s;p)]^{r-p}[Z_p]^{s-r}$$

$$\Rightarrow Z_p \geq [W(s;p)]^{\frac{p-r}{s-r}}[W(r;p)]^{\frac{s-p}{s-r}}$$

which is required.

(ii) for $-\infty < r < p < s < \infty$ (3.5) becomes

$$(\Gamma(s))^{p-r} \leq (\Gamma(p))^{s-r}(\Gamma(r))^{p-s}; \tag{3.12}$$

by setting $d = d_p$ in (3.12), we get (ii) by simple calculation.

□

Theorem 3.3. *Let* x_i, p_i, $d_p \in \mathbb{R}^+$ *(i=1,...,n),* $P_n = \sum_{i=1}^{n} p_i > 0$, *where* $d_p = \frac{\sum_{i=1}^{n} p_i x_i \psi_p^{'}(x_i)}{\sum_{i=1}^{n} p_i\psi_p^{'}(x_i)}$. *Then for every* $n \in \mathbb{N}$ *and for every* $\zeta_j \in \mathbb{R}$, $j \in \{1, 2, 3, \ldots, n\}$, *the matrices* $[W(\frac{\zeta_j+\zeta_k}{2}, \zeta_1)]_{j,k=1}^{n}$, $[W(\frac{\zeta_j+\zeta_k}{2}, \frac{\zeta_1+\zeta_2}{2})]_{j,k=1}^{n}$ *are positive semi-definite matrices. Particularly*

$$det[W(\frac{\zeta_j + \zeta_k}{2}, \zeta_1)]_{j,k=1}^{n} \geq 0, \tag{3.13}$$

$$det[W(\frac{\zeta_j + \zeta_k}{2}, \frac{\zeta_1 + \zeta_2}{2})]_{j,k=1}^{n} \geq 0, \tag{3.14}$$

where $W(s,t)$ *is defined by* (3.10).

Proof. By setting $d = d_{\zeta_1}$ and $d = d_{\frac{\zeta_1+\zeta_2}{2}}$ in Theorem 3.1(ii), we get the required results. □

Remark 3.1. We note that $W(p,p) = Z_p$. So by setting $n = 2$ in (3.13), we have special case of (3.8) for $p = \zeta_1$, $r = \frac{\zeta_1+\zeta_2}{2}$, $s = \zeta_2$ if $\zeta_1 < \zeta_2$ and for $p = \zeta_1$, $r = \zeta_2$, $s = \frac{\zeta_1+\zeta_2}{2}$ if $\zeta_2 < \zeta_1$. Similarly by setting $n = 2$ in (3.14), we have special case of (3.9) for $r = \zeta_1$, $s = \zeta_2$, $p = \frac{\zeta_1+\zeta_2}{2}$ if $\zeta_1 < \zeta_2$ and for $r = \zeta_2$, $s = \zeta_1$ $p = \frac{\zeta_1+\zeta_2}{2}$ if $\zeta_2 < \zeta_1$.

Remark 3.2. Related results for convex function have been given in [1,2].

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